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Mathematics

ICSE 10

Focused Theory | Topic Exercises | Chapter Exercises | Past Exams' Questions | Sample Papers

Edition 2022-23

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ARIHANT PRAKASHAN (School Division Series)

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A WORD
With The Readers

All in One ICSE Mathematics Class 10 has been written keeping in mind the needs of students studying in 10th ICSE. This book has been made in such a way that students will be fully guided to prepare for the exam in the most effective manner, securing higher grades.

The purpose of this book is to aid any ICSE student to achieve the best possible grade in the exam. This book will give you support during the course as well as advice you on revision and preparation for the exam itself. The material given in this book is presented in a clear & concise form and there are lots of questions for practice.

KEY FEATURES

- To make the students understand the chapter completely, each chapter has been divided into Individual Topics and each such topic has been treated as a separate chapter. Each topic has detailed theory, supported by Solved Examples, Notes, Tables, Figures, etc. and **Topic Exercise.**
- For the students to check their understanding of the chapter, **Chapter Exercise** has been given alongwith Topical Exercises.
- **Challengers** includes some special questions based on the pattern of Olympiad and other competitions to give the students a taste of the questions asked in competitions, these are not meant for school examinations.
- For complete practice of the examination **5 Sample Question Papers** based on latest pattern & syllabus have been given.
- At the end of the book **Latest ICSE Specimen Paper Semester I & II** and **ICSE Solved Paper Semester I 2021-22** have been given.

All in One Mathematics for ICSE 10th has all the material required for examination and will surely guide students to the Way to Success.

We are highly thankful to ARIHANT PRAKASHAN, MEERUT for giving us such an excellent opportunity to write this book. The role of Arihant DTP Unit and Proof Reading team is praise worthy in making of this book.

Huge efforts have been made from our side to keep this book error free, but inspite of that if any error or whatsoever is skipped in the book then that is purely incidental, apology for the same, please write to us about that so that it can be corrected in the further edition of the book. Suggestions for further improvement of the book will also be welcomed & incorporated in further editions.

In the end, we would like to say BEST OF LUCK to our readers!

Authors

PREVIEW

TOPICAL DIVISION

To make the student understand the chapter completely, each chapter has been divided into Individual Topics and each such topic has been treated as a separate chapter. Each topic has detailed theory, supported by Solved Examples, Notes, Tables, Figures, etc.

CHAPTER EXERCISE

It contains questions in that format in which these are asked in the examinations, i.e. 3 Marks Questions & 4 Marks Questions. All the Questions have Hints and Answers. Students can use these questions for practice and assess their understanding & recall of the chapter.

for ICSE 10th Examination is a complete book which can give you all Study, Practice & Assessment. It is hoped that this book will reinforce and extend your ideas about the subject and finally will place you in the ranks of toppers.

CHALLENGERS

It includes some special questions based on the pattern of Olympiad and other competitions to give the students a taste of the questions asked in competitions, these are not meant for school examinations.

CHALLENGERS*

A Set of Brain Teasing Questions for Your Mind Exercise

1. If α and β are the roots of the equation $x^2 + x - 6 = 0$ such that $\beta > \alpha$, then the product of the matrices $\begin{bmatrix} 0 & \alpha \\ \alpha & \beta \end{bmatrix}$ and $\begin{bmatrix} \beta+1 & 0 \\ -\beta & \alpha \end{bmatrix}$ is

(a) $\begin{bmatrix} 6 & 9 \\ -13 & -6 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 4 \\ 9 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 6 & 13 \\ 9 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} -5 & 4 \\ -9 & -2 \end{bmatrix}$

2. The matrices A and B, such that $AB = O$, but $A \neq O$ and $B \neq O$, are

(a) $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then the value of matrix A^5 is

(a) $\begin{bmatrix} 62 & 149 \\ -149 & -87 \end{bmatrix}$ (b) $\begin{bmatrix} 87 & 149 \\ 149 & 62 \end{bmatrix}$

(c) $\begin{bmatrix} -62 & -149 \\ 149 & 87 \end{bmatrix}$ (d) $\begin{bmatrix} 87 & 149 \\ 149 & -62 \end{bmatrix}$

4. If both $A + B$ and AB are defined, then which one of the following is true?

(a) A and B are rectangular matrices of same order
(b) A and B are square matrices of same order
(c) A and B are rectangular matrices of different order
(d) A and B are square matrices of different order

5. If $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then A^n (where, n is a natural number) is equal to

(a) $\begin{bmatrix} 3n & 0 \\ 0 & 3n \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $3^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $I_{2 \times 2}$

SAMPLE QUESTION PAPER 1

A HIGHLY SIMULATED SAMPLE QUESTION PAPER FOR ICSE CLASS X

MATHEMATICS

GENERAL INSTRUCTIONS

1. You will not be allowed to write during the first 15 minutes. This time is to be spent in reading the questions paper.
2. The time given at the head of this paper is the time allowed for writing the answers.
3. Attempt **all questions** from **Section A** and **any 4 questions** from **Section B**.

Time : 2.5 Hrs **Max. Marks : 80**

Section A [40 *Marks*]

1. (a) If both $(x-2)$ and $\left(x-\frac{1}{2}\right)$ are factors of $px^2 + 5x + r$, then show that $p = r$. [3]

(b) Ajay opened a cumulative time deposit account with OBC Bank for $\frac{3}{2}$ yr. If the rate of interest is 10% per annum and the bank pays ₹4662 on maturity, then find how much did Ajay deposit each month? [3]

(c) A shopkeeper buys a camera at a discount of 20% from the wholesaler, the printed price of the camera being ₹1600 and rate of sales tax in

2. (a) Solve the following inequation and represent the solution set on the number line.

$$4x - 19 < \frac{3x}{5} - 2 \leq \frac{-2}{5} + x, \forall x \in R$$ [3]

(b) The sum of the 5th and 7th terms of an AP is 52 and its 10th term is 46. Find the AP. [3]

(c) Prove that the angle between two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line segments joining the points of contact at the centre. [4]

ICSE EXAMINATION PAPER 2022

MATHEMATICS

GENERAL INSTRUCTIONS

1. All Questions are compulsory.
2. The marks intended for questions are given in brackets [ ].
3. Select the correct option for each of the following questions.

Time : 1 Hr 30 Min **Max. Marks : 40**

Section A (16 Marks) [16×1]

1. If $(x+2)$ is a factor of the polynomial $x^3 - kx^2 - 5x + 6$, then the value of k is

(a) 1 (b) 2
(c) 3 (d) -2

2. The solution set of the inequation $x - 3 \geq -5$, $x \in R$ is

(a) $\{x : x > -2, x \in R\}$ (b) $\{x : x \leq -2, x \in R\}$
(c) $\{x : x \geq -2, x \in R\}$ (d) $\{-2, -1, 0, 1, 2\}$

3. The product AB of two matrices A and B is possible if

(a) A and B have the same number of rows.
(b) the number of columns of A is equal to the number of rows of B
(c) the number of rows of A is equal to the number of columns of B
(d) A and B have the same number of columns.

4. If 70, 75, 80, 85 are the first four terms of an Arithmetic Progression, then the 10th terms is

(a) 35 (b) 25
(c) 115 (d) 105

5. The selling price of a shirt excluding GST is ₹ 800. If the rate of GST is 12%, then the total price of the shirt is

(a) ₹ 704 (b) ₹ 96
(c) ₹ 896 (d) ₹ 848

6. Which of the following quadratic equations has 2 and 3 as its roots?

(a) $x^2 - 5x + 6 = 0$ (b) $x^2 + 5x + 6 = 0$
(c) $x^2 - 5x - 6 = 0$ (d) $x^2 + 5x - 6 = 0$

LATEST ICSE SPECIMEN PAPER

A SAMPLE QUESTION PAPER FOR ICSE CLASS X (SEMESTER I)
ISSUED BY COUNCIL OF INDIAN SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS

GENERAL INSTRUCTIONS

1. All Questions are compulsory.
2. The marks intended for questions are given in brackets [ ].
3. Select the correct option for each of the following questions.

Time : 1 Hr 30 Min **Max. Marks : 40**

Section A (16 Marks) [16×1]

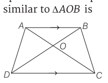
1. If matrix A is of order 3×2 and matrix B is of order 2×2, then the matrix AB is of order

(a) 3×2 (b) 3×1
(c) 2×3 (d) 1×3

2. The percentage share of SGST of total GST for an Intra-State sale of an article is

(a) 25% (b) 50%
(c) 75% (d) 100%

3. $ABCD$ is a trapezium with AB parallel to DC. Then, the triangle similar to ΔAOB is



(a) ΔADB (b) ΔACB
(c) ΔCOD (d) ΔCOB

SAMPLE QUESTION PAPERS, LATEST ICSE SPECIMEN PAPER AND 2021-22 SOLVED PAPER (Semester I)

To make the students practice in the real sense, we have provided 5 Sample Question Papers, exactly based on the latest pattern. Alongwith Latest ICSE Specimen Paper and Solved Paper have also been given.

CONTENTS

COURSE STRUCTURE

*There will be **one** paper of **two and a half** hours duration carrying 80 Marks and Internal Assessment of 20 Marks. The paper will be divided into two sections, **Section I** (40 Marks), **Section II** (40 Marks).*

***Section I** Will consists of compulsory short answer questions.*

***Section II** Candidates will be required to answer **four** out of **seven** questions.*

Commercial Mathematics

(i) Goods and Service Tax (GST)

Computation of tax including problems involving discounts, list-price, profit, loss, basic/cost price including inverse cases.
Candidates are also expected to find price paid by the consumer after paying State Goods and Service Tax (SGST) and Central Goods and Service Tax (CGST). The different rates as in vogue on different types of items will be provided. Problems based on corresponding inverse cases are also included.

(ii) Banking

Recurring Deposit Accounts : computation of interest and maturity value using the formula

$$I = P\,\frac{n\,(n+1)}{2\times 12}\times\frac{r}{100}$$

$$\text{MV} = P\times n + I$$

(iii) Shares and Dividends

(a) *Face/Nominal Value, Market Value, Dividend, Rate of Dividend, Premium.*

(b) *Formulae*

- *Income = Number of shares x Rate of dividend x FV.*
- *Return = (Income/Investment) x 100.*

Note** : Brokerage and fractional shares **not included.

Algebra

(i) Linear Inequations

Linear Inequations in one unknown for $x \in$ N, W, Z, R. Solving

- *Algebraically and writing the solution in set notation form.*
- *Representation of solution on the number line.*

(ii) Quadratic Equations in one variable.

(a) *Nature of roots*

- *Two distinct real roots, if $b^2 - 4ac > 0$.*
- *Two equal real roots, if $b^2 - 4ac = 0$.*
- *No real roots, if $b^2 - 4ac > 0$.*

(b) Solving Quadratic equations by

- *Factorisation.*
- *Using Formula.*

(c) Solving simple quadratic equation problems.

(iii) Ratio and Proportion

(a) *Proportion, Continued proportion, mean proportion*
(b) *Componendo, dividendo, alternendo, invertendo properties and their combinations.*
(c) *Direct simple applications on proportions only.*

(iv) Factorisation of polynomials

(a) *Factor Theorem.*
(b) *Remainder Theorem.*
(c) *Factorising a polynomial completely after obtaining one factor by factor theorem.*

Note : *f (x)* not to exceed degree 3.

(v) Matrices

(a) *Order of a matrix. Row and column matrices.*
(b) *Compatibility for addition and multiplication.*
(c) *Null and Identity matrices.*
(d) *Addition and subtraction of 2×2 matrices.*
(e) *Multiplication of a 2×2 matrix by*

- *a non-zero rational number.*
- *a matrix.*

(vi) Arithmetic and Geometric Progression

- *Finding their General term.*
- *Finding sum of their first 'n' terms.*
- *Simple Applications.*

(vii) Coordinate Geometry

(a) *Reflection*

- *Reflection of a point in a line : $x = 0, y = 0, x = a, y = a$, the origin.*
- *Reflection of a point in the origin.*
- *Invariant points.*

(b) *Coordinates expressed as (x, y), Section formula, Mid-point formula, Concept of slope, equation of a line, Various forms of straight lines.*

(i) *Section and Mid-point formula (Internal section only, coordinates of the centroid of a triangle included).*
(ii) *Equation of a line*

- *Slope-intercept form $y = mx + c$.*

- *Two-point form $(y - y_1) = m (x - x_1)$ Geometric understanding of 'm' as slope/gradient/tan θ where θ is the angle the line makes with the positive direction of the X-axis.*
  Geometric understanding of 'c' as the y-intercept/the ordinate of the point, where the line intercepts the Y-axis/the point on the line where x=0.
- *Conditions for two lines to be parallel or perpendicular.*

Simple applications of all the above.

Geometry

(a) Similarity

Similarity, conditions of similar triangles.

(i) As a size transformation.

(ii) Comparison with congruency, keyword being proportionality.

(iii) Three conditions : SSS, SAS, AA. Simple applications (proof not included).

(iv) Applications of Basic Proportionality Theorem.

(v) Areas of similar triangles are proportional to the squares of corresponding sides.

(vi) Direct applications based on the above including applications to maps and models.

(b) Loci

Loci : Definition, meaning, Theorems and constructions based on Loci.

(i) The locus of a point at a fixed distance from a fixed point is a circle with the fixed point as centre and fixed distance as radius.

(ii) The locus of a point equidistant from two intersecting lines is the bisector of the angles between the lines.

(iii) The locus of a point equidistant from two given points is the perpendicular bisector of the line joining the points.

Proofs not required

(c) Circles

(i) Angle Properties

- *The angle that an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circle.*
- *Angles in the same segment of a circle are equal (without proof).*
- *Angle in a semi-circle is a right angle.*

(ii) Cyclic Properties

- *Opposite angle of a cyclic quadrilateral are supplementary.*
- *The exterior angles of a cyclic quadrilateral is equal to the opposite interior angle (without proof).*

(iii) Tangent and Secant Properties

- *The tangent at any point of a circle and the radius through the point are perpendicular to each other.*
- *If two circles touch, the point of contact lies on the straight line joining their centres.*
- *From any point outside a circle two tangents can be drawn and they are equal in length.*
- *If two chords intersect internally or externally, then the product of the lengths of the segments are equal.*
- *If a chord and a tangent intersect externally, then the product of the lengths of segments of the chord is equal to the square of the length of the tangent from the point of contact to the point of intersection.*
- *If a line touches a circle and from the point of contact, a chord is drawn, the angles between the tangent and the chord are respectively equal to the angles in the corresponding alternate segments.*

Note : **Proofs of the theorems given above are to be taught unless specified otherwise.**

(iv) Constructions

- *Construction of tangents to a circle from an external point.*
- *Circumscribing and inscribing a circle on a triangle and a regular hexagon.*

Mensuration

Area and volume of solids-Cylinder, Cone and Sphere.

Three-dimensional solids-right circular cylinder, right circular cone and sphere. Area (total surface and curved surface) and Volume. Direct application problems including cost, Inner and Outer volume and melting and recasting method to find the volume or surface area of a new solid. Combination of solids included.

Note : **Problems on Frustum are not included.**

Trigonometry

(a) Using Identities to solve/prove simple algebraic trigonometric expressions

$sin^2 A + cos^2 A = 1$

$1 + tan^2 A = sec^2 A$

$1 + cot^2 A = cosec^2 A ; 0 \le A \le 90^\circ$

(b) Heights and distances : Solving 2-D problems involving angles of elevation and depression using trigonometric tables.

Note : Cases involving more than two right angled triangles excluded.

Statistics

Statistics-basic concepts, Mean, Median, Mode. Histograms and Ogive.

(a) Computation of

Measures of Central Tendency : Mean, median, mode for raw and arrayed data. Mean, median class and modal class for grouped data. (both continuous and discontinuous).*

**Mean by all 3 methods included*

Direct : $\frac{\Sigma fx}{\Sigma f}$

Shortcut : $A + \frac{\Sigma fd}{\Sigma f}$, where $d = x - A$

Step-deviation : $A + \frac{\Sigma ft}{\Sigma f} \times i$, where $t = \frac{x-A}{i}$

(b) Graphical Representation, Histograms and Less than Ogive.

- *Finding the mode from the histogram, the upper quartile, lower quartile and median etc. from the ogive.*
- *Calculation of inter Quartile range.*

Probability

- *Random experiments*
- *Sample space*
- *Events*
- *Definition of probability*
- *Simple problems on single events*

Note : SI units, signs, symbols and abbreviations

1. Agreed conventions

(a) *Units may be written in full or using the agreed symbols, but no other abbreviation may be used.*

(b) *The letters' is never added to symbols to indicate the plural form.*

(c) *A full stop is not written after symbols for units unless it occurs at the end of a sentence.*

(d) *When unit symbols are combined as a quotient, e.g. metre per second it is recommended that it should be written as m/s or as* ms^{-1}.

(e) *Three decimal signs are in common international use : the full point, the mid-point and the comma. Since, the full point is sometimes used for multiplication and the comma for spacing digits in large numbers, it is recommended that the mid-point be used for decimals.*

2. Names and symbols

In general			
Implies that	$\Rightarrow$	is logically equivalent to	$\Leftrightarrow$
Identically equal to	$\equiv$	is approximately equal to	>>
In set language			
Belongs to	$\in$	does not belong to	$\notin$
is equivalent to	$\leftrightarrow$	is not equivalent to	$\not\leftrightarrow$
union	$\cup$	intersection	$\cap$
universal set	ξ	is contained in,	$\subset$
natural (counting) number	N	the empty set	ϕ
		whole numbers	W
integers	Z	real numbersR	R
In measures			
Kilometre	km	Metre	m
Centimetre	cm	Millimetre	mm
Kilogram	kg	Gram	g
Litre	L	Centilitre	cl
Square kilometre	km^2	Square metre	m^2
Square centimetre	cm^2	Hectare	ha
cubic metre	m^3	Cubic centimetre	cm^3
kilometres per hour	km/h	Metres per second	m/s

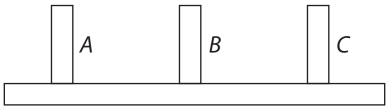
Internal Assessment

The minimum number of assignments :
Two assignments as prescribed by the teacher.

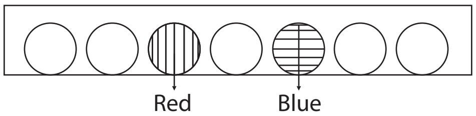
Suggested Assignments

- *Comparative newspaper coverage of different items.*
- Survey of various types of Bank accounts, rates of interest offered.
- Planning a home budget.

- Conduct a survey in your locality to study the mode of conveyance/Price of various essential commodities/favourite sports. Represent the data using a bar graph/histogram and estimate the mode.
- To use a newspaper to study and report on shares and dividends.
- Set up a dropper with ink in it vertical at a height say 20 cm above a horizontally placed sheet of plain paper. Release one ink drop; observe the pattern, if any, on the paper. Vary the vertical distance and repeat. Discover any pattern of relationship between the vertical height and the ink drop observed.
- You are provided (or you construct a model as shown)-three vertical sticks (size of a pencil) stuck to a horizontal board. You should also have discs of varying sizes with holes (like a doughnut). Start with one disc; place it on (in) stick *A*. Transfer it to another stick (*B* or *C*); this is one move (*m*). Now try with two discs placed in *A* such that the large disc is below and the smaller disc is above (number of discs = *n* = 2 now). Now transfer them one at a time in *B* or *C* to obtain similar situation (larger disc below). How many moves? Try with more discs (*n* = 1, 2, 3, etc.) and generalise.



- The board has some holes to hold marbles, red on one side and blue on the other. Start with one pair. Interchange the positions by making one move at a time. A marble can jump over another to fill the hole behind. The move (*m*) equal 3. Try with 2 (*n* = 2) and more. Find the relationship between *n* and *m*.



- Take a square sheet of paper of side 10 cm. Four small squares are to be cut from the corners of the square sheet and then the paper folded at the cuts to form an open box. What should be the size of the squares cut so that the volume of the open box is maximum?
- Take an open box, four sets of marbles (ensuring that marbles in each set are of the same size) and some water. By placing the marbles and water in the box, attempt to answer the question : do larger marbles or smaller marbles occupy more volume in a given space?
- An eccentric artist says that the best paintings have the same area as their perimeter (numerically). Let us not argue whether such sizes increase the viewer's appreciation, but only try and find what sides (in integers only) a rectangle must have if its area and perimeter are to be equal (Note there are only two such rectangles).
- Find by construction the centre of a circle, using only a 60-30 setsquare and a pencil.
- Various types of "cryptarithm".

Evaluation

The assignments/project work are to be evaluated by the subject teacher and by an External Examiner. (The External Examiner may be a teacher nominated by the Head of the school, who could be from the faculty, **but not teaching the subject in the section/class.** For example, a teacher of Mathematics of Class VIII may be deputed to be an External Examiner for class X, Mathematics projects).

The Internal Examiner and the External Examiner will assess the assignments independently.

Award of marks (20 Marks)

Subject Teacher (Internal Examiner)	:	10 Marks
External Examiner	:	10 Marks

The total marks obtained out of 20 are to be sent to the Council by the Head of the school.

The Head of the school will be responsible for the online entry of marks on Council's CAREERS portal by the due date.

TOP TIPS TO SCORE HIGHEST MARKS in class 10th exam. So, check out and apply these in your exams.

TOP TIPS

to Score the HIGHEST MARKS

ALWAYS FEEL POSITIVE

Positive attitude is the key to solve many of the problems which you face in your life. During exam time, this is an important feature to have in you for success and crack your exams with flying colours. Attitude is important for all students because it reflects your personality as well as your confidence or self-confidence. It always takes you to the top of everything, whether it is for exams or interviews or for your life. Positive attitude will take you through the door of success and make you feel full of self-confidence.

PLAN WELL FOR STUDYING

You must make a schedule for your studies followed by strict implementation of that schedule. Make that schedule detailing days or even hours when your exams are really close or it is high time for your exam.You must interact with your teachers for the important topics or topics which need more hard work or more time than other topics. Use last years' exam papers or sample papers for making a proper schedule for your studies.

You must study more or give more attention to the topics in which you feel you are not up to the mark or which your teachers recommended you to study more. You should study these topics first during your exam preparation.

JUST BEFORE THE EXAM

Never try to read anything or to study or cram just before the exam time, even if your friend asks you for some topic he has missed or left during preparation for the exams. Close your book an hour before the exam starts and feel relaxed and worry free and full of self-confidence. Also get up early in the morning and take another review of the important topics and make yourself filled with confidence, as confidence is the main key to score well. The night before the exam you should sleep as soon as possible to make your brain as well as body relax a bit and to be well prepared for the exams, as our brain too needs a rest to be fresh for the exam.

DISCUSS WHAT YOU LEARN

Find a friend or relative who has similar interests or who would enjoy hearing about your studies and let them know what's going on in your class.

CHART YOUR PROGRESS

Design your map of studying and you would see a certain satisfaction coming after watching your goals being accomplished. When times get hard, you can always turn to your chart and see how far you have reached.

DURING EXAM TIME

Check out all the things you require during exam time i.e., pen, pencil, sketch pens, rubber, sharpener. Each and every thing, whether it is small or big, matters a lot during your exam time. Read all the instructions carefully before starting the paper and keep them in your mind during exam time. Don't make any foolish mistake regarding your exam paper instructions.

ATTEMPTING THE EXAMINATION PAPER

Read out all the questions carefully before writing anything on the answer sheet and always start your answering from the questions which will carry maximum marks as well as which you think are tougher or need much time to think. When you start the exam from small questions, you will always feel the problem of questions left.

So, that's why time management is very much important during exams. You can write small questions even in the last 30 minutes but you will never be able to write enough for the large questions at the end, which will eventually result in sadness.

WHEN YOU FEEL STUCK DURING THE EXAM

There will also come a moment in your exam when you feel stuck with some questions or a single question. You just need to be relaxed and calm, don't panic in that situation and make yourself confident and try to think about the answer with a cool mind.

If you are not feeling like giving that answer at that time, make any sign or mark that question with pen and move on to the next question and try doing that question after you finish your paper but are still left with time. Never try to think about the 'stuck' question when you are writing the answer to any other question. This will reduce your concentration and when you feel no way out, just make a guess and attempt that question. This will leave you with something in the space you left for that question.

SOLUTION COPY SHOULD BE NEAT & CLEAN

Handwriting matters a lot for good or highest marks during your exam, as your writing makes the first impression on the checker's mind and makes your solution copy more filled with a glow for the examiner or checker of the answer sheet.

Underline the lines you feel important and want to attract the examiner's attention towards so that he/she can be able to make a right mindset about the answer given and also reward you with the full or maximum marks.

AFTER COMPLETION OF EXAM PAPER

When you end the exam paper, don't feel like running out of the examination hall. Sit there and review each and every answer before depositing your answer book with the invigilator. Also, look for the questions you left during answering or in which you got stuck. Search for the mistakes you have done during writing and turn towards the hardest question you think and also feel uncomfortable in answering. Review it and look to add any other important lines you missed in that.

ART OF WRITING ANSWERS : GIVE YOUR BEST SHOT

- One can practice answering in previous test papers or sample papers to get used to the manner in which one has to write answers in the exams.
- Make sure that you answer the question asked and not answer what you hoped or wished the question would be.
- Examiners expect to the point and correct answers. Resist the temptation to write everything or writing beyond limits.
- Keep your answer stepwise. Some of you will be surprised to know that the board gives rather detailed dictates on how to evaluate the answer sheet. Try not to exceed the word limit.
- Write your answers in a logical systematic manner. Use examples, facts, figures, quotations, tables etc. wherever necessary to substantiate your answers. Give appropriate heading where necessary.
- Add a touch of class by putting extra information that indicate your being very knowledgeable and put this separately near the end, so that it is read just before giving the marks, especially in an essay type question like new trends etc.

CHAPTER

1

Goods and Service Tax

The Central (or State) Government provides various types of facilities to public, such as construction and maintenance of roads, schools, hospitals, etc. In order to provide above facilities, the government needs money which is collected by imposing different types of taxes on peoples and organisations such as, income tax, goods and service tax, etc.

Chapter Objectives

- Goods and Services Tax (GST)
- Different Types of Taxes Used in GST
- Objectives and Advantages of GST

Goods and Services Tax (GST)

It is an indirect tax levied by Central Government. In this tax there are so many tax merged such as Excise Duty, Custom Duty, Service Tax, Value Added Tax, Entertainment Tax and Lottery Tax etc. The purpose of GST is to make a one nation and one tax.

Important Terms Related to GST

There are various terms related to GST, which are as follow

(i) **Manufacturer** The person who produce goods for sale, is called manufacturer.

(ii) **Dealer** The person, who purchases goods for resale, is known as a dealer (trader).

(iii) **Turnover** The total amount received from sale of goods (excluding tax) by a dealer during any fixed period, is called turnover.

(iv) **Intra State Sales** Suppose a person do a business and they sales their goods (items) and provide their services in the same state (or union territory), then it is said to be Intra state sales.

(v) **Inter State Sales** Suppose a person do a business and they sales their goods (items) and provide their services outside the state (or union territory), then it is said to be Inter state sales.

(vi) **Input GST** The tax paid by a dealer on his/her purchase of goods and providing services, is called Input GST.

(vii) **Output GST** The tax charged by a dealer on his/her sales of goods and providing services is called output GST.

Different Types of Taxes Used in GST

In this system, there are three taxes applicable, which are given below

(i) **State (or Union Territory) Goods and Service Tax** (SGST or UTGST) This tax is collected by the State (or union territory) Government on an intra-state sale.

e.g. Suppose any goods is manufacturing in Uttar Pradesh and sales this goods also in Uttar Pradesh, then tax on this transaction is said to be SGST.

(ii) **Central Goods and Services Tax** (**CGST**) This tax is collected by the Central Government on an intra state sale.

Both SGST (or UTGST) and CGST are levied on intra state (i.e. same state) sales of goods and services. In intra-state sales, GST is equally divided between Central and State Governments.

e.g. Suppose a dealer of Tamil Nadu sells some goods at the rate of ₹ 4000 to the consumer in the same state (i.e. Tamil Nadu). Suppose GST is charged at the rate of 18% on that goods, the GST will comprises of CGST at the rate 9% and SGST at the rate of 9%. Therefore, the seller will collect the CGST amount of ₹ 360 (i.e. 9% of ₹ 4000) and this amount goes to the account of Central Government and seller will also collect the SGST amount of ₹ 360 (i.e. 9% of ₹ 4000) which will go the account of State Government of Tamil Nadu.

Hence, the dealer collects the 18% GST amount on ₹ 4000 (i.e. ₹ 720), which will equally (i.e. 360 each) distribute in State (i.e. Tamil Nadu) and Central Government.

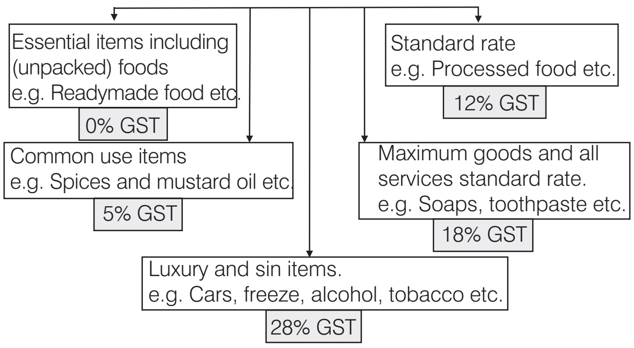
(iii) **Integrated Goods and Services Tax** (IGST) It is levied on inter state sales of goods and services outside the state. This tax is also levied on import of goods and services from one country to another country.

This tax is collected only by the Central Government for inter state sale.

e.g. Suppose a dealer from Gujarat sells goods worth of ₹ 8000 to a another dealer of Rajasthan. Suppose rate of GST is 12% on the goods, then the seller will collect 12% of ₹ 8000 (i.e. 960) under as IGST and the whole amount of IGST will go the Central Government.

Rate Structure for GST

Generally, structure of goods and services are divided into 5 slabs, which are given below



Please note that rates of GST given here are according to the latest information, these rates may be changed by the Government.

Input and Output GST Credit

When any supply of services or goods is supplied to a taxable person, then GST is charged, which is known as input tax, it is changed against of the output GST (i.e. GST collected) as follows

Input GST credit		Output GST
(i)	CGST credit	First CGST, then IGST
(ii)	SGST/UTGST credit	First SGST/UTGST, then IGST
(iii)	IGST credit	First IGST, second CGST, then SGST/UTGST

Both Central Government and State Government distribute the set off input GST against off output GST. While saling and purchasing of goods and services in each deal, dealer has to pay net GST (output GST – input GST).

Objectives and Advantages of GST

Some of the objectives and advantages are given below

(i) The important advantage is that to reduce the multiplicity of taxes and make a unified common national market.

(ii) Through GST, the tax system becomes more transparent, regular and predictable.

(iii) **Ease of doing business** It means their are some administrative rule set up between the Central Government and State Government, so that their is no interruption of doing business.

(iv) **Reduce Tax Evasion** Some of the persons are evading tax. So reduce tax evasion, use GST. In this procedure each tax payer registered under GST to make a GST return file electronically for each transaction (either purchase or sale) and this file should match with the input GST credit against the output GST and lastly paid net GST.

Basic Terms Required to Solve the Questions

There are various basic terms, which are as follow

(i) **Cost Price** (CP) The price, at which an article is bought, is called its cost price. All the overhead expenses in the transaction like freight, damage, etc., are added to the cost price.

(ii) **Selling Price** (SP) The price, at which an article is sold, is called its selling price.

(iii) **Profit** When selling price of a commodity is more than its cost price, then we are in profit and it is given by Profit = SP − CP and Profit% $= \dfrac{\text{Profit}}{\text{CP}} \times 100$

(iv) **Loss** When selling price of a commodity is less than its cost price, then we are in loss and it is given by

$$\text{Loss} = \text{CP} - \text{SP and Loss\%} = \frac{\text{Loss}}{\text{CP}} \times 100$$

(v) **Marked Price** (MP) The price of article excluding tax, is known as marked price. It is also, known as list price, quoted price, printed price, catalogue price and basic price.

(vi) **Discount** The reduction in price of object given by shopkeeper to the customer, is known as discount.

$$\therefore \quad \text{Discount} = \frac{\text{Rate of discount} \times \text{Marked price}}{100}$$

or Discount = Marked price − Selling price

Example 1. Mr. Sharma goes to a shop and buy a Jacket having cost ₹ 1180 (list price). The rate of GST 18%. He tells the shopkeeper to reduce the price such an extent that he has to pay ₹ 1180 inclusive of GST. Find the reduction needed in the price of the jacket.

Sol. Let the reduced price of jacket be ₹ x.

Then, amount of GST on ₹ x = 18% of $x = \dfrac{18}{100} \times x$

$\therefore$ Mr. Sharma pays the amount for jacket

$$= ₹\, x + ₹\, \frac{18}{100} x = ₹ \left(1 + \frac{18}{100}\right) x$$

$$= ₹ \left(1 + \frac{9}{50}\right) x = ₹ \left(\frac{59}{50}\right) x$$

According to the given condition,

$$\frac{59}{50} x = 1180$$

$$\Rightarrow \quad x = \frac{1180 \times 50}{59} = ₹\, 1000$$

$\therefore$ Reduced price of the jacket = ₹ 1000

Thus, the reduction needed in the price of jacket

$$= ₹\,(1180 - 1000) = ₹\, 180$$

Example 2. The price of a motorcycle is ₹ 44880 including tax (under GST) at the rate of 18% on its listed price. A buyer asks for a discount on the listed price so that after charging GST, the selling price of motorcycle becomes equal to the listed price. Find the discount amount in which the seller has to allow for the deal.

Sol. Let the listed price of motorcycle be ₹ x and the discount be ₹ y.

Amount of GST on ₹ x = 18% of ₹ x

$$= \frac{18}{100} \times x = \frac{9}{50} x$$

Selling price of the motorcycle including tax

$$= ₹ \left(x + \frac{9}{50} x\right) = ₹ \left(\frac{59}{50} x\right)$$

According to the given condition,

$$\frac{59}{50} x = ₹\, 44880$$

$$\Rightarrow \quad x = \frac{44880 \times 50}{59} = ₹\, 38034$$

$\therefore$ List price of the motorcycle = ₹ 38034

Now, the reduced price of the motorcycle

$$= ₹\,(38034 - y)$$

Amount of GST on ₹ $(38034 - y)$

$$= 18\% \text{ of } (38034 - y)$$

$$= \frac{18}{100} \times (38034 - y) = \frac{9}{50}(38034 - y)$$

$\therefore$ New selling price of the motor cycle including GST

$$= (38034 - y) + \frac{9}{50}(38034 - y)$$

$$= \left(1 + \frac{9}{50}\right)(38034 - y)$$

$$= \frac{59}{50}(38034 - y)$$

According to the given condition,

Selling price of motorcycle including GST

= Listed price of motorcycle

$$\therefore \quad \frac{59}{50}(38034 - y) = 38033$$

$$\Rightarrow \quad 2244006 - 59y = 1901650$$

$$\Rightarrow \quad 59y = 342356 \Rightarrow y = ₹\, 5803$$

Hence, the amount of discount is ₹ 5803.

Example 3. A shopkeeper buys an dinning table whose list price is ₹ 30000 at some rate of discount from a wholesaler. He sells the article to a consumer at the list price and charges GST at the rate of 12%. If the sales are intra state and the shopkeeper has to pay tax (under GST) of ₹ 36, to the State Government, find the rate of discount at which he bought the article from the wholesaler.

Sol. We have sales are intra state and rate of GST is 12%, so GST comprises of CGST at 6% and SGST at 6%.

Let the amount of discount be ₹ P. Then, selling price of the article (excluding tax) by the wholesaler

$$= \text{Listed price} - \text{Discount} = ₹(30000 - P)$$

Now, amount of GST received by the wholesaler from the shopkeeper,

$$\text{CGST} = 6\% \text{ of } ₹\,(30000 - P) = \frac{6}{100} \times (30000 - P)$$

$$= \frac{3}{50}(30000 - P) = ₹ \left(1800 - \frac{3P}{50}\right)$$

and SGST = 6% of $(30000 - P) = ₹ \left(1800 - \dfrac{3P}{50}\right)$

Thus, input GST amount for shopkeeper,

$$\text{Input CGST} = ₹ \left(1800 - \frac{3P}{50}\right)$$

and input SGST $= ₹ \left(1800 - \dfrac{3P}{50}\right)$

The shopkeeper sells the article to a consumer at the list price i.e. ₹ 30000.

Amount of GST collected by the shopkeeper from the consumer,

$$\text{CGST} = 6\% \text{ of } ₹\, 30000 = \frac{6}{100} \times 30000 = ₹\, 1800$$

and $\text{SGST} = 6\% \text{ of } ₹\, 30000 = \frac{6}{100} \times 30000 = ₹\, 1800$

Output GST amount for shopkeeper,

Output CGST = ₹ 1800

and output SGST = ₹ 1800

$\therefore$ Amount of tax (under GST) paid by the shopkeeper to the State Government

$$= \text{Output SGST} - \text{Input SGST}$$

$$= ₹\, 1800 - ₹\left(1800 - \frac{3P}{50}\right) = ₹\, \frac{3P}{50}$$

According to the given condition,

$$\frac{3P}{50} = 36 \Rightarrow P = ₹\, 600$$

$\therefore$ Amount of discount = ₹ 600

Hence, the shopkeeper gets a discount of ₹ 600 from the wholesaler.

$$\therefore \text{ Rate of discount} = \left(\frac{\text{Discount amount}}{\text{Article printed price}} \times 100\right)\%$$

$$= \left(\frac{600}{30000} \times 100\right)\% = 2\%$$

Example 4. Manufacturer Suresh sells a television to a dealer Anil for ₹ 20000. The dealer Anil sells it to a consumer at a profit of ₹ 2000. If the sales are intra state and the rate of GST is 12%, find

(i) the amount of tax (under GST) paid by the dealer Anil to the Central Government.

(ii) the amount of tax (under GST) received by the State Government.

(iii) the amount that the consumer pays for the television.

Sol. We have sales are intra state and the rate of GST is 12%, so GST comprises of CGST at 6% and SGST at 6%.

Also, given manufacturer Suresh sells the television to dealer Anil for ₹ 20000.

The GST amount collected by manufacturer Suresh from dealer Anil are

$$\text{CGST} = 6\% \text{ of } ₹\, 20000 = \frac{6}{100} \times 20000 = ₹\, 1200$$

and $\text{SGST} = 6\% \text{ of } ₹\, 20000 = \frac{6}{100} \times 20000 = ₹\, 1200$

$\therefore$ Amount of input GST for dealer Anil,

input CGST = ₹ 1200, input SGST = ₹ 1200

Thus, manufacturer Suresh will pay ₹ 1200 for CGST and ₹ 1200 for SGST.

As the dealer Anil sells the television to a consumer at a profit of ₹ 2000, the selling price of television by dealer Anil (or cost price of television for the consumer)

$$= ₹\, 20000 + ₹\, 2000 = ₹\, 22000$$

The amount of GST collected by dealer Anil (or paid by consumer)

$$\text{CGST} = 6\% \text{ of } ₹\, 22000 = \frac{6}{100} \times ₹\, 22000 = ₹1320$$

and $\text{SGST} = 6\% \text{ of } ₹\, 22000 = \frac{6}{100} \times ₹\, 22000 = ₹\, 1320$

$\therefore$ Amount of output GST for dealer Anil,

Output CGST = = ₹ 1320

Output SGST = ₹ 1320

(i) Amount of tax (under GST) paid by dealer Anil to the Central Government

= CGST paid by dealer Anil to the Central Government

= Output CGST – Input CGST

= ₹ 1320 – ₹ 1200 = ₹ 120

(ii) Amount of SGST paid by dealer Anil

= Output SGST – Input SGST

= ₹ 1320 – ₹ 1200 = ₹ 120

(iii) Amount that the customer pays for the television

= Cost price of television for customer + CGST paid by consumer + SGST paid by consumer

= ₹ 22000 + ₹ 1320 + ₹ 1320 = ₹ 24640

Example 5. A shopkeeper buys an item whose printed price is ₹ 3000 from a wholesaler at a discount of 15% and sells it to a consumer at the printed price. If the sales are intra state and the rate of GST is 12%, find

(i) the price of the item inclusive of GST at which the shopkeeper bought it.

(ii) the amount of tax (under GST) paid by the shopkeeper to the State Government.

(iii) the amount of tax (under GST) received by the Central Government.

(iv) the amount which the consumer pays for the item.

Sol. We have sales are intra state and the rate of GST is 12%, so GST comprises of CGST at 6% and SGST at 6%.

(i) Also, given printed price of an item = ₹ 3000

and rate of discount = 15%

$\therefore$ Amount of discount = 15% of ₹ 3000

$$= \frac{15}{100} \times 3000 = ₹\, 450$$

$\therefore$ Shopkeeper paid the amount to the wholesaler

$$= ₹\, 3000 - ₹\, 450 = ₹\, 2550$$

$\therefore$ Shopkeeper paid the amount of GST to the wholesaler,

$$\text{CGST} = 6\% \text{ of } ₹\, 2550$$

$$= \frac{6}{100} \times 2550 = ₹\, 153$$

and $\text{SGST} = 6\% \text{ of } ₹\, 2550 = \frac{6}{100} \times 2550 = ₹\, 153$

$\therefore$ Price of an item inclusive GST at which the shopkeeper bought it = ₹ 2550 + ₹ 153 + ₹153 = ₹ 2856

(ii) Since, the wholesaler will pay ₹ 153 to the CGST and ₹ 153 to the SGST.

$\therefore$ Amount of input GST for shopkeeper, CGST = ₹ 153 and SGST = ₹ 153

Since, the shopkeeper sells the item to a consumer at printed price of ₹ 3000.

$\therefore$ Shopkeeper collect the GST amount (or paid by consumer),

CGST = 6% of ₹ 3000 = $\frac{6}{100} \times 3000$ = ₹ 180

and SGST = 6% of ₹ 3000 = $\frac{6}{100} \times 3000$ = ₹180

$\therefore$ The shopkeeper has amount for output GST,

CGST = ₹ 180 and SGST = ₹ 180

Thus, amount of tax (under GST) paid by the shopkeeper to the State Government = Shopkeeper paid the SGST amount to the State Government

= Output SGST–Input SGST

= ₹ 180 – ₹ 153 = ₹ 27

(iii) Now, shopkeeper paid the CGST amount

= Output CGST – Input CGST

= ₹ 180 – ₹153 = ₹ 27

$\therefore$ Amount of tax (under GST) received by the Central Government

= Wholesaler pays CGST + Shopkeeper pays CGST

= ₹ 153 + ₹ 27 = ₹ 180

(iv) Now, amount which the consumer pays for the item

= Cost price of item for consumer + Consumer pays GST

= ₹ 3000 + Consumer pays CGST + Consumer pays SGST

= ₹ 3000 + ₹ 180 + ₹180 = ₹ 3360

Example 6. The printed price of a panasonic air conditioner (PAC) is ₹ 35000. The wholesaler allows a discount of 14% to a dealer. The dealer sells the air conditioner to a consumer at a discount of 6% on the marked price. If the sales are intra state and rate of GST is 18%, then find how much

(i) the dealer pays the amount of tax (under GST) to the Central and State Governments?

(ii) Central and State Governments receive the amount of tax (under GST)?

(iii) the consumer pays the total amount (inclusive of tax) for the panasonic AC?

Sol. We have, sales are intra state and the rate of GST is 18%, so GST comprises of CGST at 9% and SGST at 9%.

Also given, the wholesaler sells the panasonic AC to a dealer at a discount of 14%.

$\therefore$ Selling price of panasonic AC by the wholesaler (excluding tax)

= ₹ (1 – 14%) of ₹ 35000

$= \left(1 - \frac{14}{100}\right) \times 35000$ = ₹ $\frac{86}{100} \times 35000$ = ₹30100

It implies that, cost price of Panasonic AC to the dealer (excluding tax) = ₹ 30100

$\therefore$ The wholesaler collect the GST amount (or dealer pays to wholesaler),

CGST = 9% of ₹ 30100 = $\frac{9}{100} \times 30100$ = ₹ 2709

and SGST = 9% of ₹ 30100 = $\frac{9}{100} \times 30100$ = ₹ 2709

Therefore, the wholesaler will pay ₹ 2709 amount of CGST to the Central Government and ₹ 2709 amount of SGST to the State Government.

Amount of input GST of the dealer,

CGST = ₹ 2709 and SGST = ₹ 2709

Since, the dealer sells the Panasonic AC to a consumer at a discount of 6% on the marked price, therefore the dealer sales the panasonic AC (excluding tax)

$= \left(1 - \frac{6}{100}\right) \times 35000 = \left(1 - \frac{3}{50}\right) \times 35000$

$= \frac{47}{50} \times 35000$ = ₹ 32900

$\therefore$ Cost price of panasonic AC to the consumer (excluding tax) = ₹ 32900

$\therefore$ Dealer collects the GST amount (or consumer pay the amount to the dealer)

CGST = 9% of 32900

$= \frac{9}{100} \times 32900$ = ₹ 2961

and SGST = 9% of ₹ 32900

$= \frac{9}{100} \times 32900$ = ₹ 2961

Amount of output GST for the dealer,

CGST = ₹ 2961 and SGST = ₹ 2961

$\therefore$ Dealer pays the amount of tax (under GST) to the Central Government = Output CGST – Input CGST

= ₹ 2961 – ₹ 2709 = ₹ 252

(i) Dealer pays the amount of tax (under GST) to the State Government = Output SGST – Input SGST

= ₹ 2961 – ₹ 2709 = ₹ 252

(ii) Now, Central Government receive the amount of tax (under GST)

= Wholesaler pays CGST + Dealer pays CGST

= ₹ 2709 + ₹ 252 = ₹ 2961

And the State Government receive the amount of tax (under GST)

= Wholesaler pays SGST + Dealer pays SGST

= ₹ 2709 + ₹ 252 = ₹ 2961

(iii) The customer pays the total amount (inclusive of tax) for the panasonic AC

= Cost of panasonic AC + Consumer pays GST

= ₹ 32900 + Consumer pays CGST + Consumer pays SGST

= ₹ 32900 + ₹ 2961 + ₹ 2961 = ₹ 38822

Example 7. The printed price of an item is ₹ 50000. The manufacturer allows a discount of 20% to a dealer Tarun. The dealer Tarun sells the item to another dealer Sachin at a discount of 5% on the marked price. The dealer Sachin sells is to a consumer at 2% above the printed price. Suppose all the sales are intra state and the rate of GST is 12%.

(i) How much the dealer Tarun pays the price of the item inclusive tax (under GST)?

(ii) How much the dealer Sachin pays the price of an item inclusive tax (under GST)?

(iii) How much the consumer pays the amount for an item?

(iv) Find the amount of tax (under GST) paid by dealer Tarun to the Central Government.

(v) The amount of tax (under GST) paid by dealer Sachin to the State Government.

(vi) How much tax amount (under GST) collects by the Central Government?

Sol Here, sales are intra-state and rate of GST is 12%, so GST comprises of CGST at 6% and SGST at 6%.

(i) Also given, the printed price of an item is ₹ 50000.

Since, the manufacturer sells the item to the dealer Tarun at 20% discount. Therefore, the selling price of an item by manufacturer to the dealer Tarun

$$= ₹\left(1-\frac{20}{100}\right)\times 50000$$

$$= \left(1-\frac{1}{5}\right)\times 50000 = \frac{4}{5}\times 50000$$

$$= ₹\ 40000$$

∴ Manufacturer collect the GST amount from dealer Tarun (or dealer Tarun pays to the manufacturer),

$$\text{CGST} = 6\%\text{ of } ₹\ 40000 = \frac{6}{100}\times 40000 = ₹\ 2400$$

$$\text{and SGST} = 6\%\text{ of } ₹\ 40000 = \frac{6}{100}\times 40000 = ₹\ 2400$$

∴ Dealer Tarun pays the price of the item inclusive tax (under GST) = Cost price of an item for dealer Tarun + Dealer Tarun pays GST (i.e. CGST and SGST)

$$= ₹\ 40000 + ₹\ 2400 + ₹\ 2400 = ₹\ 44800$$

(ii) Since, dealer Tarun sells the item to dealer Sachin at a discount of 5% on the marked price. Therefore, the selling price of an item by dealer Tarun to dealer Sachin.

$$= ₹\left(1-\frac{5}{100}\right)\times 50000 = \left(1-\frac{1}{20}\right)\times 50000$$

$$= \frac{19}{20}\times 50000 = ₹\ 47500$$

∴ GST amount collected by dealer Tarun from dealer Sachin (or dealer Sachin pays to dealer Tarun),

$$\text{CGST} = 6\%\text{ of } ₹\ 47500$$

$$= \frac{6}{100}\times 47500 = ₹\ 2850$$

$$\text{and SGST} = 6\%\text{ of } ₹\ 47500$$

$$= \frac{6}{100}\times 47500 = ₹\ 2850$$

∴ Dealer Sachin pays the price of an item inclusive tax (under GST) = Cost price of an item for dealer Sachin + Dealer Sachin pays GST (i.e. CGST and SGST)

$$= ₹\ 47500 + ₹\ 2850 + ₹\ 2850 = ₹\ 53200$$

(iii) Since, the dealer Sachin sells the item to the consumer at 2% above the printed price. Therefore, the selling price of an item by dealer Sachin to a consumer

$$= ₹\left(1+\frac{2}{100}\right)\times 50000 = \left(1+\frac{1}{50}\right)\times 50000$$

$$= \frac{51}{50}\times 50000 = ₹\ 51000$$

∴ Dealer Sachin collects the GST amount from consumer (or consumer pays the dealer Sachin)

$$\text{CGST} = 6\%\text{ of } ₹\ 51000 = \frac{6}{100}\times 51000 = ₹\ 3060$$

$$\text{and SGST} = 6\%\text{ of } ₹\ 51000 = \frac{6}{100}\times 51000 = ₹\ 3060$$

∴ Consumer pays the amount for an item

= Cost price of an item for consumer + Consumer pays GST (i.e. SGST and CGST)

$$= ₹\ 51000 + ₹\ 3060 + ₹\ 3060 = ₹\ 57120$$

(iv) ∵ Dealer Tarun has input CGST = ₹ 2400

and Tarun dealer has output CGST = ₹ 2850

∴ Amount of tax (under GST) paid by dealer Tarun to the Central Government.

$$= \text{Output CGST} - \text{Input CGST}$$

$$= 2850 - 2400 = ₹\ 450$$

(v) ∵ Dealer Sachin has input SGST = ₹ 2850

and dealer Sachin has output SGST = ₹ 3060

∴ Amount of tax (under GST) paid by dealer Sachin to the State Government

$$= \text{Output SGST} - \text{Input SGST}$$

$$= 3060 - 2850 = ₹\ 210$$

(vi) Tax amount (under GST) collects by the Central Government.

= Manufacturer collects CGST amount + Dealer Tarun pays the CGST amount to the Central Government + Dealer Sachin pays the CGST amount to the Central Government

$$= ₹\ 2400 + ₹\ 450 + ₹\ 210 = ₹\ 3060$$

Example 8. The printed price of modular table is ₹ 2000. A wholesaler in Uttar Pradesh buys the modular table from a manufacturer in Rajasthan at a discount of 10% on the printed price. The wholesaler sells the modular table to a retailer in Madhya Pradesh at 20% above the marked price. Suppose, the rate of GST on modular table is 5%. Find

(i) the price of modular table inclusive tax (under GST) in which the wholesaler bought.

(ii) the price of modular table inclusive tax (under GST) in which the retailer bought.

(iii) how much the tax (under GST) paid by the wholesaler to the Central Government?

(iv) the total tax (under GST) collected by the Central Government.

Sol Given, printed price of modular table = ₹ 2000 and rate of GST on sales or purchase of modular table = 5%

Here, we see that both the given sales from manufacturer to wholesaler and wholesaler to retailer are inter state, so IGST is levied on these sales at 5%.

(i) Also given, wholesaler buys the modular table from a manufacturer = 10%. Discount on the printed price.

∴ Cost price of the modular table to the wholesaler

$$= ₹\left(1-\frac{10}{100}\right)\times 2000 = \left(1-\frac{1}{10}\right)\times 2000$$

$$= \frac{9}{10}\times 2000 = ₹\ 1800$$

Manufacturer collects the IGST amount from wholesaler (or wholesaler pays to manufacturer)

$$= 5\%\text{ of } ₹\ 1800 = \frac{5}{100}\times 1800 = ₹\ 90$$

It implies that input IGST for wholesaler is ₹ 90.

$\therefore$ Price of modular table inclusive tax (under GST) in which the wholesaler bought = Cost of modular table for wholesaler + Wholesaler pays IGST.

$$= ₹\ 1800 + ₹\ 90 = ₹\ 1890$$

(ii) Since, the wholesaler sells the modular table to the retailer at 20% above the marked price. Therefore, the selling price of the modular table to the wholesaler

$$= ₹\left(1+\frac{20}{100}\right)\times ₹\ 2000$$
$$=\left(1+\frac{1}{5}\right)\times 2000$$
$$=\frac{6}{5}\times 2000 = ₹\ 2400$$

$\therefore$ Cost price (excluding tax) of the modular table to the retailer = ₹ 2400

$\therefore$ Wholesaler collects the GST amount from retailer (or retailer pays to wholesaler)

$$= 5\% \text{ of } ₹\ 2400$$
$$=\frac{5}{100}\times 2400 = ₹\ 120$$

$\therefore$ Price of modular table inclusive tax (under GST) in which the retailer bought = Cost of modular table for retailer + Retailer pays IGST

$$= ₹\ 2400 + ₹\ 120 = ₹\ 2520$$

(iii) $\because$ Wholesaler has output GST = ₹ 120

$\therefore$ Tax paid by the wholesaler to the Central Government

$$= \text{Output IGST} - \text{Input IGST}$$
$$= ₹\ 120 - ₹\ 90 = ₹\ 30$$

(iv) Total tax (under GST) collected by the Central Government = Manufacturer collects the IGST amount + Wholesaler collects the IGST amount

$$= ₹\ 90 + ₹\ 30 = ₹\ 120$$

Note In this transaction, none of the State Government gets any tax (under GST). The total tax goes to the Central Government.

Example 9. A shopkeeper who lives in Rajasthan buys an article at the printed price of ₹ 35000 from a wholesaler who lives in Gujarat. The shopkeeper sells the article to a consumer in Rajasthan at a profit of 22% on the basic cost price. If the rate of GST is 12%, find

(i) the price of an article inclusive tax (under GST) in which the shopkeeper bought it.

(ii) how much the amount of tax (under GST) paid by the shopkeeper to Governments?

(iii) the amount of tax (under GST) received by Gujarat Government.

(iv) the amount of tax (under GST) received by Central Government.

(v) the amount which the consumer pays for the article.

Sol. Given, shopkeeper buys an article from wholesaler

$$= ₹\ 35000$$

and rate of GST = 12%

Here, transaction is one state to another state, so IGST is levied on sale.

$\therefore$ Wholesaler collects the IGST amount from shopkeeper

$$= 12\% \text{ of } ₹\ 35000$$
$$=\frac{12}{100}\times 35000 = ₹\ 4200$$

(i) Price of an article inclusive tax (under GST) in which the shopkeeper bought = Cost price of article for shopkeeper + Shopkeeper pays GST amount

= ₹ 35000 + Shopkeeper pays the IGST amount to the wholesaler

$$= ₹\ 35000 + ₹\ 4200$$
$$= ₹\ 39200$$

(ii) Since, the wholesaler will pay IGST amount ₹ 4200 to the Central government.

$\therefore$ Shopkeeper has input IGST amount = ₹ 4200

As the shopkeeper sells the article to a consumer at a profit of 22% on the basic cost price, therefore the selling price of the article by the shopkeeper

$$=\left(1+\frac{22}{100}\right)\times 35000$$
$$=\frac{122}{100}\times 35000 = ₹\ 42700$$

Since, the shopkeeper sells the article to a consumer in Rajasthan, so this sale is intra state. Therefore, GST comprises of CGST at 6% and SGST at 6%.

$\therefore$ Shopkeeper collects the GST amount from consumer,

$$\text{CGST} = 6\% \text{ of } ₹\ 42700$$
$$=\frac{6}{100}\times 42700 = ₹\ 2562$$

and SGST = 6% of ₹ 42700

$$=\frac{6}{100}\times 42700 = ₹\ 2562$$

Amount of output GST for the shopkeeper,

CGST = ₹ 2562 and SGST = ₹ 2562

Amount of tax (under GST) paid by the shopkeeper to the Central Government;

First set off ₹ 4200 input IGST against ₹ 2562 output CGST.

Then, set off the balance ₹ (4200 − 2562) i.e. ₹ 1638 input IGST against output SGST.

$\therefore$ SGST paid by the shopkeeper to the State Government (Rajasthan) = Output SGST − Balance of Input IGST = ₹ 2562 − ₹ 1638 = ₹ 924

$\therefore$ Net tax (under GST) paid by shopkeeper to the Central Government = No tax paid to the Central Government.

$\therefore$ Tax paid to Rajasthan Government = ₹ 924

(iii) Amount of tax (under GST) received by Gujarat Government is nil.

(iv) Amount of tax (under GST) received by the Central Government = IGST received from wholesaler + CGST received from shopkeeper

$$= ₹\ 4200 + \text{Nil} = ₹\ 4200$$

(v) Amount which the consumer pays for the article

= Cost price of article to consumer + Consumer pays GST (i.e. CGST and SGST)

$$= ₹\ 42700 + ₹\ 2562 + ₹\ 2562 = ₹\ 47824$$

CHAPTER EXERCISE

a 3 Marks Questions

1. Miss Anjali goes to a mall to purchase a saree whose cost is ₹ 885 (list price). She tells the shopkeeper to reduce the price in such an extent that she has to pay ₹ 885, inclusive GST which is at the rate of 18%, find the reduction of price needed in the saree.

2. Sandeep purchased a digital camera for ₹ 25488, which includes 10% rebate on the list price and 18% tax (under GST) on the remaining price. Find the marked price of the digital camera.

3. The price of a spider toy is ₹ 3136 inclusive tax (under GST) at the rate of 12% on its listed price. A buyer asks for a discount on the listed price, so that after charging GST, the selling price becomes equal to the listed price. Find the amount of discount which the seller has to allow for the deal.

4. A shopkeeper buys an item whose list price is ₹ 8000 at some rate of discount from a wholesaler. He sells the item to a consumer at the list price. The sales are intra state and the rate of GST is 18%. Suppose, the shopkeeper pay a tax (under GST) of ₹ 72 to the State government. Find the rate of discount in which he bought the item from the wholesaler.

b 4 Marks Questions

5. Manufacturer Anshul sells a LG freeze to a dealer Lalit for ₹ 12500. The dealer Lalit sells it to a consumer at a profit of ₹ 1500. If the sales are intra state and the rate of GST is 12%, find

(i) the amount of tax (under GST), paid by the dealer Lalit to the Central Government.

(ii) how much the tax amount (under GST) received by the State Government?

(iii) the amount that the consumer pays for the LG Freeze.

6. A manufacturer sells a Hercules cycle to a dealer for ₹ 18000 and the dealer sells it to a consumer at a profit of ₹ 1500. If the sales are intra state and the rate of GST is 12%, find

(i) how much GST amount paid by the dealer to the State Government?

(ii) the GST amount received by the Central Government.

(iii) the GST amount received by the State Government.

(iv) the amount that the consumer pays for the Hercules cycle.

7. A shopkeeper buys an item whose printed price is ₹ 4000 from a wholesaler at a discount of 20% and sells it to a consumer at the printed price. If the sales are intra state and the rate of GST is 12%, find

(i) the price of the article inclusive GST in which the shopkeeper bought it.

(ii) the amount of tax (under GST) paid by the shopkeeper to the State Government.

(iii) the amount of tax (under GST) received by the Central Government.

(iv) how much the consumer pays the amount to the article?

8. A shopkeeper buys a dressing table at a discount of 20% from a wholesaler, the printed price of dressing table being ₹ 1600. The shopkeeper sells it to a consumer at the printed price. If the sales are intra state and the rate of GST is 12%, find

(i) GST paid by the shopkeeper to the Central Government.

(ii) GST received by the Central Government.

(iii) GST received by the State Government.

(iv) the amount at which the consumer bought the dressing table.

9. A retailer buys a AC from a wholesaler for ₹ 40000. He marks the price of the AC 15% above his cost price and sells it to a consumer at 5% discount on the marked price. If the sales are intra state and the rate of GST is 12%, find

(i) the marked price of the AC.

(ii) the amount which the consumer pays for the AC.

(iii) the amount of tax (under GST) paid by the retailer to the Central Government.

(iv) the amount of tax (under GST) received by the State Government.

10. The printed price of bicycle is ₹ 60000. The manufacturer allows a discount of 25% to a dealer Sanjeev. The dealer Sanjeev sells the article to another dealer Mohit at a discount of 12% on the marked price. The dealer Mohit sells it to a consumer at 5% above the printed price. If all the sales are intra state and the rate of GST is 18%, find

(i) the price of bicycle inclusive of tax (under GST) paid by dealer Sanjeev.

(ii) the price of bicycle inclusive of tax (under GST) paid by dealer Mohit.

(iii) the amount in which the consumer pays for the bicycle.

(iv) the amount of tax (under GST) paid by dealer Sanjeev to the Central Government.

(v) the amount of tax (under GST) paid by dealer Mohit to the State Government.

(vi) the amount of tax (under GST) received by the Central Government.

11. A manufacturer marks an article at ₹ 5000. He sells it to a wholesaler at a discount of 25% on the marked price and the wholesaler sells it to a retailer at a discount of 15% on the marked price. The retailer sells it to a consumer at the marked price. If all the sales are intra state and the rate of GST is 12%, find

(i) the amount inclusive of tax (under GST) which the wholesaler pays for the article.

(ii) the amount inclusive of tax (under GST) which the retailer pays for the article.

(iii) the amount of tax (under GST) which the wholesaler pays to the Central Government.

(iv) the amount of tax (under GST) which the retailer pays to the State Government.

12. The printed price of a bedsheet is ₹ 2500. A wholesaler in Karnataka buys the bedsheet from a manufacturer in Punjab at a discount of 12% on the printed price. The wholesaler sells the bedsheet to a retailer in Jharkhand at 32% above the marked price. If the rate of GST on the bedsheet is 5%, find

(i) the price inclusive of tax (under GST) which the wholesaler bought the bedsheet.

(ii) the price inclusive of tax (under GST) at which the retailer bought the bedsheet.

(iii) the tax (under GST) paid by the wholesaler to the Government.

(iv) the tax (under GST) received by the Central Government.

13. The printed price of a washing machine is ₹ 40000. A wholesaler in Chhattisgarh buys the washing machine from a manufacturer in Delhi at a discount of 10% on the printed price. The wholesaler sells the washing machine to a retailer in Haryana at 5% above the printed price. If the rate of GST on the washing machine is 18%, find

(i) the amount inclusive of tax (under GST) paid by the wholesaler for the washing machine.

(ii) the amount inclusive of tax (under GST) paid by the retailer for the washing machine.

(iii) the amount of tax (under GST) paid by the wholesaler to the Central Government.

(iv) the amount of tax (under GST) received by the Central Government.

14. A shopkeeper in Tamil Nadu buys an bullet bike at the printed price of ₹ 40000 from a wholesaler in West Bengal. The shopkeeper sells the bullet bike to a consumer in Tamil Nadu at a profit of 25% on the basic cost price. If the rate of GST is 18%, find

(i) the price of the bullet bike inclusive of tax (under GST) in which the shopkeeper bought it.

(ii) how much the shopkeeper tax amount pays to the Central Government?

(iii) the amount of tax (under GST) received by West Bengal Government.

(iv) the amount of tax (under GST) received by Central Government.

(v) the amount which the consumer pays for the bullet bike.

15. A shopkeeper in Punjab buys scooty at the printed price of ₹ 24000 from a wholesaler in Himachal. The shopkeeper sells scooty to a consumer in Punjab at a profit of 15% on the basic cost price. If the rate of GST is 12%, find

(i) the price inclusive of tax (under GST) at which the wholesaler bought the scooty.

(ii) the amount which the consumer pays for the scooty.

(iii) the amount of tax (under GST) received by the State Government of Punjab.

(iv) the amount of tax (under GST) received by the Central Government.

Hints and Answers

1. Similar as Example 1. **Ans.** ₹ 135

2. Similar as Example 1. **Ans.** ₹ 24000

3. Similar as Example 2. **Ans.** ₹ 300

4. Similar as Example 3. **Ans.** 10%

5. Similar as Example 4.
Ans. (i) ₹ 90 (ii) ₹ 90 (iii) ₹ 15680

6. Similar as Example 4.
Ans. (i) ₹ 90 (ii) ₹ 1170 (iii) ₹ 1170 (iv) ₹ 21840

7. Similar as Example 5.
Ans. (i) ₹ 3584 (ii) ₹ 48 (iii) ₹ 240 (iv) ₹ 4480

8. Similar as Example 5.
Ans. (i) ₹19.20 (ii) ₹ 96 (iii) ₹ 96 (iv) ₹ 1792

9. Similar as Example 6.
Ans. (i) ₹ 46000 (ii) ₹ 48944 (iii) ₹ 222 (iv) ₹ 2622

10. Similar as Example 7.
Ans. (i) ₹ 53100 (ii) ₹ 62304 (iii) ₹ 74340 (iv) ₹ 702 (v) ₹ 918 (vi) ₹ 5670

11. Similar as Example 7.
Ans. (i) ₹ 4200 (ii) ₹ 4760 (iii) ₹ 30 (iv) ₹ 45

12. Similar as Example 8.
Ans. (i) ₹ 110 (ii) ₹ 3465 (iii) ₹ 55 (iv) ₹ 165

13. Similar as Example 8.
Ans. (i) ₹ 42480 (ii) ₹ 49560 (iii) ₹ 1080 (iv) ₹ 7560

14. Similar as Example 9.
Ans. (i) ₹ 47200 (ii) ₹ 1800 (iii) ₹ Nil (iv) ₹ 7200 (v) ₹ 59000

15. Similar as Example 9.
Ans. (i) ₹ 26880 (ii) ₹ 30912 (iii) ₹ 432 (iv) ₹ 2880

CHAPTER

2

Banking

In simple words, Banking can be defined as the business activity of accepting and safe guarding money owned by individuals and other entities, and then lending out this money to other individuals or entities in order to earn a profit. In this activity, a sum paid (or charged) for the use of money (or for lending money), is called the interest and it is expressed as a percentage of the amount borrowed or lend over a given period.

Generally, the activity of banking is performed by Banks. Banks can be defined as an institution that mainly carries on the business of getting deposits and lending money.

Chapter Objectives

- Different Types of Bank Accounts
- Calculations of Interest and Maturity Value on Recurring Deposit

Different Types of Bank Accounts

Traditionally banks have four types of deposit account, which are discussed below

1. Savings Bank Account

It is a bank account which can be opened by any person with a small amount, where the minimum balance vary from bank-to-bank and place-to-place. In this account, bank pay interest on the amount deposited by the person. Such bank account encourages the savings habits among the people of small and middle groups.
The bank issues a passbook for the savings accounts in which datewise entries regarding the deposits, withdrawals and the interest earned are recorded.

2. Current Account

It is a account which is mainly used for business transactions. For the current account, there is no limit for number of transactions (deposit and withdrawls). No interest is payable on deposits made in this account. Bank issues passbook, debit card, credit card etc., for current account.

3. Fixed Deposit Account

The account which is opened for a fixed period (time) by depositing a particular amount, is known as fixed deposit account. In this account, the amount can be deposited only once.

4. Recurring (or Cumulative) Deposit Account

This is a special type of bank account, in which a depositor deposits fixed amount every month for a fixed period of time. At the end of the period depositor gets all the principal sum, alongwith the interest earned during that period.

In this chapter, we will learn to calculate the interest and maturity value for recurring deposit.

Basic Terms Related to Recurring Deposit Account

(i) **Principal** The amount that the depositor deposits each month, is called principal.

(ii) **Maturity period** The fixed period for which the recurring deposit account is opened, is called maturity period.

(iii) **Maturity value** Total amount (Deposited amount + Interest) received by the depositor at the end of the maturity period, is called maturity value.

Calculation of Interest and Maturity Value on Recurring Deposit

Let P be the principal, r be the annual rate of interest and n be the number of months. Then,

(i) Simple interest (or interest) $I = \dfrac{P \times \left(\dfrac{n(n+1)}{2}\right) \times r}{12 \times 100}$

$$= \frac{Pr\,n(n+1)}{2400}$$

(ii) Maturity (or Matured) value

$$= \text{Total amount deposited} + \text{Interest}$$

$$= Pn + \frac{Pr\,n(n+1)}{2400}$$

or $\quad MV = P \times n + I$

Example 1. Kiran deposited ₹ 200 for 36 months in a bank Recurring Deposit Account. If the bank pays interest at the rate of 11% per annum, then find the amount she gets on maturity. ***[2012]***

Sol. Given, money deposited per month (P) = ₹ 200,

number of months $(n) = 36$ months and

rate of interest$(r) = 11\%$ per annum

$\therefore$ Maturity amount $= Pn\left[1 + \dfrac{r(n+1)}{2400}\right]$

$$= 200 \times 36\left[1 + \frac{11(36+1)}{2400}\right]$$

$$= 7200\left(\frac{2400+407}{2400}\right)$$

$$= 3(2807) = ₹\ 8421$$

Hence, Kiran will get the amount ₹ 8421 at the time of maturity.

Example 2. Katrina opened a Recurring Deposit Account with a nationalised bank for a period of 2 yr. If the bank pays interest at the rate of 6% per annum and the monthly instalment is ₹ 1000, then find the

(i) interest earned in 2 yr.

(ii) matured value. ***[2015]***

Sol.

(i) Given, monthly instalment (P) = ₹ 1000,

number of monthly instalments $(n) = 2 \times 12 = 24$ months

and rate of interest $(r) = 6\%$ per annum

Now, $\quad I = \dfrac{Pr\,n(n+1)}{2400}$

$$= \frac{1000 \times 6 \times 24 \times (24+1)}{2400} = ₹\ 1500$$

(ii) Matured value $= P \times n + I$

$$= 1000 \times 24 + 1500$$

$$= 24000 + 1500$$

$$= ₹\ 25500$$

Example 3. Shahrukh opened a Recurring Deposit Account in a bank and deposited ₹ 800 per month for $1\frac{1}{2}$ yr. If he received ₹ 15084 at the time of maturity, then find the rate of interest per annum. ***[2014]***

Sol. Given, money deposited per month (P) = ₹ 800,

time $(n) = 1\frac{1}{2}$ yr $= 18$ months

and maturity value = ₹ 15084

Let the rate of interest be $r\%$ per annum.

$\because$ Maturity value $= Pn\left[1 + \dfrac{(n+1)r}{2400}\right]$

$$\therefore \quad 15084 = 800 \times 18\left[1 + \frac{(18+1)r}{2400}\right]$$

$$\Rightarrow \quad 15084 = 14400\left(\frac{2400+19r}{2400}\right)$$

$$\Rightarrow \quad 15084 = 6(2400+19r)$$

$$\Rightarrow \quad \frac{15084}{6} = 2400 + 19r$$

$$\Rightarrow \quad 2514 = 2400 + 19r$$

$$\Rightarrow \quad 19r = 114$$

$$\Rightarrow \quad r = \frac{114}{19} = 6$$

Hence, the required rate of interest is 6% per annum.

Example 4. Mr. Gupta opened a Recurring Deposit Account in a bank. He deposited ₹ 2500 per month for 2 yr. At the time of maturity, he got ₹ 67500. Find

(i) the total interest earned by Mr. Gupta.

(ii) the rate of interest per annum. ***[2010]***

Sol. Given, Mr. Gupta deposits per month (P) = ₹ 2500,

time period $(n) = 2$ yr $= 2 \times 12 = 24$ months

and maturity value = ₹ 67500

(i) Now, total interest earned by Mr. Gupta

$$= \text{Maturity value} - \text{Total deposit}$$

$$= 67500 - nP = 67500 - 24 \times 2500$$

$$= 67500 - 60000 = ₹\ 7500$$

(ii) Let the rate of interest be $r\%$ per annum.

$\because \quad I = \dfrac{Prn(n+1)}{2400}$

$\therefore \quad 7500 = \dfrac{2500 \times r \times 24\,(24+1)}{2400}$

$\Rightarrow \quad 7500 = 25r(25)$

$\Rightarrow \quad 625r = 7500$

$\Rightarrow \quad r = \dfrac{7500}{625} = 12$

Hence, the rate of interest is 12% per annum.

Example 5. Priyanka has a recurring deposit account of ₹1000 per month at 10% per annum. If she gets ₹5550 as interest at the time of maturity, find the total time for which the account was held. ***[2018]***

Sol. Given, $P =$ ₹ 1000, $r = 10$ and $I =$ ₹ 5550

$\because \quad I = P \times \dfrac{n(n+1)}{2 \times 12} \times \dfrac{r}{100}$

$\therefore \quad 5550 = 1000 \times \dfrac{n(n+1)}{24} \times \dfrac{10}{100}$

$\Rightarrow \quad 5550 = 100 \times \dfrac{n(n+1)}{24}$

$\Rightarrow \quad 55.5 \times 24 = n(n+1)$

$\Rightarrow \quad n^2 + n - 1332 = 0$

$\Rightarrow \quad n^2 + 37n - 36n - 1332 = 0$

$\Rightarrow \quad n(n+37) - 36(n+37) = 0$

$\Rightarrow \quad (n+37)(n-36) = 0$

$\Rightarrow \quad n = -37 \text{ or } 36 \Rightarrow n = 36$

Hence, total time $(n) = 36$ months

[since, month cannot be negative]

Example 6. Mr. Britto deposits a certain sum of money each month in a Recurring Deposit Account of a bank. If the rate of interest is of 8% per annum and Mr. Britto gets ₹ 8088 from the bank after 3 yr, then find the value of his monthly instalment. ***[2013]***

Sol. Given, rate of interest $(r) = 8\%$ per annum,

Maturity value (MV) = ₹8088 and time $(n) = 3$ yr $= 36$ months

Let the monthly instalment (P) be ₹ x.

$\because$ Maturity value $= Pn\left(1 + \dfrac{r(n+1)}{2400}\right)$

$\therefore \quad 8088 = 36x\left(1 + \dfrac{8 \times 37}{2400}\right)$

$\Rightarrow \quad 8088 = 36x\left(\dfrac{2400 + 296}{2400}\right)$

$\Rightarrow \quad 8088 = 36x\left(\dfrac{2696}{2400}\right)$

$\Rightarrow \quad 8088 = \dfrac{97056\,x}{2400} \Rightarrow x = 200$

Hence, the monthly instalment is ₹ 200.

Example 7. Mr. Amit opens a Recurring Deposit Account of ₹ 300 per month at 8% simple interest per annum. On maturity, he gets ₹ 9930. Find the period for which he continued with the account.

Sol. Given, Mr. Amit deposits per month $(P) =$ ₹ 300

rate of interest $(r) = 8\%$ per annum

and maturity value = ₹ 9930

Let n be the number of months.

$\because$ Maturity value $= Pn\left(1 + \dfrac{r(n+1)}{2400}\right)$

$\therefore \quad 9930 = 300n\left(1 + \dfrac{8(n+1)}{2400}\right)$

$\Rightarrow \quad 9930 = 300n\left(1 + \dfrac{(n+1)}{300}\right)$

$\Rightarrow \quad 9930 = 300n + (n^2 + n)$

$\Rightarrow \quad 9930 = 300n + n^2 + n$

$\Rightarrow \quad n^2 + 301n - 9930 = 0$

$\Rightarrow \quad n^2 + 331n - 30n - 9930 = 0$

$[\because -9930 = 331 \times (-30) \text{ and } 301 = 331 - 30]$

$\Rightarrow \quad n(n+331) - 30(n+331) = 0$

$\Rightarrow \quad (n+331)(n-30) = 0$

Either $n + 331 = 0$

$\Rightarrow \quad n = -331$ [which is not possible]

or $\quad n - 30 = 0 \Rightarrow n = 30$

$\therefore$ Period = 30 months $= 2\frac{1}{2}$ yr

CHAPTER EXERCISE

a 3 Marks Questions

1. Sonia had a Recurring Deposit Account in a bank and deposited ₹600 per month for $2\frac{1}{2}$ yr. If the rate of interest was 10% per annum, find the maturity value of this account. *[2018]*

2. If Mrs. Goswami deposits ₹ 1000 every month in a Recurring Deposit Account for 3 yr at 8% interest per annum, then find the matured value. *[2009]*

3. Joseph deposits ₹ 600 per month in a Recurring Deposit Account in a post office for $4\frac{1}{2}$ yr. Find the amount payable to him on maturity, if the rate of interest is 9% per annum.

4. Ashit has Recurring Deposit Account. He deposits ₹ 950.00 per month for a period of 24 months. At the time of maturity, he gets an interest of ₹ 2256.25. Find the rate of interest.

5. Sahil opened a Recurring Deposit Account in a bank and deposits ₹ 150 per month for 8 months. At the time of maturity, he received ₹ 1236. Find the rate of interest.

6. David opened a Recurring Deposit Account in a bank and deposited ₹ 300 per month for 2 yr. If he received ₹ 7725 at the time of maturity, then find the rate of interest per annum. *[2008]*

7. Mr. Aditya has a 4 yr cumulative term Deposit Account in Axis Bank and deposits ₹ 650 per month. If he receives ₹ 36296 at the time of maturity, then find the rate of interest.

8. Ahmed has a Recurring Deposit Account in a bank. He deposits ₹ 2500 per month for 2 yr. If he gets ₹ 66250 at the time of maturity, then find
   (i) the interest paid by the bank.
   (ii) the rate of interest. *[2011]*

9. Mr. Anand opened a cumulative Deposit Account of monthly instalment ₹ 200 at 9% per annum simple interest. She earned a total interest of ₹ 1764. How many instalments did she pay?

10. Ms. Mamta opened a cumulative Deposit Account of monthly instalment of ₹ 1200 at 9% per annum simple interest. She earned a total interest of ₹ 5994. How many instalments did she pay?

11. A man has a Recurring Deposit Account in a bank for $3\frac{1}{2}$ yr. If the rate of interest is 12% per annum and the man gets ₹ 30618 on maturity, then find the value of monthly instalment.

12. Kamal has a Recurring Deposit Account in a bank for $3\frac{1}{2}$ yr at 9.5% per annum. If he gets ₹ 58978, at the time of maturity, then find the monthly instalment.

13. Mr. Bisla has a Recurring Deposit Account for 2 yr at 6% interest per annum. He receives ₹ 975 as interest on maturity. Find
   (i) the monthly instalment amount.
   (ii) the maturity amount.

14. Mohan has a Recurring Deposit Account in a bank for 2 yr at 6% per annum simple interest. If he gets ₹ 1200 as interest at the time of maturity, then find
   (i) the monthly instalment.
   (ii) the amount of maturity. *[2016]*

b 4 Marks Questions

15. Ms. Arora deposits ₹ 100 per month in a cumulative Deposit Account for a period of 5 yr. After the end of the period, she will receive ₹ 7220. (Consider the interest rate to be simple). Find
   (i) the rate of interest per annum.
   (ii) the total interest that Ms. Arora will earn.

16. Shilpa has a 4 yr Recurring Deposit Account in Bank of Maharashtra and deposits ₹ 800 per month. If she gets ₹ 48200 at the time of maturity, then find
   (i) the rate of (simple) interest.
   (ii) the total interest earned by Shilpa.

17. Mr. Richard has a Recurring Deposit Account in a bank for 3 yr at 7.5% per annum simple interest. If he gets ₹ 8325 as interest at the time of maturity, then find

(i) the monthly deposit.

(ii) the maturity value. *[2017]*

18. Rajiv Bhardwaj has a Recurring Deposit Account in a bank of ₹ 600 per month. If the bank pays simple interest of 7% per annum and he gets ₹ 15450 as maturity amount, then find the total time for which the Account was held.

19. Mrs. Sarkar deposits ₹ 1600 per month in a Recurring Deposit Account at 9% per annum simple interest. If she gets ₹ 65592 at the time of maturity, then find the total time for which the account was held.

20. Samita has a Recurring Deposit Account in a bank of ₹ 2000 per month at the rate of 10% per annum. If she gets ₹ 83100 at the time of maturity, then find the total time for which the account was held.

Hints and Answers

1. Do same as Example 1. **Ans.** ₹ 20325

2. Do same as Example 1. **Ans.** ₹ 40440

3. Do same as Example 1. **Ans.** ₹ 39082.50

4. **Hint** Use the formula, $I = \dfrac{Prn(n+1)}{2400}$

Ans. 9.5%

5. Do same as Example 3. **Ans.** 8%

6. Do same as Example 3. **Ans.** 7%

7. Do same as Example 3. **Ans.** 8%

8. Do same as Example 4. **Ans.** (i) ₹ 6250 (ii) 10%

9. **Hint** Let n be the number of instalments.

Then, time period = n months

∵ Now, use the formula

$$\text{Interest} = \frac{Prn(n+1)}{2400}$$ **Ans.** 48

10. Do same as Q. 9. **Ans.** 36

11. Do same as Example 6. **Ans.** ₹ 600

12. Do same as Example 6. **Ans.** ₹ 1200

13. (i) **Hint** Let the monthly instalment be ₹ P.

Then, interest $= \dfrac{Prn(n+1)}{2400}$

Ans. ₹ 650

(ii) **Hint** Maturity value $= P \times n +$ Interest

Ans. ₹ 16575

14. Do same as Q. 13.

Ans. (i) ₹ 800 (ii) ₹ 20400

15. Do same as Example 4. **Ans.** (i) 8% (ii) 1220

16. Do same as Example 4.

Ans. (i) 12.5% per annum (ii) ₹ 9800

17. (i) **Hint** Let monthly deposit be ₹ P.

Then, $I = \dfrac{Prn(n+1)}{2400}$

Ans. ₹ 2000

(ii) **Hint** Maturity value $= 36P + 8325$

Ans. ₹ 80325

18. Do same as Example 7. **Ans.** 2 yr

19. Do same as Example 7. **Ans.** 3 yr

20. Do same as Example 7. **Ans.** 3 yr

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1 Sonia had a Recurring Deposit Account in a bank and deposited ₹600 per month for $2\frac{1}{2}$ yr. If the rate of interest was 10% per annum, find the maturity value of this account.

2 Priyanka has a recurring deposit account of ₹1000 per month at 10% per annum. If she gets ₹5550 as interest at the time of maturity, find the total time for which the account was held.

2017

3 Mr. Richard has a Recurring Deposit Account in a bank for 3 yr at 7.5% per annum simple interest. If he gets ₹ 8325 as interest at the time of maturity, then find

(i) the monthly deposit. (ii) the maturity value.

2016

4 Mohan has a Recurring Deposit Account in a bank for 2 yr at 6% per annum simple interest. If he gets ₹ 1200 as interest at the time of maturity, then find

(i) the monthly instalment. (ii) the amount of maturity.

2015

5 Katrina opened a Recurring Deposit Account with a nationalised bank for a period of 2 yr. If the bank pays interest at the rate of 6% per annum and the monthly instalment is ₹ 1000, then find the

(i) interest earned in 2 yr. (ii) matured value.

2014

6 Shahrukh opened a Recurring Deposit Account in a bank and deposited ₹ 800 per month for $1\frac{1}{2}$ yr. If he received ₹ 15084 at the time of maturity, then find the rate of interest per annum.

2013

7 Mr. Britto deposits a certain sum of money each month in a Recurring Deposit Account of a bank. If the rate of interest is of 8% per annum and Mr. Britto gets ₹ 8088 from the bank after 3 yr, then find the value of his monthly instalment.

2012

8 Kiran deposited ₹ 200 for 36 months in a bank Recurring Deposit Account. If the bank pays interest at the rate of 11% per annum, then find the amount she gets on maturity.

2011

9 Ahmed has a Recurring Deposit Account in a bank. He deposits ₹ 2500 per month for 2 yr. If he gets ₹ 66250 at the time of maturity, then find

(i) the interest paid by the bank. (ii) the rate of interest.

* All these questions are completely covered in chapter either as solved examples or in chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. Sneha opened a cumulative time deposit account with a bank and deposits ₹ 500 per month for $1\frac{1}{4}$ yr. If she receives ₹ 300 as interest, then find the rate of interest per annum.

(a) 6% (b) 8%
(c) 7.5% (d) 10%

2. Dinesh opened a Recurring Deposit Account with State Bank of India. He deposited ₹ 900 per month at 7.5% per annum. He received ₹ 1687.5 as interest. Then, the time period of his deposit will be

(a) 3 yr (b) 2 yr
(c) 5 yr (d) 3.5 yr

3. A Recurring Deposit Account of ₹ 1200 per month has a maturity value of ₹ 12440. If the rate of interest is 8% and the interest is calculated at the end of every month, then find the time of this Recurring Deposit Account.

(a) 10 months (b) 20 months
(c) 30 months (d) 50 months

4. Vishal has a Recurring Deposit Account in a finance company for 1 yr at 8% per annum. If he gets ₹ 9390 at the time of maturity, then how much amount per month has been invested by Vishal?

(a) ₹ 650 (b) ₹ 700
(c) ₹ 750 (d) ₹ 800

5. Ravi deposited ₹ 400 per month in a Recurring Deposit Account for 2 yr at 9% per annum. Sonu deposited ₹ 400 per month in a Recurring Deposit Account for $2\frac{1}{2}$ yr at 12% per annum. Which of the following can be concluded about the interests to be paid to them on maturities of their deposits?

(a) Ravi must be paid an interest of ₹ 860 less than that of Sonu
(b) Ravi must be paid an interest of ₹ 960 less than that of Sonu
(c) Ravi must be paid an interest of ₹ 916 less than that of Sonu
(d) Ravi must be paid an interest of ₹ 816 less than that of Sonu

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 393.

CHAPTER

3

Shares and Dividends

Chapter Objectives

- Some Basic Definitions
- Dividend

While starting or establishing a company a large sum of money is required, generally it is not possible for a single/few person(s) to arrange such a huge amount of money on their own, so the company raises its capital from the public by issue of shares. A 'share' is one of the units into which the capital of the company is divided.

Some Basic Definitions

(i) **Nominal/Face value** The original value of a share is called its nominal (printed or face) value, which is fixed by the company.

(ii) **Shareholder** An individual who purchases/possesses the share(s) of the company is called a shareholder of the company.

(iii) **Market value** The price of a share at any time in the market is called market value of the share. The market value of a share is not fixed, it varies from time-to-time.

(iv) **At par** When shares are issued at their face value, the shares are said to have been issued at par, e.g. shares with a face value of ₹10 is issued at ₹10.

(v) **At premium** When shares are issued at a value which is higher than the face value of shares, the shares are said to have been issued at premium, i.e. issue price is more than face value, e.g. share with a face value of ₹10 is issued at ₹12, here ₹ 2 being premium.

(vi) **At discount** When shares are issued at a value which is lower than the face value of shares, the shares are said to have been issued at discount, i.e. issue price is less than face value, e.g. share with a face value of ₹10 is issued at ₹8, here ₹2 being discount.

Important Formulae Related to Share

(i) Total face value $=$ Number of shares $\times$ Nominal value of one share

(ii) Money invested $=$ Number of shares $\times$ Market value of one share

(iii) When shares are purchased from the company, then

$$\text{Number of shares} = \frac{\text{Total investment}}{\text{Face value of a share}}$$

(iv) When shares are purchased from the market, then

$$\text{Number of shares} = \frac{\text{Total investment}}{\text{Market value of a share}}$$

Note In share market, share is purchased and sold on market value.

Example 1. A company issued shares at 10% premium. Lalit applied for 1000 shares, but was allotted 500 shares of this company. Find his investment, if the face value of a share is ₹ 100.

Sol. Given, face value of a share = ₹ 100
and premium percentage = 10%
∴ Premium on one share = ₹ 10
Now, market value of one share = 100 + 10 = ₹110
∴ Price of 500 shares = 500 × 110 = ₹ 55000
[∵ money invested = number of shares × market value of one share]
Hence, Lalit's investment in the company is ₹55000.

Example 2. A man invests ₹ 15840 in buying shares of face value ₹ 24 each and selling at 10% premium. Find the number of shares bought by him.

Sol. Given, total investment = ₹ 15840
and face value of one share = ₹ 24
Now, premium on one share = 10% of ₹ 24

$$= \frac{10}{100} \times 24 = ₹\ 2.40$$

and investment on one share = 24 + 2.40 = ₹26.40

$$\therefore \text{Number of shares bought} = \frac{\text{Total investment}}{\text{Investment on one share}} = \frac{15840}{26.40} = 600$$

Example 3. Akansha invested ₹ 8000 in buying ₹ 50 shares of a company. She sold half of these when they were at a premium of ₹ 10 and the remaining at a discount of ₹ 5. Find her gain or loss on the whole transaction.

Sol. Given, investment = ₹ 8000
and nominal value of one share = ₹ 50

$$\therefore \text{Number of shares} = \frac{\text{Investment}}{\text{face value of one share}} = \frac{8000}{50} = ₹160$$

Selling price of a share at premium = 50 + 10 = ₹60

Number of shares sold at this rate = $\frac{1}{2}$ of 160 = 80

Amount received on selling 80 shares at ₹60 = 60 × 80 = ₹ 4800

Selling price of a share at discount = 50 – 5 = ₹ 45
Number of shares sold at this rate = 160 – 80 = 80
Amount received on selling 80 shares at ₹45 = 45 × 80 = ₹3600

Total amount obtained by selling all the shares = 4800 + 3600 = ₹ 8400
∵ Amount invested = ₹ 8000
∴ Gain in total transaction = 8400 – 8000 = ₹ 400

Dividend

A part of profit of the company which a shareholder gets for his investment in shares from the company, is called dividend.

It is always expressed as the percentage of the face value of the share, which is called **rate of dividend**,

i.e. $$\text{Dividend on one share} = \frac{d}{100} \times \text{Nominal value}$$

where, d is the rate of dividend.

The company always gives a dividend on the face value (Nominal value) of the share irrespective of the market value of the share.

Annual Income

Total income received by a shareholder in a year, by way of dividend is referred to its annual income.

Yield Percentage

Yield percentage is the percentage return that a shareholder gets on the investment made by purchasing the shares of a company, with respect to the dividend declared by the company.

Some Useful Cases

20% of ₹ 100 shares at ₹ 140 means that

(i) the face value (nominal value) of 1 share = ₹ 100
(ii) the market value of 1 share = ₹ 140
(iii) the dividend on 1 share = 20% of ₹ 100 = ₹ 20 per annum
(iv) the income on ₹ 140 is ₹ 20 for one year
(v) the rate of return (or yield) per annum

$$= \left(\frac{20}{140} \times 100\right)\% = 14\frac{2}{7}\%$$

Similarly, '10% of ₹ 50 shares at a discount ₹ 10' means that

(i) the face value of 1 share = ₹ 50
(ii) the market value of 1 share = 50 – 10 = ₹ 40
(iii) the dividend on 1 share = 10% of 50

$$= \frac{10}{100} \times 50 = ₹\ 5 \text{ per annum}$$

(iv) the income on ₹ 40 = ₹ 5 for one year
(v) the rate of return per annum

$$= \left(\frac{5}{40} \times 100\right)\% = 12.5\%$$

Important Formulae Related to Dividend

(i) Annual income (or annual dividend) per share

$$= \text{Nominal value of a share} \times \frac{\text{Rate of dividend}}{100}$$

(ii) Total annual income = Number of shares

$$\times \text{Nominal value of a share} \times \frac{\text{Rate of dividend}}{100}$$

(iii) Number of shares $= \dfrac{\text{Annual income}}{\text{Income on one share}}$

(iv) Rate of dividend (or income) on investment

$$= \frac{\text{Dividend}}{\text{Investment}} \times 100\%$$

(v) Rate of return (or yield) $= \dfrac{\text{Annual income}}{\text{Total investment}} \times 100\%$

(vi) Dividend percentage $\times$ Nominal value

$$= \text{Percentage return} \times \text{Market value}$$

Example 4. A man invests ₹ 12800 in a company paying 18% dividend at a time when its ₹ 100 share can be bought at a premium of ₹ 60, find his annual income from these shares.

Sol. Given, face value of one share = ₹ 100

Premium for one share = ₹ 60

Rate of dividend = 18%

and total investment = ₹ 12800

Investment on one share = 100 + 60 = ₹ 160

Now, number of shares bought

$$= \frac{\text{Total investment}}{\text{Investment on one share}} = \frac{12800}{160} = 80$$

Now, annual income on one share = 18% of ₹ 100

$$\left[\because \text{annual income per share} = \text{nominal value of a share} \times \frac{\text{rate of dividend}}{100}\right]$$

$$= \frac{18}{100} \times 100 = ₹\ 18$$

$\therefore$ Annual income from 80 shares = 80×18 = ₹1440

Example 5. A man buys ₹ 10 share at a premium of ₹ 5 per share. If the company pays 9% dividend, then find the percentage return on his investment in buying 200 shares.

Sol. Given, face value of one share = ₹ 10

and premium = ₹ 5

Investment on one share = 10 + 5 = ₹ 15

$\therefore$ Investment in buying 200 shares = 200×15 = ₹ 3000

Also, given rate of dividend = 9%

Now, annual income on 1 share = 9% of ₹ 10

$$= \frac{9}{100} \times 10 = ₹\ \frac{9}{10}$$

$\therefore$ Annual income on 200 shares $= 200 \times \dfrac{9}{10}$ = ₹ 180

Now, percentage return on his investment

$$= \frac{\text{Annual income}}{\text{Total investment}} \times 100\%$$

$$= \frac{180}{3000} \times 100\% = 6\%$$

Example 6. Salman invests a sum of money in ₹ 50 shares, paying 15% dividend quoted at 20% premium. If his annual dividend is ₹ 600, then calculate

(i) the number of shares, he bought.

(ii) his total investment.

(iii) the rate of return on his investment. ***[2014]***

Sol. Given, nominal value of one share = ₹ 50

and rate of dividend = 15%

Now, dividend on one share

$$= \frac{15}{100} \times 50 = ₹\ 7.50$$

Total dividend that Salman got from the company = ₹ 600

(i) Number of shares Salman bought

$$= \frac{\text{Annual income}}{\text{Income on one share}} = \frac{600}{7.50} = 80$$

(ii) Premium on one share = 20% of ₹ 50 $= \dfrac{20}{100} \times 50$ = ₹ 10

$\therefore$ Market value of one share = 50 + 10 = ₹ 60

Total investment for 80 shares = 80×60 = ₹ 4800

(iii) Rate of return on his investment

$$= \frac{\text{Total dividend amount}}{\text{Total investment}} \times 100\%$$

$$= \frac{600}{4800} \times 100\% = 12.5\%$$

Example 7. Mr. Prakash invested ₹ 52000 on ₹ 100 shares at a discount of ₹ 20, paying 8% dividend. At the end of one year, he sells the shares at a premium of ₹ 20. Find

(i) the annual dividend.

(ii) the profit earned including his dividend. ***[2011]***

Sol. Given, money invested by Mr. Prakash = ₹ 52000

Nominal value of one share = ₹ 100 and dividend = 8%

Market value of one share = 100 − 20 = ₹ 80

$\therefore$ Number of shares $= \dfrac{\text{Total investment}}{\text{Market value of one share}}$

$$= \frac{52000}{80} = 650$$

(i) Annual dividend $= \dfrac{\text{Rate of dividend}}{100}$

$$\times \text{Number of shares} \times \text{Nominal value of one share}$$

$$= \frac{8}{100} \times 650 \times 100 = ₹\ 5200$$

(ii) As Mr. Prakash sells his shares at a premium of ₹ 20.

$\therefore$ Selling price of one share = 100 + 20 = ₹ 120

Now, selling value of his 650 shares

= 650 × selling price of one share

= 650 × 120 = ₹ 78000

$\therefore$ Profit earned = Total selling value + Annual dividend − Money invested

= 78000 + 5200 − 52000 = ₹ 31200

Example 8. Vivek invests ₹ 4500 in 8%, ₹ 10 shares at ₹ 15. He sells the shares, when the price rises to ₹ 30 and invests the proceeds in 12%, ₹ 100 shares at ₹ 125.

Calculate

(i) the sale proceeds.

(ii) the number of ₹ 125 shares, he buys.

(iii) the change in his annual income from dividend. ***[2010]***

Sol. Given, money invested by Vivek = ₹ 4500

Nominal value of one share = ₹ 10

Market value of one share = ₹ 15

$\therefore$ Number of shares he bought

$$= \frac{\text{Money invested}}{\text{Market value of one share}}$$

$$= \frac{4500}{15} = 300$$

Total nominal value = Number of shares × Nominal value of one share

= 300 × 10 = ₹ 3000

$\therefore$ Dividend = 8% of total nominal value

$$= \frac{8}{100} \times 3000 = ₹\ 240$$

(i) Sale proceed = Amount received on selling 300 shares

= Number of shares × Market value

= 300 × 30 = ₹ 9000

(ii) If he bought a share at ₹ 125, then the number of shares he bought $= \frac{9000}{125} = 72$

(iii) Total nominal value of 72 shares = 72 × 100 = ₹ 7200

$\therefore$ Dividend = 12% of total nominal value

$$= \frac{12}{100} \times 7200 = ₹\ 864$$

$\therefore$ Change in his annual income = 864 − 240 = ₹ 624

Example 9. Rohit invested ₹ 9600 on ₹ 100 shares at ₹ 20 premium paying 8% dividend. Rohit sold the shares, when the price rises to ₹ 160. He invested the proceeds (excluding dividend) in 10%, ₹ 50 shares at ₹ 40. Find

(i) the original number of shares.

(ii) the sale proceeds.

(iii) the new number of shares.

(iv) the change in the two dividends. ***[2015]***

Sol. Given, money invested by Rohit = ₹ 9600

Nominal value of one share = 100

Premium on one share = ₹ 20

$\therefore$ Market value of one share = 100 + 20 = ₹ 120

(i) Original number of shares

$$= \frac{\text{Investment}}{\text{Market value}} = \frac{9600}{120} = 80$$

(ii) Sale proceeds = 80 × 160 = ₹ 12800

(iii) New number of shares $= \frac{12800}{40} = 320$

(iv) Change in two dividends

= 10% of (320 × 50) − 8% of (80 × 100)

= 1600 − 640 = 960

Example 10. A person invested 20%, 30% and 40% of his savings in buying shares of 3 different companies A, B and C which declared dividends of 10%, 12% and 15%, respectively. If his total income from dividends is ₹ 1450, then find his savings and the amount which he invested in each company.

Sol. Let his savings be ₹ x.

Then, investment in company A = 20% of x = ₹ $\frac{20x}{100}$

Investment in company B = 30% of x = ₹ $\frac{30x}{100}$

and investment in company C = 40% of x = ₹ $\frac{40x}{100}$

Now, income from $A = \frac{20x}{100} \times \frac{10}{100} = ₹\ \frac{20x}{1000}$

Income from $B = \frac{30x}{100} \times \frac{12}{100} = ₹\ \frac{36x}{1000}$

and income from $C = \frac{40x}{100} \times \frac{15}{100} = ₹\ \frac{60x}{1000}$

$\therefore$ Total income $= \frac{20x}{1000} + \frac{36x}{1000} + \frac{60x}{1000} = ₹\ \frac{116x}{1000}$

But total income = ₹ 1450 [given]

$$\therefore \quad \frac{116x}{1000} = 1450$$

$$\Rightarrow \quad x = 1450 \times \frac{1000}{116} = ₹\ 12500$$

$\therefore$ Investment in company $A = \frac{20}{100} \times 12500 = ₹\ 2500$

Investment in company $B = \frac{30}{100} \times 12500 = ₹\ 3750$

and investment in company $C = \frac{40}{100} \times 12500 = ₹\ 5000$

CHAPTER EXERCISE

a 3 Marks Questions

1. Find the dividend received on 60 shares of ₹ 20 each, if 9% dividend is declared.

2. A man bought 500 shares, each of face value ₹ 10 of a certain business concern and during the first year after purchase, received ₹ 400 as dividend on his shares. Find the rate of dividend on shares.

3. A man buys shares at the par value of ₹ 10, yielding 8% dividend at the end a year. Find the number of shares bought, if he receives a dividend of ₹ 300.

4. A man invests ₹ 8800 in a company paying 6% dividend, when a share of face value ₹ 100 is selling at ₹ 60 premium. What is the percentage return on his investment?

5. Sachin buys ₹ 100 shares at ₹ 20 premium in a company paying 15% dividend. Find
(i) the market value of 200 shares.
(ii) his annual income.
(iii) his percentage income.

6. A man invests ₹22500 in ₹50 shares available at 10% discount. If the dividend paid by the company is 12%, calculate
(i) the number of shares purchased.
(ii) the annual dividend received.
(iii) the rate of return he gets on his investment. Give your answer correct to the nearest whole number. ***[2018]***

7. A man invests ₹ 9600 on ₹ 100 shares at ₹ 80. If the company pays him 18% dividend, then find
(i) the number of shares, he buys.
(ii) his total dividend.
(iii) his percentage return on the shares. ***[2012]***

8. Amit Kumar invests ₹ 36000 in buying ₹ 100 shares at ₹ 20 premium. The dividend is 15% per annum. Find
(i) the number of shares, he buys.
(ii) his yearly dividend.
(iii) the percentage return on his investment.
Give your answer correct to the nearest whole number. ***[2009]***

9. How much should a man invest in ₹ 50 shares selling at ₹ 60 to obtain an income of ₹ 450, if the rate of dividend declared is 10%. Also, find his yield per cent, to the nearest whole number. ***[2017]***

10. What sum should a person invest in ₹ 25 shares, selling at ₹ 36, to obtain an income of ₹ 720, if the dividend declared is 12%? Find
(i) the number of shares bought by him.
(ii) the percentage return on his income.

11. Mr. Somil desires to have an annual income of ₹ 36000 from 18%, ₹ 125 shares available in the market at a premium of 20%. How much should he invest?

12. Mr. Jaipal invested a sum of money in ₹ 250 shares paying 24% dividend and quoted at 20% premium. If the annual dividend received by him is ₹ 6960, calculate
(i) total investment.
(ii) the rate of return on his investment.

13. Karan buys 125 shares of ₹ 100 each of 'Reliance Technologies Ltd,' which pays a dividend of 6%. If his annual income is 4% of his total investment. Then, for what price did Karan buy all the shares?

14. Ashok invested ₹ 26400 on 12%, ₹ 25 shares of a company. If he receives a dividend of ₹ 2475, find the
(i) number of shares he bought.
(ii) market value of each share. ***[2016]***

15. A company pays a dividend of 15% on its ₹ 10 shares, from which it deducts tax at the rate of 22%. Find the annual income of a man, who owns 1000 shares of this company.

b 4 Marks Questions

16. Mahesh bought 600 shares of ₹ 50 each of 'Excel Computers'. He sold one-third of them when they were at a premium of ₹20 and the remaining when they were at a discount of ₹5. Find his gain or loss in the whole transaction.

17. By purchasing ₹ 25 shares, for ₹ 10 each, a man gets 4% profit on his investment. What rate per cent is company paying? What is his dividend, if he buys 80 shares?

18. Vishal wants to invest ₹ 27000 in buying shares. The shares of the following companies are available to him: ₹ 100 shares of company A at par value; ₹ 100 shares of company B at a premium of ₹ 25; ₹ 100 shares of company C at a discount of ₹ 10; ₹ 50 shares of company D at a premium of 20%. Find how many shares will he get, if he buys shares of company

(i) A ? (ii) B ? (iii) C ? (iv) D ?

19. A man sells 100, ₹ 25 shares of a company paying 10% dividend, at ₹ 50 each and invests the proceeds in ₹ 5 shares of another company at ₹ 10 each. Find his change in income, if the second company pays a dividend of 6%.

20. Mr. Dubey has 60 shares of nominal value ₹100 and he decides to sell them when they are at a premium of 60%. He invests the proceeds in shares of nominal value ₹ 50, quoted at 4% discount, paying 18% dividend annually.

Calculate

(i) the sale proceeds.

(ii) the number of shares he buys.

(iii) his annual dividend from these shares.

21. A company with 10000 shares of nominal value of ₹ 100 declares an annual dividend of 8% to the shareholders.

Calculate

(i) the total amount of dividend paid by the company.

(ii) Ramesh bought 90 shares of the company at ₹150 per share, then calculate the dividend.

(iii) Find the percentage return on the investment of Ramesh.

22. A company with 4000 shares of nominal value of ₹110 each declares on annual dividend of 15%.

Calculate

(i) the total amount of dividend paid by the company.

(ii) the annual income of Shahrukh who holds 88 shares in the company.

(iii) if he received 10% on his investment, find the price Shahrukh paid for each share. ***[2008]***

23. Divide ₹ 40608 into two parts such that, if one part is invested in 8%, ₹ 100 shares at 8% discount and the other part is invested in 9%, ₹ 100 shares at 8% premium, annual incomes from both the investments are equal. Find each part of the investments.

24. Jai Prakash wants to invest ₹ 3600. He invests ₹ 750 in ₹ 100 shares of 3.5% at ₹ 75, ₹ 1050 in ₹ 100 shares of 3% at ₹ 70 and the remaining in ₹ 100 shares of 6%. If his total yield is $5\frac{5}{9}$% of his investment, at what price did he buy the shares of 6%?

25. Pramod wants to invest ₹35000 in shares, such that the percentage return on his investment is $8\frac{1}{7}$%. He invested ₹6000 in 6%, ₹50 shares of 'Lakme' at ₹40, ₹15000 in 8%, ₹100 shares of 'Volta' at ₹125 and the remaining in 12%, ₹150 shares of 'BPL'. At what rate did he buy the 'BPL' shares?

26. Mr. Ghosh sold a certain number of ₹ 20 shares paying 8% dividend at ₹ 18 and invested the proceeds in ₹ 10 shares, paying 12% dividend at 50% premium. If the change in his annual income is ₹ 120, then find the number of shares sold by Mr. Ghosh.

27. Ramesh had ₹100 shares of 'Bihar Steel' paying 8% dividend. He sold them at a market price of ₹130 and invested the proceeds in buying ₹ 50 shares of 'Jindal Steel' available at ₹ 75 and paying 12% dividend. He thus increased his annual income by ₹ 360. How many shares did Ramesh sell?

28. Mihir and Sameer both invested ₹ 90000 each in buying shares of two companies. Mihir invested in buying ₹150 shares of 'ABC Solutions' available at a 20% premium and paying a dividend of 18%. Sameer invested in buying ₹225 shares of 'Mphasis Ltd.' available at a 20% discount and paying a dividend of 16%. Whose investment is better?

29. Ahmad had 1000 shares of a company with a face value of ₹ 50 and paying 10% dividend. He sold some of these shares at a discount of 10% and invested the proceeds in ₹ 20 shares at a premium of 50% and paying 10% dividend. If the changes in his income is ₹ 200, then find the number of shares sold by Ahmad.

30. Mr. Gupta has a choice to invest in ₹ 10 shares of two firms at ₹13 or at ₹16. If the first firm pays 5% dividend and the second firm pays 6% dividend per annum, then find

(i) which firm is paying better?

(ii) how much, in all, does Mr. Gupta invest, if he invests equally in both the firms and the difference between the returns from them is ₹30?

31. Mamta and Sonia invested equal amounts in buying shares of two different companies. Mamta bought shares of Hindalco paying a dividend of 14% on its ₹125 share available in the market for ₹140, while Sonia bought shares of HLL paying a dividend of 13% on its ₹300 share available in the market for ₹260. If Sonia's annual income is ₹1820 more than that of Mamta, then find the amount invested by each of them.

32. Amit and Richa invest ₹ 12000 each in buying shares of two companies. Amit buys 15%, ₹100 shares at a discount of ₹ 20, while Richa buys ₹ 25 shares at a premium of 20%. If both receive equal dividends at the end of the year, then find the rate per cent of the dividend declared by Richa's company.

33. Salman buys 50 shares of face value ₹ 100 available at ₹ 132.

(i) What is his investment?

(ii) If the dividend is 7.5%, then what will be his annual income?

(iii) If he wants to increase his annual income by ₹ 150, then how many extra shares should he buy? *[2013]*

34. Bhavna invested ₹ 20000 and ₹ 25000 in buying shares of 'Bharti Telecom' and 'Satyam Infoways', which later declared dividends of 10% and 12.5%, respectively. After collecting the dividends, Bhavna sells all her shares at a loss of 4% and 5% respectively on her investments. Find her total earnings.

35. Niharika invested her savings in the following manner: 25% of her savings in 12%, ₹ 100 shares available at 25% premium, 25% of her savings in 20%, ₹200 shares available at 25% premium and the balance in 18%, ₹ 50 shares available at 50% premium. If her total earnings from all these shares is ₹ 26350, then calculate her total savings and the investment made in each.

Hints and Answers

1. Hint Dividend on one share = 9% of 20

Ans. ₹ 108

2. Hint Here, total face value of shares = ₹5000

and dividend on 500 shares = ₹ 400

$\therefore$ Rate of dividend $= \left(\frac{400}{5000} \times 100\right)\%$

Ans. 8%

3. Hint Dividend = Number of shares $\times \frac{8}{100} \times 10$

Ans. 375

4. Hint Market value of 1 share = 100 + 60 = ₹160

$\therefore$ Total number of shares bought

$$= \frac{\text{Total investment}}{\text{Investment on one share}}$$

Ans. 3.75%

5. Hint

(i) Market value of one share = 100 + 20 = ₹120

(ii) Annual income

$$= \text{Number of shares} \times \frac{\text{Rate of dividend}}{100} \times \text{Nominal value of one share}$$

(iii) Percentage income $= \frac{\text{Annual income}}{\text{Total investment}} \times 100\%$

Ans. (i) ₹ 24000 (ii) ₹ 3000 (iii) 12.5%

6. Hint

(i) Number of shares purchased

$$= \frac{\text{Total investment}}{\text{Issued psice of each share}}$$

(ii) Annual dividend = Number of share purchased × Face value of a share × Rate of dividend

(iii) Rate of return $= \frac{\text{Dividend}}{\text{Invested amount}} \times 100$

Ans. (i) 500 (ii) ₹ 3000 (iii) 13%

7. Hint

(i) Number of shares $= \frac{\text{Money invested}}{\text{Market value of one share}}$

(ii) Total dividend (annual income)

$$= \text{Number of shares} \times \frac{\text{Rate of dividend}}{100} \times \text{Nominal value of one share}$$

(iii) Percentage return $= \frac{\text{Annual income}}{\text{Investment}} \times 100\%$

Ans. (i) 120 (ii) ₹ 2160 (iii) 22.5%

8. Hint (i) Number of shares $= \frac{\text{Money invested}}{\text{Market value of one share}}$

(ii) Yearly dividend = Number of shares $\times \frac{\text{Rate of dividend}}{100} \times$ Nominal value of one share

(iii) Percentage return $= \frac{\text{Annual dividend}}{\text{Investment}} \times 100$

Ans. (i) 300 (ii) ₹ 4500 (iii) 12.5%

9. Hint

(i) $\because$ Number of shares $= \dfrac{\text{Total dividend amount}}{\text{Dividend on one share}}$

$$= \frac{450}{\frac{10}{100} \times 50} = 90$$

$\therefore$ Total investment for 90 shares $= 90 \times 60$

(ii) Yield per cent $= \dfrac{\text{Annual income}}{\text{Investment}} \times 100\%$

Ans. (i) ₹ 5400 (ii) 8%

10. Hint

(i) $\because$ Number of shares $= \dfrac{\text{Total dividend amount}}{\text{Dividend on one share}}$

(ii) Percentage return $= \dfrac{\text{Annual income}}{\text{Investment}} \times 100$

Ans. (i) 240 (ii) $8\frac{1}{3}\%$

11. Do same as Example 6. **Ans.** ₹ 240000

12. Do same as Example 6.

Ans. (i) ₹ 34800 (ii) 20%

13. Hint Dividend $= 125 \times \dfrac{6}{100} \times 100 = ₹750$

$\because$ Annual income = 4% of total investment

$\therefore 750 = \dfrac{4}{100} \times$ Total investment

Ans. ₹ 18750

14. Hint

(i) Dividend = Number of shares $\times \dfrac{12}{100} \times 25$

(ii) Market price of share $= \dfrac{\text{Investment}}{\text{Number of shares}}$

Ans. (i) 825 (ii) ₹ 32

15. Hint Dividend $= 1000 \times \dfrac{15}{100} \times 10 = ₹1500$

Annual income $= ₹\left(1500 - \dfrac{22}{100} \times 1500\right)$

Ans. ₹ 1170

16. Do same as Example 3.

Ans. Gain of ₹ 2000

17. Hint Dividend on one share = 4% of ₹ 10 $= \dfrac{4}{100} \times 10$

$\therefore$ Rate of dividend $= \dfrac{\text{Dividend on one share}}{\text{Nominal value of one share}} \times 100$

Ans. 1.6% and ₹ 32

18. Hint

(i) Market value of a share of company $A = ₹100$

$\therefore$ Number of shares of company

$$A = \frac{\text{Investment}}{\text{Market value of one share}}$$

(ii) Market value of a share of company

$$B = 100 + 25 = ₹125$$

(iii) Market value of a share of company

$$C = 100 - 10 = ₹90$$

(iv) Market value of a share of company

$$D = ₹50 + 20\% \text{ of } ₹50 = ₹60$$

Ans. (i) 270 (ii) 216 (iii) 300 (iv) 450

19. Hint

(i) Sale proceeds $= 100 \times 50 = ₹5000$

(ii) Number of ₹ 5 shares $= \dfrac{5000}{10} = 500$

(iii) Total nominal value of ₹ 25 shares

$$= 100 \times 25 = ₹2500$$

(iv) Total nominal value of ₹ 5 shares

$$= 500 \times 5 = ₹2500$$

Ans. ₹ 100

20. Do same as Example 8.

Ans. (i) ₹ 9600 (ii) 200 (iii) ₹1800

21. Hint

(i) Total dividend = Number of shares

$$\times \frac{\text{Rate of dividend}}{100} \times \text{Nominal value of one share}$$

(ii) Dividend on 90 shares $= 90 \times$ Dividend on one share

(iii) Percentage return

$$= \frac{\text{Dividend}}{\text{Investment}} \times 100\% = \frac{720}{90 \times 150} \times 100$$

Ans. (i) ₹ 8000 (ii) ₹ 720 (iii) $5\frac{1}{3}\%$

22. Hint

(i) Total dividend = Number of shares

$$\times \frac{\text{Rate of dividend}}{100} \times \text{Nominal value of one share}$$

(ii) Shahrukh income $= 88 \times$ Dividend on one share

(iii) Let Shahrukh investments be x.

Then, 10% of $x = 1452$

$\Rightarrow \quad x = ₹14520$

$\therefore$ Price paid by Shahrukh for each share $= \dfrac{14520}{88}$

Ans. (i) ₹ 66000 (ii) ₹ 1452 (iii) ₹ 165

23. Hint Let the two parts be ₹ x and ₹ $(40608-x)$.

For first part,

Market value of each share = ₹100 − 8% of ₹100 = ₹92

$\therefore$ Number of shares bought $=\frac{x}{92}$

$\Rightarrow$ Total dividend = Number of share

$\times\frac{\text{Rate of dividend}}{100}\times\text{Face value}=\frac{2x}{23}$

For second part, Investment = ₹$(40608-x)$

Market value of each share = ₹100 + 8% of ₹100 = ₹108

$\therefore$ Number of shares bought $=\frac{40608-x}{108}$

$$\Rightarrow \quad \text{Total dividend}=\frac{40608-x}{108}\times\frac{9}{100}\times100$$

$$=₹\left(\frac{40608-x}{12}\right)$$

$$\therefore \quad \frac{2x}{23}=\frac{40608-x}{12}$$

Ans. ₹19872 and ₹20736

24. Hint

(i) Number of shares of 3.5% $=\frac{750}{75}=10$

$\therefore$ Annual income from 3.5% shares

$=10\times100\times\frac{3.5}{100}=₹35$

(ii) Number of shares of 3% $=\frac{1050}{70}=15$

$\therefore$ Annual income from 3% shares

$=15\times100\times\frac{3}{100}=₹45$

(iii) $\because$ Annual income

= Number of shares × Face value of one share

$\times\frac{\text{Rate of dividend}}{100}$

$\therefore$ Number of shares of 6% = 20

Now, market price of 6% shares $=\frac{\text{Investment}}{\text{Number of shares}}$

Ans. ₹90 per share

25. Do same as Q. 24.

Ans. ₹175 per share

26. Hint Let the number of shares sold by Mr. Ghosh be x.

$\therefore$ Total income on x shares = 8% of ₹ $20\times x$.

As Mr. Ghosh invested the proceed, i.e. ₹$18x$ in ₹10 shares at 50% premium and market value of each share

= ₹(10 + 50% of 10) = ₹15

$\therefore$ Number of new shares bought $=\frac{18x}{15}=\frac{6x}{5}$

Now, change (loss) in income = Total income on x shares − Total income on new shares

Ans. 750

27. Hint Let the number of shares sold by Ramesh be x.

$\therefore$ Number of shares bought

$=\frac{\text{Sum invested}}{\text{Market value of each share}}=\frac{26}{15}x$

Now, total dividend received $=\frac{26}{15}x\times\frac{12}{100}\times50=\frac{52}{5}x$

Given, change in income = 360

$$\Rightarrow \quad \frac{52}{5}x-8x=360$$ **Ans.** 150

28. Hint Income from '*ABC* solution'

$=\text{Number of shares}\times\frac{\text{Rate of dividend}}{100}$

$\times$ Face value of one share

Income from 'Mphasis Ltd.' $=\left(\frac{90000}{180}\right)\times\frac{16}{100}\times225$

Ans. Sameer's investment is better.

29. Hint Let the number of shares sold by Ahmad be x.

***Case* I** Dividend on each share = 10% of ₹50 = ₹5

$\therefore$ Dividend on x shares = ₹$5x$

Selling price of each share = ₹50 − 10% of ₹50 = ₹45

Thus, money obtained by selling x shares = ₹ $45x$

***Case* II** Some invested = ₹ $45x$

MV of each share = ₹20 + 50% of ₹20 = ₹30

$\therefore$ Number of shares bought

$=\frac{\text{Sum invested}}{\text{Market value of each share}}=\frac{45x}{30}=\frac{3x}{2}$

Now, total dividend received = Dividend on one share × Number of shares

= ₹ $3x$

Given, change in income = ₹200

$$\therefore \quad 5x-3x=200$$ **Ans.** 100

30. Hint

(i) Let the investment in each case be ₹(13×16).
Then, income from each firm

$=\text{Number of shares}\times\frac{\text{Rate of dividend}}{100}$

$\times$ Face value of one share

(ii) Let Mr. Gupta invest ₹x in each firm.

$$\text{Then,} \quad \frac{x}{13}\times\frac{5}{100}\times10-\frac{x}{16}\times\frac{6}{100}\times10=30$$

Ans. (i) First firm (ii) ₹62400

31. Hint Let the investment in each case be ₹ x.

Then, Mamta's income $= \frac{x}{140} \times \frac{14}{100} \times 125 = \frac{x}{8}$

and Sonia's income $= \frac{x}{260} \times \frac{13}{100} \times 300 = \frac{3x}{20}$

According to given condition, we have

$$\frac{3x}{20} = 1820 + \frac{x}{8}$$

Ans. ₹72800

32. Hint Annual dividend received by Amit

$= \text{Number of shares held by Amit} \times \frac{\text{Rate of dividend}}{100}$

$\times$ Face value of one share = ₹2250

$\therefore$ Annual dividend received by Richa = ₹2250

$\therefore$ Number of shares purchased by Richa

$= \frac{\text{Total invested amount by Richa}}{\text{Market value of one share purchased by Richa}}$

$= \frac{12000}{30} = 400$

Let r% be the rate of dividend declared by Richa company.

Then, $2250 = 400 \times \frac{r}{100} \times 25$

Ans. 22.5%

33. Hint

(i) Salman's investment in shares = Market value share $\times$ Number of share $= 132 \times 50$

(ii) Annual income

$= \text{Number of shares} \times \frac{\text{Rate of dividend}}{100}$

$\times$ Face value of one share

= ₹375

(iii) Let extra share he buys be x.

Then, total number of shares $= 50 + x$

According to the question, Salman's annual income increased by ₹150.

$\therefore (50 + x) \times 7.5 = 375 + 150$

Ans. (i) ₹6600 (ii) ₹375 (iii) 20

34. Net earning = Total amount earned from shares − Total investment

Ans. ₹3075

35. Do same as Example 10.

Ans. Total savings = ₹212500

Investment in first company = ₹53125

Investment in second company = ₹53125

Investment in third company = ₹ 106250

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1 A man invests ₹22500 in ₹50 shares available at 10% discount. If the dividend paid by the company is 12%, calculate

(i) the number of shares purchased. (ii) the annual dividend received.

(iii) the rate of return he gets on his investment. Give your answer correct to the nearest whole number.

2017

2 How much should a man invest in ₹ 50 shares selling at ₹ 60 to obtain an income of ₹ 450, if the rate of dividend declared is 10%. Also, find his yield per cent, to the nearest whole number.

2016

3 Ashok invested ₹ 26400 on 12%, ₹ 25 shares of a company. If he receives a dividend of ₹ 2475, find the

(i) number of shares he bought. (ii) market value of each share.

2015

4 Rohit invested ₹ 9600 on ₹ 100 shares at ₹ 20 premium paying 8% dividend. Rohit sold the shares when the price rise to ₹ 160. He invested the proceeds (excluding dividend) in 10%, ₹ 50 shares at ₹ 40. Find

(i) the original number of shares. (ii) the sale proceeds.

(iii) the new number of shares. (iv) the change in the two dividends.

2014

5 Salman invests a sum of money in ₹ 50 shares, paying 15% dividend quoted at 20% premium. If his annual dividend is ₹ 600, then calculate

(i) the number of shares, he bought. (ii) his total investment.

(iii) the rate of return on his investment.

2013

6 Salman buys 50 shares of face value ₹ 100 available at ₹ 132.

(i) What is his investment?

(ii) If the dividend is 7.5%, then what will be his annual income?

(iii) If he wants to increase his annual income by ₹ 150, then how many extra shares should he buy?

2012

7 A man invests ₹ 9600 on ₹ 100 shares at ₹ 80. If the company pays him 18% dividend, then find

(i) the number of shares, he buys. (ii) his total dividend.

(iii) his percentage return on the shares.

2011

8 Mr. Prakash invested ₹ 52000 on ₹ 100 shares at a discount of ₹ 20, paying 8% dividend. At the end of one year, he sells the shares at a premium of ₹ 20. Find

(i) the annual dividend. (ii) the profit earned including his dividend.

* All these questions are completely covered in chapter either as solved examples or in chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. Which of the following is a better investment?
(i) 10%, ₹ 100 shares at ₹ 120
(ii) 9%, ₹ 100 shares at ₹ 110
(a) Only (i)
(b) Only (ii)
(c) Both (i) and (ii)
(d) Data insufficient

2. The total investment made in buying ₹ X shares at a premium of 25% is ₹ $125X$. The number of shares bought is
(a) 500 (b) 250
(c) 125 (d) 100

3. Ms. Shriya invested some amount buying ₹ 150 shares of a company which pays dividend at the rate of 8% per annum. If she gets back 10% per annum on his investment, then the value at which she bought the share, is
(a) ₹ 100 (b) ₹ 120
(c) ₹ 180 (d) ₹ 210

4. If ₹ 25000 more invested, then 400 more shares can be purchased. The market value of each share is
(a) ₹ 62.50 (b) ₹ 77.50
(c) ₹ 85.00 (d) Can't be determined

5. Mr. Anil bought ₹ 110 shares of a company at a discount of ₹ 10. If the company pays a dividend at the rate of x% per annum and Mr. Anil earns at the rate of y% per annum on his investment, then $x : y$ is
(a) 10 : 9 (b) 9 : 10
(c) 11 : 10 (d) 10 : 11

6. Ms. Neelam made an investment of ₹ 27000 in 10%, ₹ 100 shares at ₹ 90, while Mr. Ram made an investment of ₹ 30000 in 8%, ₹ 150 shares at ₹ 120. If both of them have sold their shares at the end of a year for ₹ 110 each, then their individual earnings are, respectively
(a) ₹ 9000 and ₹ 500
(b) ₹ 500 and ₹ 9000
(c) ₹ 9000 and ₹ 9000
(d) None of the above

7. Ram invested ₹ 54000 in 18%, ₹ 200 shares of Jindal Electronics. He bought them at a rate such that the percentage return on his investment is 16%. Later he sold them, when the market price was ₹ 275 and invested the proceeds in buying 24%, ₹ 100 shares of Golmal Entertainment. If his annual income increased by ₹ 1920, then the rate, at which he bought the Golmal Entertainment shares is
(a) ₹ 120 (b) ₹ 180
(c) ₹ 150 (d) ₹ 160

8. A man has choice to invest in ₹ 100 shares of two companies X and Y. Shares of company X are available at a premium of 20% and it pays 5% dividend whereas shares of company Y are available at a discount of 10% and it pays 7% dividend. If the man invests equally in both the companies and the sum of the returns from them is ₹1290, then find how much in all does he invest?
(a) ₹ 20000 (b) ₹ 21000
(c) ₹ 21500 (d) ₹ 21600

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 393.

CHAPTER

4

Linear Inequations

In earlier classes, we study about such quantities which are bigger (greater) or smaller (lesser) with respect to each other or any third quantity. Between any two quantities (say x and y) in the universe, only three conditions hold which are '<', '>' and '=', i.e. $x < y$, $x > y$ and $x = y$. The symbols '>', '<', '≥', '≤' are called the sign of inequality or inequation.
In this chapter, we will study about the solution of inequations in one variable and their representation on number line.

Chapter Objectives

- Solving the Linear Inequations Algebraically
- Representation of Solution of Inequality on the Number Line

Topic 1
Solving the Linear Inequations Algebraically

Inequality or Inequation

We can define inequations as a statement involving variable(s) and the sign of inequality <, >, ≤ or ≥.

Or

Two real numbers or two algebraic expressions related by the symbols < (less than), > (greater than),
≤ (less than or equal to), ≥ (greater than or equal to) form **an inequality**.

Inequalities Among Real Numbers

There are two cases of inequality, which are discussed below

***Case* I For less than (or less than equal to) inequality**

Let x and y be any real numbers, then

(i) **'x is less than y'** is written as $x < y$. It is possible if and only if $y - x$ is positive. e.g. For numbers 5 and 8, 5 is less than 8, i.e. $5 < 8$. Here, $8 - 5 = 3$, which is positive.

(ii) **'x is less than or equal to y'** is written as $x \leq y$. It is possible if and only if $y - x$ is either positive or zero.

Case II For greater than (or greater than equal to) inequality

(i) **'x is greater than y'** is written as $x > y$. It is possible if and only if $x - y$ is positive. e.g. For numbers 7 and 3, 7 is greater than 3, i.e. $7 > 3$. Here, $7 - 3 = 4$, which is positive.

(ii) **'x is greater than or equal to y'** is written as $x \geq y$. It is possible if and only if $x - y$ is either positive or zero.

Example 1. Express the relation between the following numbers with the help of inequality and verify it.

(i) 5 and 7 (ii) $\frac{4}{5}$ and $\frac{1}{5}$

(iii) – 7 and – 3 (iv) 4 and – 8

Sol.

(i) Given numbers are 5 and 7.
We know that 7 is always greater than 5.
$\therefore \quad 7 > 5$
Verification $7 - 5 = 2$, which is positive.

(ii) Given rational numbers are $\frac{4}{5}$ and $\frac{1}{5}$.
We know that if denominators of two fractions are same, then fraction having greater numerator is greater.
$\therefore \quad \frac{4}{5} > \frac{1}{5}$
Verification $\frac{4}{5} - \frac{1}{5} = \frac{4-1}{5} = \frac{3}{5}$, which is positive.

(iii) Given numbers are -7 and -3.
We know that -3 is always greater than -7.
$\therefore \quad -3 > -7$
Verification $-3 - (-7) = -3 + 7 = 4$, which is positive.

(iv) Given numbers are 4 and – 8.
We know that 4 is always greater than – 8.
$\therefore \quad 4 > -8$
Verification $4 - (-8) = 4 + 8 = 12$, which is positive.

Linear Inequality (Inequation)

An inequation is said to be linear if and only if exponent of each variable occurring in it is one (or each variable occurs in first degree only) and there is no term involving the product of the variables.

e.g. $ax + b \leq 0$, $ax + by + c > 0$, $ax \leq 5$, $by \geq 10$, etc.

Linear Inequation in One Variable

A linear inequation which has only one variable, is called linear inequation in one variable and can be written as

$ax + b < 0$ or $ax + b > 0$ or $ax + b \leq 0$ or $ax + b \geq 0$, where a, b are real number and $a \neq 0$.

Example 2. Identify which of the following is linear inequation in one variable?

(i) $ax + 4 \geq 3 + ay$ (ii) $4x + 7 \geq 2$

(iii) $3y + 8 > 5 + y$ (iv) $3x < 7y$

Sol.

(i) Given, $ax + 4 \geq 3 + ay$ is a linear inequation but it is not linear inequation in one variable, because it has two variables, i.e. x and y.

(ii) Given, $4x + 7 \geq 2$ is a linear inequation in one variable, because it has only one variable, i.e. x.

(iii) Given, $3y + 8 > 5 + y$ is a linear inequation in one variable, because it has only one variable, i.e. y.

(iv) Given, $3x < 7y$ is a linear inequation but it is not linear inequation in one variable, because it has two variables, i.e. x and y.

Replacement and Solution Sets

The set from which values of the variable involved in the inequality are chosen, is called the **replacement set**. Generally, we use either N (set of natural numbers; 1, 2, 3, 4, ...) or W (set of whole numbers; 0, 1, 2, 3, ...) or I (set of integers; ... – 3, – 2, – 1, 0, 1, 2, 3, ...) or R (real number) as the replacement set.

Also, the solution to an inequation is a number which, when substituted for the variable, makes the inequation true and the set of all solutions of the given inequation is called the **solution set** of the inequation. Solution set is the subset of replacement set.

Example 3. If {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} is the replacement set, then find the solution set of the inequality $x > 5$.

Sol. Given, $x > 5$

$\therefore$ The solution set is {6, 7, 8, 9, 10, 11}.

Example 4. If {– 3, – 2, – 1, 0, 1, 2, 3, 4, 5} is the replacement set, then find the solution set of $x \leq 3$.

Sol. Given, $x \leq 3$

$\therefore$ The solution set is {– 3, – 2, – 1, 0, 1, 2, 3}.

Rules for Solving Linear Inequations in One Variable

Some rules for solving linear inequations in one variable are given below

(i) **Addition (or Subtraction) Rule** Equal numbers or expressions may be added to or subtracted from both sides of an inequation, produces an equivalent inequation.

e.g. $x - 3 > 4$ is equivalent to $x - 3 + 2 > 4 + 2$

or $x + 3 > 5$ is equivalent to $x + 3 - 1 > 5 - 1$.

(ii) **Multiplication (or Division) Rule** There are following two types of multiplication (or division) rule

(a) Multiplication (or division) on both sides of an inequation with same positive number produces an equivalent inequation.

e.g. $5x < 6$ is equivalent to $\frac{5x}{5} < \frac{6}{5}$

or $\frac{x}{7} > 2$ is equivalent to $\frac{x}{7} \times 7 > 2 \times 7$.

(b) Multiplication (or division) on both sides of an inequation with negative number produces an equivalent inequation, if the symbol of inequality is reversed.

e.g. $x > 9$ is equivalent to $(-1)(x) < (9)(-1)$.

(iii) If $ax + b > c$, then it can be written as $c < ax + b$.

(iv) If $x > y$, then $\frac{1}{x} < \frac{1}{y}$ and if $x < y$, then $\frac{1}{x} > \frac{1}{y}$.

(v) The square root of any real number is always greater than or equal to zero.

(vi) If y and t are of same sign, then

$\frac{x}{y} > \frac{s}{t} \Rightarrow \frac{y}{x} < \frac{t}{s}$ or $xt > sy$ and $\frac{x}{y} < \frac{s}{t} \Rightarrow \frac{y}{x} > \frac{t}{s}$ or $xt < sy$.

(vii) If a, b, c and d are four real numbers, such that

(a) $a > b$ and $c > d$, then $a \pm c > b \pm d$

(b) $a < b$ and $c < d$, then $a \pm c < b \pm d$

Example 5. Show that the sign of inequality remains same, if we add and subtract 3 and 2 respectively from the following inequalities.

(i) $7 < 10$ (ii) $-5 > -7$

Sol.

(i) Given as, $7 < 10 \Rightarrow 7 + 3 < 10 + 3$ [adding 3 on both sides]

$\Rightarrow 10 < 13 \Rightarrow 10 - 2 < 13 - 2$ [subtracting 2 from both sides]

$\Rightarrow 8 < 11$, which is true.

Hence, the sign of the given inequality remains same.

(ii) Given as, $-5 > -7$

$\Rightarrow -5 + 3 > -7 + 3$ [adding 3 on both sides]

$\Rightarrow -2 > -4$

$\Rightarrow -2 - 2 > -4 - 2$ [subtracting 2 from both sides]

$\Rightarrow -4 > -6$, which is true.

Hence, the sign of the given inequality remains same.

Example 6. Show that the sign of inequality remains same, if we multiply and divide the inequality 15 > 6 by 2 and 6, respectively.

Sol. Given inequality is

$15 > 6$

$\Rightarrow 15 \times 2 > 6 \times 2$ [multiplying both sides by 2]

$\Rightarrow 30 > 12$

$\Rightarrow \frac{30}{6} > \frac{12}{6}$ [dividing both sides by 6]

$\Rightarrow 5 > 2$, which is true.

Hence, this shows that the sign of inequality remains same, if we multiply or divide on both sides of inequality by same positive number.

Example 7. Show that the sign of inequality 18 >12 changes, if we multiply by negative number (say) – 3.

Sol. We have, $18 > 12$

On multiplying both sides by -3, we get

$18 \times (-3) < 12 \times (-3) \Rightarrow -54 < -36$

Hence, this shows that sign of inequality changes, when both sides of the inequality multiplied by same negative number.

Example 8. Check whether the sign of inequality 7 <14 changes, if divide by negative number (say) –7.

Sol. We have, $7 < 14$

On dividing both sides by -7, we get

$\frac{7}{(-7)} > \frac{14}{(-7)} \Rightarrow -1 > -2$, which is true.

Hence, this shows that the sign of inequality changes, when both sides of the inequality divided by any negative number.

Example 9. Check which of the following inequalities are correct?

(i) If $a > b$ and $4 > 2$, then $3a + 4 < 3b + 2$.

(ii) If $x > y$, then $2x + 5 > 2y + 4$.

(iii) If $5x \geq 4$ and $7y \leq 4$, then $5x = 7y$.

(iv) If $\frac{4}{x} > \frac{3}{y}$, then $\frac{x}{4} > \frac{y}{3}$.

Sol.

(i) Given, $a > b$ and $4 > 2$

Now, $a > b \Rightarrow 3a > 3b$ [multiplying both sides by 3]

Also, we have $4 > 2$

$\therefore 3a + 4 > 3b + 2$

But $3a + 4 < 3b + 2$ is given in the question.

Hence, the given statement is incorrect.

(ii) Given, $x > y \Rightarrow 2x > 2y$ [multiplying both sides by 2]

We know that, $5 > 4$

$\therefore 2x + 5 > 2y + 4$

Hence, this statement is correct.

(iii) Given, $5x \geq 4$ and $7y \leq 4$

$\therefore 7y \leq 4 \leq 5x \Rightarrow 7y \leq 5x$

Hence, the given statement is incorrect.

(iv) Given, $\frac{4}{x} > \frac{3}{y}$ and its reciprocal is $\frac{x}{4} < \frac{y}{3}$.

Hence, the given statement is not incorrect.

Method to Solve a Linear Inequation in One Variable

There are following steps used to solve the linear inequation in one variable.

Step I Remove the fractions (or decimals) by multiplying both the sides by an appropriate factor.

Step II On putting all variable terms on one side and all constants on the other side.

Step III Make the coefficient of the variable 1.

Step IV Choose the solution set from the replacement set.

Note When replacement set is not given, then we take R (real number) as replacement set.

Example 10. Write the solution of the following inequation in the set notation form $5x - 10 \le 2x + 2$.

Sol. Given, $5x - 10 \le 2x + 2$

$\Rightarrow \quad 5x - 10 + 10 \le 2x + 2 + 10$ [adding 10 on both sides]

$\Rightarrow \quad 5x \le 2x + 12 \Rightarrow 5x - 2x \le 2x + 12 - 2x$ [subtracting $2x$ from both sides]

$\Rightarrow \quad 3x \le 12 \Rightarrow \frac{3x}{3} \le \frac{12}{3}$ [dividing both sides by 3]

$\Rightarrow \quad x \le 4$

Hence, the solution set is $\{x : x \le 4, x \in R\}$ or $(-\infty, 4]$.

Example 11. Rohit needs a minimum of 360 marks in four tests in a Mathematics course to obtain an A grade. On his first three tests, he scored 88, 96, 79 marks. What should his score be in the fourth test so that he can make an A grade?

Sol. Let Rohit score x marks in the fourth test. Then, the sum of Rohit's test scores should be greater than or equal to 360,

i.e. $\quad 88 + 96 + 79 + x \ge 360 \Rightarrow 263 + x \ge 360$

$\Rightarrow \quad 263 + x - 263 \ge 360 - 263$ [subtracting 263 from both sides]

$\Rightarrow \quad x \ge 97$

Hence, Rohit should score 97 marks or greater than 97 marks in the fourth test to obtain A grade.

Example 12. An integer is such that one-third of the next integer is atleast 2 more than one-fourth of the previous integer. Find the smallest value of the integer.

Sol. Let the integer be x, then one-third of the next integer is $\frac{x+1}{3}$ and one-fourth of the previous integer is $\frac{x-1}{4}$.

According to the question,

$$\frac{x+1}{3} \ge \frac{x-1}{4} + 2 \Rightarrow \frac{12(x+1)}{3} \ge \frac{12(x-1)}{4} + 2 \times 12$$

[multiplying both sides by 12]

$\Rightarrow \quad 4(x+1) \ge 3(x-1) + 24$

$\Rightarrow \quad 4x + 4 \ge 3x - 3 + 24$

$\Rightarrow \quad 4x + 4 - (3x + 4) \ge 3x + 21 - (3x + 4)$ [subtracting $(3x + 4)$ from both sides]

$\Rightarrow \quad 4x + 4 - 3x - 4 \ge 3x + 21 - 3x - 4$

$\Rightarrow \quad x \ge 17$

Hence, the smallest value of x is 17.

Example 13. P is the solution set of $8x - 1 > 5x + 2$ and Q is the solution set of $7x - 2 \ge 3(x + 6)$, where $x \in N$. Find the set $P \cap Q$.

Sol. Given, $8x - 1 > 5x + 2$

$\Rightarrow \quad 8x - 1 + (-5x + 1) > 5x + 2 + (-5x + 1)$ [adding $(-5x + 1)$ on both sides]

$\Rightarrow \quad 3x > 3 \Rightarrow x > 1$ [dividing both sides by 3]

But $x \in N$

$\therefore \quad P = \{2, 3, 4, 5, \ldots\}$

Also, $7x - 2 \ge 3(x + 6) \Rightarrow 7x - 2 \ge 3x + 18$

$\Rightarrow \quad 7x - 2 + (-3x + 2) \ge 3x + 18 + (-3x + 2)$ [adding $(-3x + 2)$ on both sides]

$\Rightarrow \quad 4x \ge 20 \Rightarrow x \ge 5$ [dividing both sides by 4]

But $x \in N$

$\therefore \quad Q = \{5, 6, 7, 8, \ldots\}$

Hence, the set $P \cap Q$ is $\{5, 6, 7, 8, \ldots\}$.

Topic Exercise 1

1. Express the relation between the following numbers with the help of the less than or greater than inequalities. Also, verify it

   (i) 7 and 10 (ii) -8 and 3 (iii) -11 and -9

   (iv) $\frac{1}{5}$ and $\frac{3}{5}$ (v) $\frac{-7}{3}$ and $\frac{4}{3}$

2. Identify which of the following is linear inequation in one variable?

   (i) $2x + y \ge 4$ (ii) $3x + 11 \ge 5$

   (iii) $x + 7 \ge 8 + 4y$ (iv) $10t + 3 \le 7s + 10$

3. Check which of the following inequations are correct?

   (i) If $s > t$, then $3s + 5 > 3t + 2$

   (ii) If $\frac{1}{x} < \frac{2}{y}$, then $\frac{x}{5} < \frac{y}{2}$

   (iii) If $7x \ge 5$ and $3y \le 5$, then $7x = 3y$

   (iv) If $x > y$ and $9 > 6$, then $4x + 9 > 4y + 6$.

4. Given $x \in \{-2, -3, -4, -5, -6, -7\}$ and $9 \le 1 - 2x$, find the solution set of the given inequation.

5. Write the solution of inequation $13t + 3 \le 16, \forall x \in R$ in the set notation form.

6. Write the solution of inequation $4x + 1 < 13, \forall x \in N$ in the set notation form.

7. Solve $-12x > 30$, when

   (i) x is a natural number. (ii) x is an integer.

8. Solve $3x + 8 > 2$, when

   (i) x is an integer. (ii) x is a real number.

9. Solve the inequality for real x, $4x + 3 < 6x + 7$.

10. Solve the inequality $5x - 3 < 3x + 1$, when x is an integer.

11. Solve the inequality $3(x - 1) \le 2(x - 3)$ for real x.

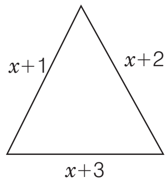
12. Solve the inequality $2(2x + 3) - 10 < 6(x - 2)$ for real x.

13. If P is the solution set of $-3x + 4 < 2x - 3, x \in N$ and Q is the solution set of $4x - 5 < 12, x \in W$, then find $Q \cap P$.

14. If $x \in R$, A is the solution set of $2(x-1) < 3x - 1$ and B is the solution set of $4x - 4 \le 8 + x$, then $A \cup B$.

15. Let $A = \{x : 11x - 5 > 7x + 3, x \in R\}$ and $B = \{x : 18x - 9 \ge 15 + 12x, x \in R\}$, then find the solution set of $A \cap B$.

16. Consider the following figure



If the perimeter of triangle is less than equal to 33 and greater than equal to 24. Then, find the solution set, where $x \in N$.

17. One-third of a bamboo pole is buried in mud, one-sixth of it is in water and the part above the water is greater than or equal to 3 units. Find the length of the shortest pole.

18. Find the greatest integer which is such that, if 7 is added to its double, then the resulting number becomes greater than three times the integer.

19. Find all pairs of consecutive even positive integers, both of which are larger than 5, such that their sum is less than 23.

20. The sum of three consecutive integer must not be more than 12. What is the minimum positive integer?

Hints and Answers

1. Do same as Example 1.

Ans. (i) $7 < 10$ (ii) $-8 < 3$ (iii) $-11 < -9$

(iv) $\frac{1}{5} < \frac{3}{5}$ (v) $\frac{-7}{3} < \frac{4}{3}$

2. Do same as Example 2.

Ans. Only (ii) is linear inequation in one variable.

3. Do same as Example 9.

Ans. (i) Correct (ii) Incorrect (iii) Incorrect (iv) Correct

4. Hint Adding both sides $2x - 9$ in $9 \le 1 - 2x$.

Ans. $\{-4, -5, -6, -7\}$

5. Hint Subtracting 3 from both sides. **Ans.** $(-\infty, 1]$

6. Hint Given, $4x + 1 < 13 \Rightarrow 4x + 1 - 1 < 13 - 1$

$\Rightarrow 4x < 12 \Rightarrow \frac{4x}{4} < \frac{12}{4} \Rightarrow x < 3$ **Ans.** $\{1, 2\}$

7. Hint $-12x > 30 \Rightarrow x < -\frac{30}{12}$

[dividing both sides by – 12]

Ans. (i) No solution (ii) $\{\ldots, -4, -3\}$

8. Hint $3x + 8 > 2 \Rightarrow 3x + 8 - 8 > 2 - 8$

$\Rightarrow 3x > -6 \Rightarrow x > -2$

(i) $\{-1, 0, 1, 2, \ldots\}$ (ii) $(-2, \infty)$

9. Hint Subtracting $(4x + 7)$ from both sides of $4x + 3 < 6x + 7$. **Ans.** $(-2, \infty)$

10. Hint Subtracting $(3x - 3)$ from both sides of $5x - 3 < 3x + 1$.

Ans. $\{\ldots, -4, -3, -2, -1, 0, 1\}$

11. Hint $3x - 3 \le 2x - 6$

On adding both sides $-2x + 3$, we get $x \le -3$

Ans. $(-\infty, -3]$

12. $4x + 6 - 10 < 6x - 12$

$\Rightarrow 4x - 4 < 6x - 12$

On adding both sides $(-4x + 12)$, we get

$8 < 2x$ **Ans.** $(4, \infty)$

13. Hint The solution set of $-3x + 4 < 2x - 3, x \in N$ is $P = \{2, 3, 4, \ldots\}$ and the solution set of $4x - 5 < 12, x \in W$ is $Q = \{0, 1, 2, 3, 4\}$.

Ans. $Q \cap P = \{2, 3, 4\}$

14. Hint $A = (-1, \infty)$ and $B = (-\infty, 4]$

Ans. $A \cup B = (-\infty, \infty)$

15. Do same as Q. 13. **Ans.** $A \cap B = [4, \infty)$

16. Hint $24 \le$ Perimeter of triangle ≤ 33

$\Rightarrow 24 \le 3x + 6 \le 33 \Rightarrow 6 \le x \le 9$

Ans. $\{6, 7, 8, 9\}$

17. Hint Let length of the bamboo pole $= x$ units

Length of bamboo pole above water

$$= x - \left(\frac{x}{3} + \frac{x}{6}\right) = \frac{x}{2}$$

According to the question, $\frac{x}{2} \ge 3$

Ans. Length of shortest pole $= 6$ units

18. Hint Let greatest integer $= x$

$\therefore$ $7 + 2x > 3x$ **Ans.** 6

19. Hint Let pair of consecutive even positive integers be x and $x + 2$.

$\therefore$ $x > 5$, $x + 2 > 5$ and $2x + 2 < 23$

Now, find the values of x which simultaneously satisfy all the inequations.

Ans. (6, 8) or (8, 10) or (10, 12)

20. Hint Let the three consecutive integers be x, $(x + 1)$ and $(x + 2)$.

Then, $x + (x + 1) + (x + 2) \le 12$

$\therefore$ $x \le 3$

Ans. 1

Topic 2
Representation of Solution of Inequality on the Number Line

Representation of Solution Set of Linear Inequation in One Variable on Number Line

Use the following rules to represent the solution of a linear inequation in one variable on number line

(i) If the inequation involves $\geq$ or $\leq$, then draw filled circle or dark circle (•) on the number line to show that, the number corresponding to the filled circle or dark circle is included in the solution set.

(ii) If the inequation involves $>$ or $<$, then draw unfilled circle or blank circle (◦) on the number line to show that, the number corresponding to the unfilled circle or blank circle is excluded from the solution set.

Example 1. Solve the inequation $37-(3x+5) \geq 9x-8\,(x-3)$ and represent the solution set on the number line.

Sol. Given, $37-(3x+5) \geq 9x-8(x-3)$

$\Rightarrow \quad 37-3x-5 \geq 9x-8x+24 \Rightarrow 32-3x \geq x+24$

$\Rightarrow \quad 32-3x-32 \geq x+24-32$ [subtracting 32 from both sides]

$\Rightarrow \quad -3x \geq x-8$

$\Rightarrow \quad -3x-x \geq x-8-x$ [subtracting x from both sides]

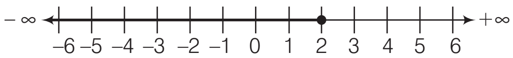
$\Rightarrow \quad -4x \geq -8$

$\Rightarrow \quad \dfrac{-4x}{-4} \leq \dfrac{-8}{-4}$ [dividing both sides by – 4]

$\Rightarrow \quad x \leq 2$

$\therefore$ Solution set $= (-\infty, 2]$

On the number line, it can be represented as



Here, the dark portion on the number line represents the solution of the given inequation.

Example 2. Solve the inequation $2x-7<4$, $x \in \{1, 2, 3, 4, 5, 6, 7\}$ and represent the solution set on the number line.

Sol. Given, $2x-7<4$

$\Rightarrow \quad 2x-7+7<4+7$ [adding 7 on both sides]

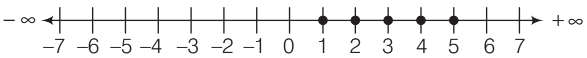
$\Rightarrow \quad 2x<11 \Rightarrow \dfrac{2x}{2}<\dfrac{11}{2}$ [dividing both sides by 2]

$\Rightarrow \quad x<5.5$

But it is given that, $x \in \{1, 2, 3, 4, 5, 6, 7\}$.

Hence, the solution set is $\{1, 2, 3, 4, 5\}$.

On number line, it can be represented as



Example 3. Solve the following inequation and represent the solution set on a number line.

$$-8\frac{1}{2}<-\frac{1}{2}-4x \leq 7\frac{1}{2}, x \in I$$ ***[2017]***

Sol. Given, $\quad -8\dfrac{1}{2}<-\dfrac{1}{2}-4x \leq 7\dfrac{1}{2}$

$\Rightarrow \quad -\dfrac{17}{2}<-\dfrac{1}{2}-4x \leq \dfrac{15}{2}$

$\Rightarrow \quad -\dfrac{17}{2}+\dfrac{1}{2}<\dfrac{-1}{2}-4x+\dfrac{1}{2} \leq \dfrac{15}{2}+\dfrac{1}{2}$

$\left[\text{adding } \dfrac{1}{2} \text{ on each types}\right]$

$\Rightarrow \quad -\dfrac{16}{2}<-4x \leq \dfrac{16}{2}$

$\Rightarrow \quad -8<-4x \leq 8$

$\Rightarrow \quad \dfrac{-8}{-4}>\dfrac{-4x}{-4} \geq \dfrac{8}{-4}$ [dividing each term by (–4)]

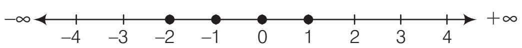
$\Rightarrow \quad 2>x \geq -2 \quad$ or $\quad -2 \leq x<2$

But it is given that $x \in I$.

$\therefore$ The solution set $=\{x:-2 \leq x<2, \; x \in I\}$

$=\{-2,-1,0,1\}$

$\therefore$ The graphical representation of the solution set is as follows



Example 4. Find the value of x, which satisfy the inequation $-2\dfrac{5}{6}<\dfrac{1}{2}-\dfrac{2x}{3} \leq 2, \forall\, x \in W$ and also represent the solution set on the number line. ***[2014]***

Sol. Given, $\quad -2\dfrac{5}{6}<\dfrac{1}{2}-\dfrac{2x}{3} \leq 2 \Rightarrow -\dfrac{17}{6}<\dfrac{3-4x}{6} \leq 2$

$\Rightarrow \quad \dfrac{-17}{6} \times 6<\dfrac{3-4x}{6} \times 6 \leq 2 \times 6$

[multiplying each term by 6]

$\Rightarrow \quad -17<3-4x \leq 12$

$\Rightarrow \quad -17-3<3-4x-3 \leq 12-3$

[subtracting 3 from each term]

$\Rightarrow \quad -20<-4x \leq 9$

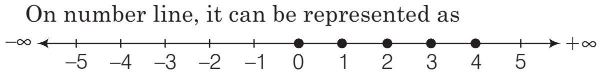
$\Rightarrow \quad \dfrac{-20}{-4}>\dfrac{-4x}{-4} \geq \dfrac{9}{-4}$ [dividing each term by (– 4)]

$\Rightarrow \quad 5>x \geq \dfrac{-9}{4} \Rightarrow \dfrac{-9}{4} \leq x<5$

Since, $x \in W$, i.e. $x \in \{0, 1, 2, 3, 4, \ldots\}$

$\therefore$ The solution set is $\{0, 1, 2, 3, 4\}$.

On number line, it can be represented as



Example 5. Solve the following inequation and write the solution set. Also, represent the solution set on the number line.

$$\frac{-x}{3} \leq \frac{x}{2} - 1\frac{1}{3} < \frac{1}{6},\ x \in R$$ **[2013]**

Sol. Given inequation is $-\frac{x}{3} \leq \frac{x}{2} - 1\frac{1}{3} < \frac{1}{6}$

On splitting the above inequation, we get

$$\frac{-x}{3} \leq \frac{x}{2} - 1\frac{1}{3} \text{ and } \frac{x}{2} - 1\frac{1}{3} < \frac{1}{6}$$

Consider $\frac{-x}{3} \leq \frac{x}{2} - 1\frac{1}{3} \Rightarrow -\frac{x}{3} \leq \frac{x}{2} - \frac{4}{3}$

$\Rightarrow \quad -\frac{x}{3} \times 6 \leq \left(\frac{x}{2} - \frac{4}{3}\right) \times 6$ [multiplying both sides by 6]

$\Rightarrow \quad -2x \leq 3x - 8 \Rightarrow -2x + 2x + 8 \leq 3x - 8 + 2x + 8$ [adding $(2x + 8)$ on both sides]

$\Rightarrow \quad 8 \leq 5x \Rightarrow \frac{8}{5} \leq \frac{5x}{5}$ [dividing both sides by 5]

$\Rightarrow \quad \frac{8}{5} \leq x$ ... (i)

and $\quad \frac{x}{2} - 1\frac{1}{3} < \frac{1}{6} \Rightarrow \frac{x}{2} - \frac{4}{3} < \frac{1}{6}$

$\Rightarrow \quad 6\left(\frac{x}{2} - \frac{4}{3}\right) < \frac{1}{6} \times 6$ [multiplying both sides by 6]

$\Rightarrow \quad 3x - 8 < 1$

$\Rightarrow \quad 3x - 8 + 8 < 1 + 8$ [adding 8 on both sides]

$\Rightarrow \quad 3x < 9$

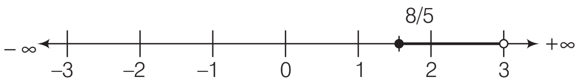
$\Rightarrow \quad \frac{3x}{3} < \frac{9}{3}$ [dividing both sides by 3]

$\Rightarrow \quad x < 3$ ...(ii)

From Eqs. (i) and (ii), we get $\frac{8}{5} \leq x < 3$

$\therefore$ Solution set $= \left\{x : \frac{8}{5} \leq x < 3, x \in R\right\} = x \in \left[\frac{8}{5}, 3\right)$

On the number line, it can be represented as



Here, the dark portion on the number line represents the solution set of the given inequation.

Example 6. Solve the following inequation, write the solution set and represent it on the number line.

$$-3(x - 7) \geq 15 - 7x > \frac{x + 1}{3},\ x \in R$$ **[2016]**

Sol. Given inequation is $-3(x - 7) \geq 15 - 7x > \frac{x+1}{3}, x \in R$

$\therefore \quad -3(x - 7) \geq 15 - 7x$ and $15 - 7x > \frac{x + 1}{3}$

[splitting into two inequations]

Consider $-3(x - 7) \geq 15 - 7x$

$\Rightarrow \quad -3x + 21 \geq 15 - 7x$

$-3x + 21 + 7x - 21 \geq 15 - 7x + 7x - 21$

[adding $(7x - 21)$ on both sides]

$\Rightarrow 4x \geq -6 \Rightarrow x \geq -\frac{6}{4} \Rightarrow x \geq -\frac{3}{2}$ [dividing both sides by 4]

$\Rightarrow \quad x \geq -1.5$ ...(i)

and $\quad 15 - 7x > \frac{x + 1}{3}$

$\Rightarrow \quad 3(15 - 7x) > x + 1$ [multiplying both sides by 3]

$\Rightarrow \quad 45 - 21x > x + 1$

$\Rightarrow \quad 45 - 21x + 21x - 1 > x + 1 + 21x - 1$

[adding $(21x - 1)$ on both sides]

$\Rightarrow \quad 44 > 22x$ or $22x < 44$

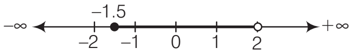
$\Rightarrow \quad x < 2$ [dividing both sides by 22] ...(ii)

From Eqs. (i) and (ii), we get

$-1.5 \leq x < 2$

$\therefore$ The solution set $= \{x : -1.5 \leq x < 2, x \in R\} = x \in [-1.5, 2)$

The graph of the solution set on the number line is shown by dark line



Example 7. Solve $\frac{2x + 1}{3x - 2} \geq \frac{3}{5}, x > \frac{2}{3}$.

Graph the solution set on the number line.

Sol. Given, $\frac{2x + 1}{3x - 2} \geq \frac{3}{5} \Rightarrow \frac{(2x + 1)(3x - 2)}{(3x - 2)} \geq \frac{3(3x - 2)}{5}$

$\left[\text{multiplying both sides by } (3x - 2), \text{ as } x > \frac{2}{3} \Rightarrow 3x - 2 > 0\right]$

$\Rightarrow \quad (2x + 1) \geq \frac{3(3x - 2)}{5} \Rightarrow 5(2x + 1) \geq \frac{5 \times 3(3x - 2)}{5}$

[multiplying both sides by 5]

$\Rightarrow \quad 5(2x + 1) \geq 3(3x - 2) \Rightarrow 10x + 5 \geq 9x - 6$

$\Rightarrow \quad 10x + 5 - 9x \geq 9x - 6 - 9x$

[subtracting $9x$ from both sides]

$\Rightarrow \quad x + 5 \geq -6 \Rightarrow x + 5 - 5 \geq -6 - 5$

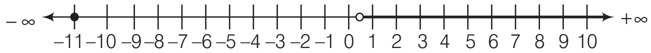
[subtracting 5 from both sides]

$\Rightarrow \quad x \geq -11$

But $x > \frac{2}{3}$.

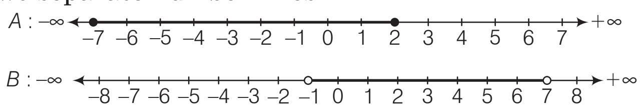
$\therefore$ The solution set is $\{x : x > \frac{2}{3}, \forall x \in R\}$.

On the number line, it can be represented as



Here, the dark portion on the number line represents the solution of the given inequation.

Example 8. If solution sets A and B are shown by two separate number lines



(i) Write down the set-builder form of A and B.

(ii) Represent each of the following sets on different number lines.

(a) $A \cup B$ (b) $A \cap B$ (c) $A' \cap B$ (d) $A \cap B'$

Sol. (i) From the figure,

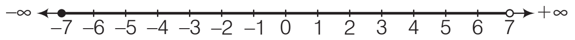
Solution set $A = [-7, 2]$

and solution set $B = (-1, 7)$

$\therefore$ Set-builder form of $A = \{x : -7 \le x \le 2,\ x \in R\}$
and set-builder form of $B = \{x : -1 < x < 7, x \in R\}$

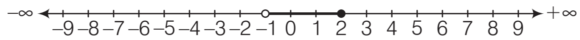
(ii) (a) $A \cup B = \{x : -7 \le x < 7,\ x \in R\}$

Representation of $A \cup B$ on number line is as



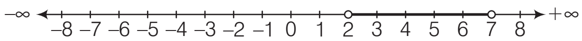
(b) $A \cap B = \{x : -1 < x \le 2,\ x \in R\}$

Representation of $A \cap B$ on the number line is as



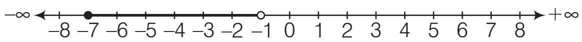
(c) $A' \cap B = B - A = \{x : 2 < x < 7, x \in R\}$

Representation of $A' \cap B$ on the number line is as



(d) $A \cap B' = A - B = \{x : -7 \le x < -1, x \in R\}$

Representation of $A \cap B'$ on the number line is as



Topic Exercise 2

1. Solve the following linear inequation for the variable involved and also represent the solution set on the number line, $\frac{1}{4}(2t-1) - t < \frac{t}{6} - \frac{1}{3},\ \forall\ t \in R$

2. Solve the following inequation and represent the solution set on the number line.

$$x - 3 - \frac{1}{7}(5x - 17) > 0,\ \forall\ x \in R$$

3. Solve the inequation $1 - \frac{4}{25}(2x-1) < 1,\ \forall\ x \in N$ and $x < 5$. Also, represent the solution set on the number line.

4. Solve the following inequation and represent the solution set on the number line.

$$-3 < -\frac{1}{2} - \frac{2x}{3} \le \frac{5}{6},\ x \in R$$

5. Solve the following inequation and write the solution set. $13x - 5 < 15x + 4 < 7x + 12,\ x \in R$

Represent the solution on a real number line. ***[2015]***

6. Solve the following equation, write down the solution set and represent it on the real number line.

$-2 + 10x \le 13x + 10 < 24 + 10x,\ x \in Z$ ***[2018]***

7. Solve the following inequation and graph the solution set on the number line.

$2y - 3 < y + 1 \le 4y + 7,\ \forall\ y \in R$ ***[2008]***

8. Solve the following system of linear inequations and represent the solution set on real number line

$$4x + 5 > 3x \text{ and } -(x+3) + 4 \le -2x + 5$$

9. Given, $S = \{x : -(35 + 2x) < 4x - 5 \le 7 - 2x, \forall\ x \in R\}$ and $T = \{x : 2 - 4x \le 2 - 3x < 7 - 4x,\ \forall\ x \in R\}$.

(i) Represent solution set of S and T on the separate number lines.

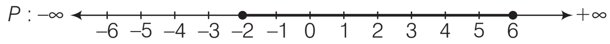
(ii) Find $S \cup T$, $S \cap T$ and $S - T$.

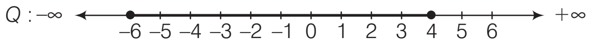
10. Solve the following inequations.

$$\frac{x+3}{x-2} \le 2, x > 2$$

Also, represent the solution set on the number line.

11. If solution sets P and Q are shown by two separate number lines





(i) Write down the set-builder form of P and Q.

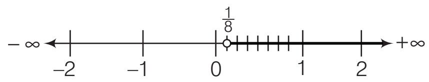
(ii) Represent each of the following sets on different number lines

(a) $P \cup Q$ (b) $P \cap Q$ (c) $P' \cap Q$

Hints and Answers

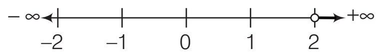
1. Hint $\frac{t}{2} - \frac{1}{4} - t < \frac{t}{6} - \frac{1}{3} \Rightarrow \frac{t}{2} - t - \frac{t}{6} < -\frac{1}{3} + \frac{1}{4} \Rightarrow t > \frac{1}{8}$

Ans. $\{t : t > \frac{1}{8}, t \in R\}$, i.e. $\left(\frac{1}{8}, \infty\right)$



2. Hint $x - 3 - \frac{5x}{7} + \frac{17}{7} > 0 \Rightarrow \frac{2x}{7} - \frac{4}{7} > 0 \Rightarrow x > 2$

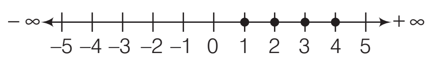
Ans. $\{x : x > 2, x \in R\}$, i.e. $(2, \infty)$



3. Hint $1 - \frac{8x}{25} + \frac{4}{25} < 1 \Rightarrow x > \frac{1}{2}$ and we have $x < 5$

$$\therefore \quad \frac{1}{2} < x < 5$$

Ans. $\{x : \frac{1}{2} < x < 5, x \in N\}$ or $\{1, 2, 3, 4\}$



4. Hint $-3 < \frac{-3-4x}{6} \le \frac{5}{6}$

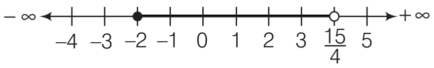
$$\Rightarrow -3 \times 6 < \frac{(-3-4x) \times 6}{6} \le \frac{5}{6} \times 6$$

$\Rightarrow \ -18+3<-3-4x+3\le 5+3$

$\Rightarrow \quad \dfrac{-15}{-4} > \dfrac{-4x}{-4} \ge \dfrac{8}{-4}$

$\Rightarrow \quad -2 \le x < \dfrac{15}{4}$

Ans. $\{x : -2 \le x < \dfrac{15}{4}, x \in R\}$

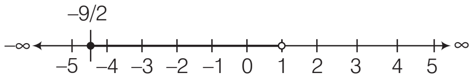


5. Hint $\quad 13x-5<15x+4$

and $\quad 15x+4<7x+12$

$\Rightarrow \quad x > -\dfrac{9}{2}$ and $x<1$

Ans. $\{x : -\dfrac{9}{2} < x < 1, x \in R\}$



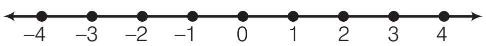
6. Hint $-2 + 10x \le 13x+10<24+10x,\ x \in Z$

$\therefore \quad -2+10x \le 13x+10$

and $\quad 13x+10<24+10x$

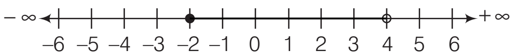
$\therefore \quad -4 \le x < 4\dfrac{2}{3}$

Ans. Solution set $=[-4, -3, -2, -1, 0, -1, 2, 3, 4]$



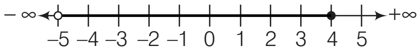
7. Hint $2y-3<y+1$ and $y+1\le 4y+7 \Rightarrow -2 \le y < 4$

Ans. $\{y : -2 \le y < 4,\ y \in R\}$



8. Hint (i) Solve both linear inequation separately and find the intersection of solution set.
(ii) Represents the solution set on the number line.

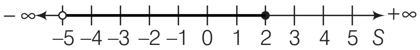
Ans. $\{x : -5 < x \le 4, x \in R\}$

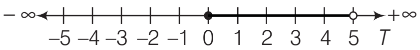


9. (i) **Hint** $S=\{x : -5 < x \le 2, x \in R\}$

and $T=\{x : 0 \le x < 5, \forall x \in R\}$

Ans.





(ii) **Hint** $S \cup T =$ Collection of all elements of S and T.

$S \cap T =$ Collection of common elements of S and T.

$S - T =$ Collection of those elements of S, which are not in T.

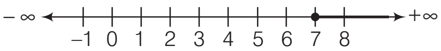
Ans. $S \cup T = \{x : -5 < x < 5, x \in R\}$

$S \cap T = \{x : 0 \le x \le 2, x \in R\}$

$S - T = \{x : x \in (-5, 0), x \in R\}$

10. Do same as Example 7.

Ans. $\{x : x \ge 7,\ \ x \in R\}$

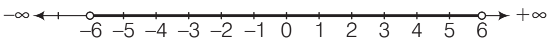


11. Do same as Example 8.

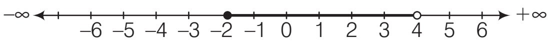
Ans. (i) $P=\{x : -2 \le x < 6, x \in R\}$

and $Q=\{x : -6 < x < 4, x \in R\}$

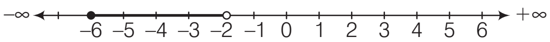
(ii) (a) $P \cup Q = \{x : -6 < x < 6, x \in R\}$



(b) $P \cap Q = \{x : -2 \le x < 4, x \in R\}$



(c) $P' \cap Q = Q - P = \{x : -6 < x < -2, x \in R\}$



CHAPTER EXERCISE

a 3 Marks Questions

1. If $25 - 4x \le 16$, then find the smallest value of x.
(i) When x is a real number.
(ii) When x is an integer.

2. Solve the inequation $3 - 2x \ge x - 12$, given that $x \in N$.

3. Solve the following inequation.
$$\frac{3}{5}x - \frac{2}{3}x + \frac{1}{3} > 1,\ \forall\ x \in W$$

4. Solve the following inequation and represent the solution set on the number line.
$$\frac{2x+1}{2} + 2(3 - x) \ge 7,\ \forall\ x \in R$$

5. If $5x - 3 \le 5 + 3x \le 4x + 2$, express it as $a \le x \le b$ and then state the values of a and b.

6. Find the largest three consecutive natural numbers, such that the sum of one-third of first, one-fourth of second and one-fifth of the third is atmost 25.

7. In the first four examinations, each of 100 marks, Mohan got 94, 73, 72 and 84 marks. If average marks greater than or equal to 80 and less than 90 is needed in fifth examination to obtain a final grade B in a course, then what range of marks in fifth (last) examination will be required, if Mohan is receiving grade B in the course?

8. If the replacement set is the set of integers lying between -4 and 8, then find the solution set of $14 - 5x \ge 3x - 40$. Also, represent the solution set on the number line.

9. Solve the following inequation and graph the solution set on the number line $x + \frac{2}{15} \le \frac{-8}{15}$, $x \in R$.

10. Solve the inequation $2x - 3 \ge 5$ and find the minimum value of x. Also, represent the solution set on the number line.

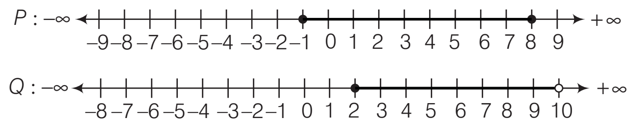
11. Solve the inequation $2x - 8 \le 17 - 3x$ and find the maximum value of x. Also, represent the solution set on the number line.

12. Solve the inequation $1 \le 3(x - 2) + 4 < 7$, $\forall\ x \in W$ and represent the solution set on the number line.

13. Solve the following inequation and represent the solution set on the number line.
$2x - 5 \le 5x + 4 < 11$, where $x \in I$. ***[2011, 2002]***

14. If $A = \{x : 11x - 5 > 7x + 3,\ \forall\ x \in R\}$ and $B = \{x : 18x - 9 \ge 15 + 12x,\ \forall\ x \in R\}$, then find the range of set $A \cap B$ and represent it on a number line. ***[2005]***

15. If P and Q are shown by two separate number lines



(i) Write down the set-builder form of P and Q.
(ii) Represent the solution set $P \cup Q$ and $P \cap Q$ on the number lines.

b 4 Marks Questions

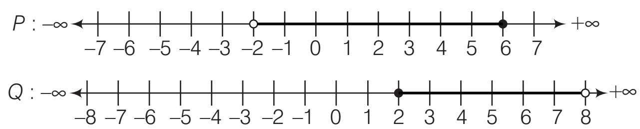
16. Solve the linear inequation $3x - 5 < x + 7$, when
(i) x is a natural number.
(ii) x is a whole number.
(iii) x is an integer.
(iv) x is a real number.

17. Find the smallest value of x for $x - 3(2 + x) < 2(3x - 1)$, when
(i) $x \in W$ (ii) $x \in I$ (iii) $x \in N$

18. Solve and graph the solution set of $3x + 6 \ge 9$ and $-5x > -15$, where $x \in R$.

19. Given, $A = \{x : 7x - 2 > 4x + 1, \forall x \in R\}$ and $B = \{x : 9x - 45 \ge 5(x - 5),\ \forall\ x \in R\}$.
Represent (i) $A \cap B$ (ii) $A - B$ on separate number lines.

20. The figure given below represents two solution sets P and Q on real number lines.



(i) Write down P and Q in set-builder form.
(ii) Represent each of the following sets on different number lines.
(a) $P \cap Q$ (b) $P' \cap Q$ (c) $P \cap Q'$

21. Solve the inequation $\frac{4}{x+1} \le 3 \le \frac{6}{x+1}, x > 0$ and represent the solution set on the number line.

22. Solve the inequation $-3 + x \le \frac{8x}{3} + 2 \le \frac{14}{3} + 2x, x \in I$ and represent the solution set on the number line.

[2009]

23. Solve the following inequation and represent the solution set on the number line.

$$4x - 19 < \frac{3x}{5} - 2 \le \frac{-2}{5} + x, \forall\, x \in R$$

[2012]

Hints and Answers

1. Hint $25 - 4x \le 16 \Rightarrow x \ge 2.25$

Ans.

(i) When x is real number, then smallest value of x is 2.25.

(ii) When x is an integer, then smallest value of $x \ge 2.25$ is $x = 3$.

2. Hint First, add both sides of inequality by $2x + 12$, then divide the result both sides by 3.

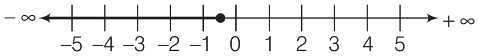
Ans. {1, 2, 3, 4, 5}

3. Hint $\frac{3}{5}x - \frac{2}{3}x > 1 - \frac{1}{3}$ $\therefore$ $x < -10$

Ans. Solution set is empty, i.e. $\{\phi\}$.

4. Hint $\frac{2x+1}{2} + 6 - 2x \ge 7 \Rightarrow -2x + 13 \ge 14 \Rightarrow x \le -\frac{1}{2}$

Ans. The solution set is $\{x : x \le -\frac{1}{2}, x \in R\}$.



5. Hint Consider $5x - 3 \le 5 + 3x$

$\Rightarrow$ $x \le 4$ ...(i)

and $5 + 3x \le 4x + 2 \Rightarrow 3 \le x$ ...(ii)

$\therefore$ $3 \le x \le 4$

Now, compare the above inequation with $a \le x \le b$.

Ans. $a = 3$ and $b = 4$

6. Hint Let three consecutive natural numbers be x, $x + 1$ and $x + 2$, respectively.

$\therefore$ $\frac{x}{3} + \frac{x+1}{4} + \frac{x+2}{5} \le 25 \Rightarrow x \le 31\frac{4}{47}$

Hence, the largest value of x is 31. **Ans.** 31, 32 and 33

7. Hint Let x be the score obtained by Mohan in the last examination.

$\therefore$ $\frac{323 + x}{5} \ge 80 \Rightarrow 323 + x \ge 400$

[multiplying both sides by 5]

$\Rightarrow$ $x \ge 77$ [subtracting 323 from both sides]

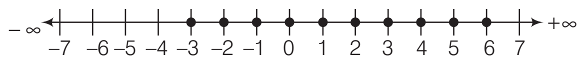
Ans. Required range is $77 \le x < 90$.

8. Hint We have, $14 - 5x \ge 3x - 40 \Rightarrow x \le 6.75$

$\because$ Replacement set is the set of integer lying between -4 and 8.

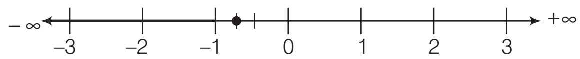
$\therefore$ $x \in \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$

Ans. The solution set is $\{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$.



9. Hint $x \le -\frac{8}{15} - \frac{2}{15}$, $x \le -\frac{2}{3}$

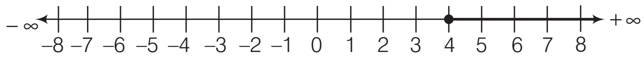
Ans. The solution set is $\{x : x \le -\frac{2}{3}, x \in R\}$.



10. Hint $2x - 3 \ge 5 \Rightarrow x \ge 4$

$\therefore$ Minimum value of $x = 4$.

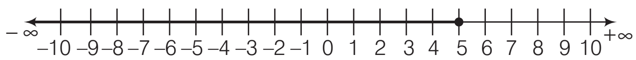
Ans. The solution set is $x \in [4, \infty)$, the minimum value of x is 4.



11. Hint $2x + 3x \le 17 + 8 \Rightarrow x \le 5$

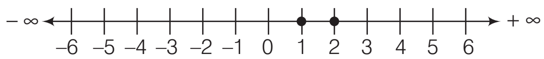
$\therefore$ Maximum value of $x = 5$.

Ans. The solution set is $x \in (-\infty, 5]$, the maximum value of x is 5.



12. Hint Add each term of inequation by 2 and then divide each term of the result by 3.

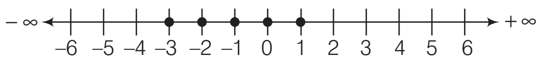
Ans. The solution set is {1, 2}.



13. Hint Consider $2x - 5 \le 5x + 4 \Rightarrow x \ge -3$

and $5x + 4 < 11 \Rightarrow x < \frac{7}{5}$

Ans. The solution set is $\{-3, -2, -1, 0, 1\}$.



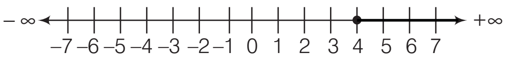
14. Hint Since, the solution set of the given inequation is A.

$\therefore \quad A = \{x : x > 2, x \in R\}$ ...(i)

Since, the solution set of the given inequation is B.

$\therefore \quad B = \{x : x \geq 4, x \in R\}$ ...(ii)

Ans. $A \cap B = \{x : x \geq 4, x \in R\}$



15. (i) **Hint** $P = [-1, 8]$ and $Q = [2, 10)$

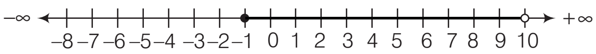
Ans. $P = \{x : -1 \leq x \leq 8, x \in R\}$

and $Q = \{x : 2 \leq x < 10, x \in R\}$

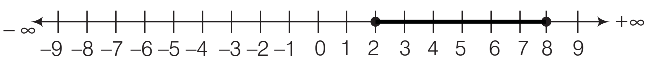
(ii) **Hint** $P \cup Q = \{x : -1 \leq x < 10, x \in R\}$

and $P \cap Q = \{x : 2 \leq x \leq 8, \in R\}$

Ans.



$P \cup Q$



$P \cap Q$

16. Hint $3x - 5 < x + 7 \Rightarrow x < 6$

Ans.

(i) When $x \in N$, then the solution set is $\{1, 2, 3, 4, 5\}$.

(ii) When $x \in W$, then the solution set is $\{0, 1, 2, 3, 4, 5\}$.

(iii) When $x \in I$, then the solution set is $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$.

(iv) When $x \in R$, then the solution set is $\{x : x < 6, x \in R\}$.

17. Hint $x - 6 - 3x < 6x - 2 \Rightarrow x > -\dfrac{1}{2}$

Ans.

(i) When $x \in W$, then the smallest value of x is 0.

(ii) When $x \in I$, then the smallest value of x is 0.

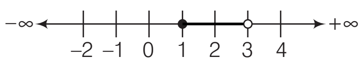
(iii) When $x \in N$, then the smallest value of x is 1.

18. Hint Consider $3x + 6 \geq 9 \Rightarrow x \geq 1$

and $-5x > -15 \Rightarrow x < 3$

$\therefore \quad \{x : x \in [1, \infty) \cap (-\infty, 3)\} = \{x : x \in [1, 3), x \in R\}$.

Ans. $\{x : x \in [1, 3), x \in R\}$



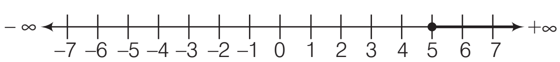
19. Hint Consider $7x - 2 > 4x + 1 \Rightarrow x > 1$

$\therefore \quad A = \{x : x > 1, x \in R\}$ ...(i)

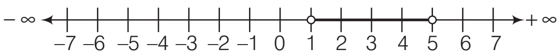
and $9x - 45 \geq 5(x - 5) \Rightarrow 4x \geq 20 \Rightarrow x \geq 5$

$\therefore \quad B = \{x : x \geq 5, x \in R\}$ ...(ii)

Ans. (i) $A \cap B = \{x : x \geq 5, x \in R\}$



(ii) $A - B = \{x : 1 < x < 5, x \in R\}$



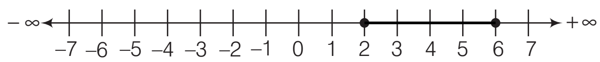
20. (i) **Hint** $P = (-2, 6]$ and $Q = [2, 8)$

Ans. $P = \{x : -2 < x \leq 6, x \in R\}$

and $Q = \{x : 2 \leq x < 8, x \in R\}$

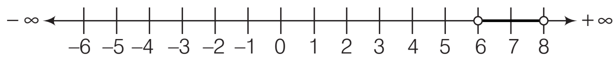
(ii) (a) **Hint** $P \cap Q = \{x : 2 \leq x \leq 6, x \in R\}$

Ans.



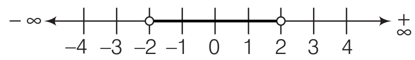
(b) **Hint** $P' \cap Q = Q - P = \{x : 6 < x < 8, x \in R\}$

Ans.



(c) **Hint** $P \cap Q' = P - Q = \{x : -2 < x < 2, x \in R\}$

Ans.



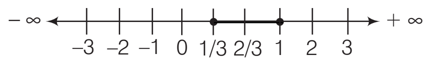
21. Hint $\dfrac{4}{x+1} \leq 3 \leq \dfrac{6}{x+1}, x > 0$

$\Rightarrow \quad 4 \leq 3x + 3 \leq 6$ [multiplying each term by $(x + 1)$]

$1 \leq 3x \leq 3$ [subtracting 3 from each term]

$\Rightarrow \quad \dfrac{1}{3} \leq x \leq 1$ [dividing each term by 3]

Ans. $\therefore$ The solution set is $\{x : \dfrac{1}{3} \leq x \leq 1, x \in R\}$.



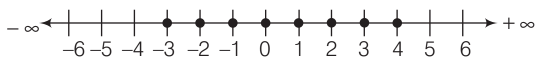
22. Hint Consider $-3 + x \leq \dfrac{8x}{3} + 2$

$\Rightarrow \quad -3 \leq x$ ...(i)

and $\dfrac{8x}{3} + 2 \leq \dfrac{14}{3} + 2x \Rightarrow x \leq 4$ ...(ii)

From Eqs. (i) and (ii), we get $-3 \leq x \leq 4$

Ans. The solution set is $\{-3, -2, -1, 0, 1, 2, 3, 4\}$.



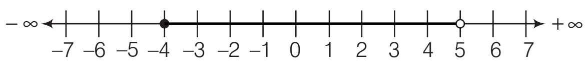
23. Hint Consider $4x - 19 < \dfrac{3x}{5} - 2 \Rightarrow -17 < \dfrac{3x - 20x}{5}$

or $\quad x < 5 \quad \left[\text{multiplying both sides by } \dfrac{-5}{17}\right]$ ...(i)

and $\dfrac{3x}{5} - 2 \leq -\dfrac{2}{5} + x \Rightarrow x \geq -4$ ...(ii)

From Eqs. (i) and (ii), we get $-4 \leq x < 5$.

Ans. $\{x : -4 \leq x < 5, \forall x \in R\}$



ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1 Solve the following equation, write down the solution set and represent it on the real number line.

$$-2+10x \le 13x+10 < 24+10x,\ x \in Z$$

2017

2 Solve the following inequation and represent the solution set on a number line.

$$-8\frac{1}{2} < -\frac{1}{2} - 4x \le 7\frac{1}{2},\ x \in I$$

2016

3 Solve the following inequation, also write the solution set and represent it on the number line.

$$-3(x-7) \ge 15 - 7x > \frac{x+1}{3},\ x \in R$$

2015

4 Solve the following inequation and write the solution set $13x - 5 < 15x + 4 < 7x + 12,\ x \in R$

Represent the solution on a real number line.

2014

5 Find the value of x, which satisfy the inequation $-2\frac{5}{6} < \frac{1}{2} - \frac{2x}{3} \le 2,\ \forall\ x \in W$ and also represent the solution set on the number line.

2013

6 Solve the following inequation and write the solution set. Also, represent the solution set on the number line.

$$\frac{-x}{3} \le \frac{x}{2} - 1\frac{1}{3} < \frac{1}{6},\ \ x \in R$$

2012

7 Solve the following inequation and represent the solution set on the number line.

$$4x - 19 < \frac{3x}{5} - 2 \le \frac{-2}{5} + x,\ \forall\ x \in R$$

2011

8 Solve the following inequation and represent the solution set on the number line.

$$2x - 5 \le 5x + 4 < 11,\ \text{where } x \in I$$

* All these questions are completely covered in chapter either as solved examples or in topic/chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. Find the range of values of x which satisfy the inequation, $(x+1)^2-(x-1)^2<6$.

(a) $\left(-\infty, -\frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$
(b) $\left(\frac{3}{2}, \infty\right)$
(c) $\left(-\infty, \frac{3}{2}\right)$
(d) None of these

2. The maximum value of $23-|2x+3|$ is

(a) 20
(b) 26
(c) 17
(d) 23

3. The solution set of the inequation $\frac{1}{5+3x} \le 0$ is

(a) $\left(-\frac{5}{3}, \infty\right)$
(b) $\left(-\infty, \frac{5}{3}\right)$
(c) $\left(\frac{5}{3}, \infty\right)$
(d) $\left(-\infty, \frac{-5}{3}\right)$

4. Solve the inequation $\left|\frac{2}{|x-4|}\right| > 1$, $x \ne 4$

(a) $\{x : x \in (-2, 4) \cup (-4, 6), x \in R\}$
(b) $\{x : x \in [-2, 4) \cup (-4, 6], x \in R\}$
(c) $\{x : x \in (2, 4) \cup (4, 6), x \in R\}$
(d) $\{x : x \in [2, 4) \cup (4, 6), x \in R\}$

5. Solve for x: $|x+1| + |x| > 3$.

(a) $x \in (-\infty, -2) \cup (1, \infty)$
(b) $x \in (-2, \infty) \cup (-1, \infty)$
(c) $x \in (-\infty, -2] \cup [1, \infty)$
(d) $x \in [-2, \infty) \cup [-1, \infty)$

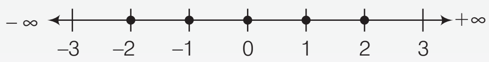
6. Graph the range of the inequation
$-2\frac{2}{3} \le x + \frac{1}{3} \le 3\frac{1}{3}, \forall\, x \in R$ on the number line. If the solution set is consider as a diagonal of a square on the number line, then the area of obtained figure, is

(a) 11 sq units
(b) 14 sq units
(c) 17 sq units
(d) 18 sq units

7. Graph the solution set of $-2 < 2x - 6$ or $-2x + 5 \ge 13$, where $x \in R$. If we shift the origin at the position 1, then the new solution set, is

(a) $\{x : x < 1$ or $x \ge -5, x \in R\}$
(b) $\{x : x > 1$ or $x \le -5, x \in R\}$
(c) $\{x : x > 1$ or $x \ge -5, x \in R\}$
(d) None of these

8. The solution sets P and Q on number line are given below



Consider the number line P on Y-axis and the number line Q on X-axis. Then, find the number of all the pairs of coordinates of the solution set.

(a) 25
(b) 24
(c) 23
(d) 22

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 394.

CHAPTER

5

Quadratic Equations in One Variable

In earlier classes, we have already studied about the polynomial. The second degree polynomial in one variable is called quadratic polynomial and if we take it equal to zero, then it becomes a quadratic equation. In this chapter, we will study quadratic equations and various ways of finding their roots. We will also discuss some application of quadratic equations based on daily life situations.

Chapter Objectives

- Quadratic Equation and Its Solution
- Nature of Roots of a Quadratic Equation
- Applications of Quadratic Equation

Topic 1
Quadratic Equation and Its Solution

Quadratic Equation

An equation of the form $ax^2 + bx + c = 0$ is called **quadratic equation** in variable x, where a, b and c are real numbers and $a \neq 0$. e.g. $2x^2 + x - 100 = 0$, $-x^2 + 1 + 300x = 0$, $4x - 3x^2 + 7 = 0$, $4x^2 - 25 = 0$ are quadratic equations.

The form $ax^2 + bx + c = 0$, $a \neq 0$, is called the **standard form of a quadratic equation**.

To express a quadratic equation in its standard form, write the equation in decreasing order of the degrees of variable. e.g. $3x^2 + x + 2 = 0$, $x^2 - 2x + 6 = 0$ are in standard form, whereas $x^2 - 3 + 4x = 0$, $x + x^2 + 8 = 0$ are not in standard form.

Method to Check Whether a Given Equation is Quadratic or Not

To check whether a given equation is quadratic or not, first write the given equation in its simplest form and then compare the equation with the standard form of a quadratic equation, i.e. $ax^2 + bx + c = 0$, $a \neq 0$.

If the given equation follows the standard form, then it is a quadratic equation, otherwise not.

Example 1. Check whether the following are quadratic equations or not.

(i) $(x-1)(x+2)=(x-3)(x+1)$

(ii) $(x+2)^2 = 4(x+3)$

Sol.

(i) Given, $(x-1)(x+2)=(x-3)(x+1)$ ...(i)

Here, in LHS, $(x-1)$ and $(x+2)$ are in product and in RHS, $(x-3)$ and $(x+1)$ are also in product. So, first we simplify their product.

LHS $=(x-1)(x+2)=x^2+2x-x-2=x^2+x-2$

RHS $=(x-3)(x+1)=x^2+x-3x-3=x^2-2x-3$

On substituting these values in Eq. (i), we get

$$x^2+x-2=x^2-2x-3$$

$\Rightarrow \quad x^2-x^2+x+2x-2+3=0$

$\Rightarrow \quad 3x+1=0$

It is not of the form $ax^2+bx+c=0, a \neq 0$.

Hence, the given equation does not represent a quadratic equation.

Note In the above example, the given equation appears to be a quadratic equation, but it is not a quadratic equation. So, simplify the given equation before deciding whether it is quadratic or not.

(ii) Given, $(x+2)^2=4(x+3)$ ...(ii)

Here, term $(x+2)$ has power 2. So, first expand it.

LHS $=(x+2)^2=x^2+4+4x$ $\quad [\because (a+b)^2=a^2+b^2+2ab]$

On substituting this value in Eq. (ii), we get

$$x^2+4+4x=4x+12$$

$\Rightarrow \quad x^2+4x-4x+4-12=0$

$\Rightarrow \quad x^2-8=0$ or $x^2+0x-8=0$

It is of the form $ax^2+bx+c=0, a \neq 0$.

Hence, the given equation represents a quadratic equation.

Solutions or Roots of a Quadratic Equation

All the values of variable which satisfy the given quadratic equation, are called **roots** or **solutions** of given quadratic equation.

In other words, a real number α is said to be a root or solution of a quadratic equation $ax^2+bx+c=0$, $a \neq 0$, if $a(\alpha)^2+b(\alpha)+c=0$.

Any quadratic equation can have atmost two roots.

Example 2. Determine whether $x=\frac{-1}{2}, x=\frac{1}{3}$ are the solutions of the given equation $6x^2-x-2=0$ or not.

Sol. Given equation is in the form $p(x)=0$, where

$$p(x)=6x^2-x-2 \quad \text{...(i)}$$

On putting $x=\frac{-1}{2}$ in Eq. (i), we get

$$p\left(\frac{-1}{2}\right)=6\left(\frac{-1}{2}\right)^2-\left(\frac{-1}{2}\right)-2$$

$$=\frac{6}{4}+\frac{1}{2}-2=\frac{6+2-8}{4}=\frac{8-8}{4}$$

$\Rightarrow \quad p\left(\frac{-1}{2}\right)=0$

So, $x=\frac{-1}{2}$ is a solution of the given equation.

Now, putting $x=\frac{1}{3}$ in Eq. (i), we get

$$p\left(\frac{1}{3}\right)=6\left(\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)-2$$

$$=6\times\frac{1}{9}-\frac{1}{3}-2=\frac{6}{9}-\frac{1}{3}-2$$

$$=\frac{6-3-18}{9}=\frac{-15}{9}\neq 0$$

$\Rightarrow \quad p\left(\frac{1}{3}\right)\neq 0$

So, $x=\frac{1}{3}$ is not a solution of the given equation.

Solution of a Quadratic Equation by Factorisation

A quadratic equation can be solved using two methods; by factorisation and by quadratic formula. Here, we will discuss the solution of quadratic equation by factorisation.

To find the solution of a quadratic equation by factorisation, we use the following steps

***Step* I** Write the given equation in standard form i.e. $ax^2+bx+c=0$ (if not given in standard form) and find the values of a, b and c.

***Step* II** Find the product of a and c and write it as a product of its two factors such that their sum is equal to b, i.e. write $ac=p\times q$ such that $p+q=b$, where p and q are factors of ac.

***Step* III** Put the value of b obtained from Step II in given equation and then write its LHS as product of two linear factors.

***Step* IV** Now, equate each factor to zero to get desired roots of given quadratic equation.

Note (i) In case of $ax^2 \pm bx + c = 0$, both p and q will be negative or positive.

(ii) In case of $ax^2 \pm bx - c = 0$, one of p and q will be negative.

Example 3. Solve the following quadratic equation by factorisation method.

(i) $x^2-14x+24=0$ (ii) $4\sqrt{3}x^2+5x-2\sqrt{3}=0$

Sol.

(i) The given quadratic equation is $x^2-14x+24=0$.

On comparing with standard form of quadratic equation i.e. $ax^2+bx+c=0$

$\therefore \quad a=1, b=-14$ and $c=24$

Here, $ac=1\times 24=24$

Now, we need to find those factors of 24 whose sum is -14. So, let $p=-12$ and $q=-2$, as $p\times q=24$ and $p+q=-14$

$\therefore \quad x^2 - 14x + 24 = 0$

$\Rightarrow \quad x^2 + (-12 - 2)x + 24 = 0$

$\Rightarrow \quad x^2 - 12x - 2x + 24 = 0$

$\Rightarrow \quad x(x - 12) - 2(x - 12) = 0$

$\Rightarrow \quad (x - 2)(x - 12) = 0$

$\Rightarrow \quad (x - 2) = 0$ or $(x - 12) = 0$

$\Rightarrow \quad x = 2$ or $x = 12$

Hence, roots of the equation $x^2 - 14x + 24$ are 2 and 12.

(ii) Given $4\sqrt{3}x^2 + 5x - 2\sqrt{3} = 0$.

On comparing with standard form of quadratic equation i.e $\quad ax^2 + bx + c = 0$

$\therefore \quad a = 4\sqrt{3}, \quad b = 5$ and $c = -2\sqrt{3}$

Here, $ac = 4\sqrt{3} \times (-2\sqrt{3}) = -24$.

Now, we need to find those factors of -24, whose sum is 5.

So, let $p = 8$ and $q = -3$, as $p \times q = -24$ and $p + q = 5$

$\Rightarrow \quad 4\sqrt{3}x^2 + (8 - 3)x - 2\sqrt{3} = 0$

$\Rightarrow \quad 4\sqrt{3}x^2 + 8x - 3x - 2\sqrt{3} = 0$

$\Rightarrow \quad 4x(\sqrt{3}x + 2) - \sqrt{3}(\sqrt{3}x + 2) = 0$

$\Rightarrow \quad (4x - \sqrt{3})(\sqrt{3}x + 2) = 0$

$\Rightarrow \quad 4x - \sqrt{3} = 0$ or $\sqrt{3}x + 2 = 0$

$\therefore \quad x = \dfrac{\sqrt{3}}{4}$ or $x = \dfrac{-2}{\sqrt{3}}$

Hence, roots of equation $4\sqrt{3}x^2 + 5x - 2\sqrt{3} = 0$ are $\dfrac{\sqrt{3}}{4}$ and $\dfrac{-2}{\sqrt{3}}$.

Example 4. Find the roots of the quadratic equation $x^2 - 8x - 20 = 0$ by factorisation method.

Sol. Given quadratic equation is $x^2 - 8x - 20 = 0$, which is already in standard form.

On comparing it with $ax^2 + bx + c = 0$, we get

$a = 1, b = -8$ and $c = -20$

Here, $ac = 1 \times (-20) = -20$

Now, let $p = -10$ and $q = 2$, as $p \times q = -20$ and $p + q = -8$

$\therefore \quad x^2 - 8x - 20 = 0$

$\Rightarrow \quad x^2 + (-10 + 2)x - 20 = 0$

$\Rightarrow \quad x^2 - 10x + 2x - 20 = 0$

$\Rightarrow \quad x(x - 10) + 2(x - 10) = 0$

$\Rightarrow \quad (x - 10)(x + 2) = 0$

$\Rightarrow \quad (x - 10) = 0$ or $x + 2 = 0$

$\Rightarrow \quad x = 10$ or $x = -2$

Thus, 10 and -2 are the required roots of the given quadratic equation.

Example 5. Solve the equation $x^2 + (3 - 2a)x - 6a = 0$ by factorisation method.

Sol. Given equation is $x^2 + (3 - 2a)x - 6a = 0$..

$\Rightarrow \quad x^2 + 3x - 2ax - 6a = 0$

$\Rightarrow \quad x(x + 3) - 2a(x + 3) = 0$

$\Rightarrow \quad (x + 3)(x - 2a) = 0 \quad$ [factorising left side]

$\Rightarrow \quad x + 3 = 0$ or $x - 2a = 0$

$\therefore \quad x = -3$ or $x = 2a$

Hence, the roots of the given equation are -3 and $2a$.

Example 6. Solve the equation $\sqrt{3}x^2 + 11x + 6\sqrt{3} = 0$ by factorisation method.

Sol. Given equation is $\sqrt{3}x^2 + 11x + 6\sqrt{3} = 0$.

$\Rightarrow \quad \sqrt{3}x^2 + 9x + 2x + 6\sqrt{3} = 0$

[$\because$ $9 + 2 = 11$ and $9 \cdot 2 = (6\sqrt{3})(\sqrt{3}) = 18$]

$\Rightarrow \quad \sqrt{3}x(x + 3\sqrt{3}) + 2(x + 3\sqrt{3}) = 0$

$\Rightarrow \quad (x + 3\sqrt{3})(\sqrt{3}x + 2) = 0 \quad$ [factorising left side]

$\Rightarrow \quad x + 3\sqrt{3} = 0$ or $\sqrt{3}x + 2 = 0$

$\therefore \quad x = -3\sqrt{3}$ or $x = \dfrac{-2}{\sqrt{3}}$

Hence, the roots of the given equation are $-3\sqrt{3}$ and $-\dfrac{2}{\sqrt{3}}$.

Example 7. Solve the quadratic equation $\dfrac{x+3}{x-2} - \dfrac{1-x}{x} = \dfrac{17}{4}$ by factorisation method.

Sol. Given equation is $\dfrac{x+3}{x-2} - \dfrac{1-x}{x} = \dfrac{17}{4}$.

$\Rightarrow \quad \dfrac{x(x+3) - (1-x)(x-2)}{x(x-2)} = \dfrac{17}{4}$

$\Rightarrow \quad 4[x^2 + 3x - (x - 2 - x^2 + 2x)] = 17(x^2 - 2x)$

$\Rightarrow \quad 4(x^2 + 3x - 3x + 2 + x^2) = 17x^2 - 34x$

$\Rightarrow \quad 4(2x^2 + 2) = 17x^2 - 34x$

$\Rightarrow \quad 8x^2 + 8 = 17x^2 - 34x$

$\Rightarrow \quad 8x^2 - 17x^2 + 34x + 8 = 0$

$\Rightarrow \quad -9x^2 + 34x + 8 = 0$

$\Rightarrow \quad 9x^2 - 34x - 8 = 0 \quad$ [multiplying by (-1)]

$\Rightarrow \quad 9x^2 + (-36 + 2)x - 8 = 0$

[$\because$ $(-36) \times 2 = -72$ and $-36 + 2 = -34$]

$\Rightarrow \quad 9x^2 - 36x + 2x - 8 = 0$

$\Rightarrow \quad 9x(x - 4) + 2(x - 4) = 0$

$\Rightarrow \quad (x - 4)(9x + 2) = 0$

$\Rightarrow \quad x - 4 = 0$ or $9x + 2 = 0$

$\Rightarrow \quad x = 4$ or $x = -\dfrac{2}{9}$

Hence, $x = 4$ and $x = -\dfrac{2}{9}$ are the required roots of given equation.

Solution of a Quadratic Equation by Quadratic Formula

Solution of quadratic equation by factorisation is somehow a lengthy process and also not applicable for all quadratic equations. So, there is a need of direct formula which can give the solution of any quadratic equation. This formula is called quadratic formula.

The roots/solutions of the quadratic equation $ax^2 + bx + c = 0$, by quadratic formula, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}$$

where, $D = b^2 - 4ac$ is called the **discriminant**. This formula is also known as **Sridharacharya's formula**.

Example 8. Solve $12x^2 + 5x - 3 = 0$ by quadratic formula.

Sol. Given quadratic equation is $12x^2 + 5x - 3 = 0$.

On comparing the given equation with $ax^2 + bx + c = 0$, we get $a = 12$, $b = 5$ and $c = -3$.

On substituting the values of a, b and c in quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we get

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4 \times 12 \times (-3)}}{2 \times 12}$$

$$\Rightarrow \quad x = \frac{-5 \pm \sqrt{25 + 144}}{24} = \frac{-5 \pm \sqrt{169}}{24} = \frac{-5 \pm 13}{24}$$

Now, $x = \frac{-5 + 13}{24} = \frac{8}{24} = \frac{1}{3}$ [taking +ve sign]

or $x = \frac{-5 - 13}{24} = -\frac{18}{24} = -\frac{3}{4}$ [taking –ve sign]

Hence, the roots of the given equation are $\frac{1}{3}$ and $-\frac{3}{4}$.

Example 9. Using the quadratic formula, solve the quadratic equation $\sqrt{3}x^2 + 11x + 6\sqrt{3} = 0$.

Sol. Given equation is $\sqrt{3}x^2 + 11x + 6\sqrt{3} = 0$.

On comparing it with $ax^2 + bx + c = 0$, we get

$$a = \sqrt{3}, \; b = 11 \text{ and } c = 6\sqrt{3}$$

On substituting the values of a, b and c in the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we get

$$x = \frac{-11 \pm \sqrt{(11)^2 - 4(\sqrt{3})(6\sqrt{3})}}{2(\sqrt{3})} = \frac{-11 \pm \sqrt{121 - 72}}{2\sqrt{3}} = \frac{-11 \pm \sqrt{49}}{2\sqrt{3}} = \frac{-11 \pm 7}{2\sqrt{3}}$$

$\Rightarrow \quad x = \frac{-11 + 7}{2\sqrt{3}} = \frac{-4}{2\sqrt{3}} = \frac{-2}{\sqrt{3}}$ [taking +ve sign]

or $x = \frac{-11 - 7}{2\sqrt{3}} = \frac{-18}{2\sqrt{3}} = \frac{-9}{\sqrt{3}}$ [taking –ve sign]

Hence, $\frac{-2}{\sqrt{3}}$ and $\frac{-9}{\sqrt{3}}$ $\left(\text{or } \frac{-2\sqrt{3}}{3} \text{ and } -3\sqrt{3}\right)$ are the required solutions of the given equation.

Example 10. Solve $(x - 1)^2 - 3x + 4 = 0$ for x, using the quadratic formula. Write your answer correct to two significant figures. ***[2014]***

Sol. Given quadratic equation is

$$(x - 1)^2 - 3x + 4 = 0$$

$\Rightarrow \quad x^2 + 1^2 - 2x - 3x + 4 = 0$ $[\because (a - b)^2 = a^2 + b^2 - 2ab]$

$$\Rightarrow \quad x^2 - 5x + 5 = 0$$

On comparing with $ax^2 + bx + c = 0$, we get

$$a = 1, \; b = -5 \text{ and } c = 5$$

Using quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 1 \times 5}}{2 \times 1} = \frac{5 \pm \sqrt{25 - 20}}{2} = \frac{5 \pm \sqrt{5}}{2} = \frac{5 \pm 2.236}{2}$$

$\Rightarrow \quad x = \frac{5 + 2.236}{2} = \frac{7.236}{2} = 3.618 \approx 3.62$ [taking '+ ve' sign]

or $x = \frac{5 - 2.236}{2} = \frac{2.764}{2} = 1.382 \approx 1.38$ [taking '– ve' sign]

Hence, the values of x are 3.62 and 1.38.

Example 11. Solve the following equation and give your answer correct to three significant figures.

$$5x^2 - 3x - 4 = 0$$ ***[2012]***

Sol. Given quadratic equation is $5x^2 - 3x - 4 = 0$.

On comparing it with $ax^2 + bx + c = 0$, we get

$$a = 5, \; b = -3 \text{ and } c = -4$$

By using Sridharacharya's formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 5 \times (-4)}}{2 \times 5} = \frac{3 \pm \sqrt{9 + 80}}{10} = \frac{3 \pm \sqrt{89}}{10}$$

$= \frac{3 \pm 9.434}{10}$ $[\because \sqrt{89} = 9.434]$

$x = \frac{3 + 9.434}{10} = \frac{12.434}{10} = 1.243$ [taking '+ ve' sign]

$x = \frac{3 - 9.434}{10} = \frac{-6.434}{10} = -0.643$ [taking '– ve' sign]

Hence, the solutions are 1.243 and –0.643.

Method to Determine an Unknown Constant Involved in a Quadratic Equation, When its Solution or Root is Given

Sometimes, given quadratic equation involves an unknown constant and its solution or root is given. Then, to find the value of unknown constant, we put the value of root or solution in given quadratic equation and simplify it to get the required unknown constant.

Example 12. In each of the following equations, determine the value of k for which the given value is a solution of the equation.

(i) $kx^2 + 2x - 3 = 0, x = 2$ (ii) $x^2 + 2ax - k = 0, x = -a$

Sol.

(i) We have, $kx^2 + 2x - 3 = 0$, where k is unknown.

Since, $x = 2$ is a solution of given equation, so it will satisfy the given equation.

On putting $x = 2$ in the given equation, we get

$$k(2)^2 + 2(2) - 3 = 0$$
$$\Rightarrow \quad 4k + 4 - 3 = 0$$
$$\Rightarrow \quad 4k + 1 = 0$$
$$\Rightarrow \quad k = \frac{-1}{4}$$

(ii) We have, $x^2 + 2ax - k = 0$, where k is unknown.

Since, $x = -a$ is a solution of given equation, so it will satisfy the given equation.

On putting $x = -a$ in the given equation, we get

$$(-a)^2 + 2a(-a) - k = 0$$
$$\Rightarrow \quad a^2 - 2a^2 - k = 0$$
$$\Rightarrow \quad -a^2 - k = 0$$
$$\Rightarrow \quad k = -a^2$$

Example 13. If $x = 2$ and $x = 3$ are roots of the equation $3x^2 - 2ax + 2b = 0$, then find the values of a and b.

Sol. Given, $3x^2 - 2ax + 2b = 0$ ...(i)

where, a and b are unknown constants.

Since, $x = 2$ and $x = 3$ are the solutions of given equation, so they will satisfy the given equation.

On putting $x = 2$ and $x = 3$ one-by-one, in Eq. (i), we get

$$3(2)^2 - 2a \times (2) + 2b = 0$$
$$\Rightarrow \quad 3 \times 4 - 4a + 2b = 0$$
$$\Rightarrow \quad 12 - 4a + 2b = 0$$
$$\Rightarrow \quad -2(2a - b - 6) = 0$$
$$\Rightarrow \quad 2a - b = 6 \qquad [\because -2 \neq 0] \quad \text{...(ii)}$$

and $\quad 3(3)^2 - 2a \times 3 + 2b = 0$

$$\Rightarrow \quad 27 - 6a + 2b = 0$$
$$\Rightarrow \quad 6a - 2b = 27 \qquad \text{...(iii)}$$

On multiplying Eq. (ii) by 2 and then subtract it from Eq. (iii), we get

$$6a - 2b - 4a + 2b = 27 - 12$$
$$\Rightarrow \quad 2a = 15$$
$$\Rightarrow \quad a = \frac{15}{2}$$

On substituting $a = \frac{15}{2}$ in Eq. (ii), we get

$$2 \times \frac{15}{2} - b = 6$$
$$\Rightarrow \quad 15 - b = 6$$
$$\Rightarrow \quad b = 15 - 6 = 9$$
$$\Rightarrow \quad b = 9$$

Hence, the required values of a and b are 15/2 and 9, respectively.

Topic Exercise 1

1. Check whether the following equations are quadratic or not.

(i) $2x^2 - 5x = x^2 - 2x + 3$

(ii) $x^2 - 3x - \sqrt{x} + 4 = 0$

(iii) $x(2x + 3) = x + 2$

(iv) $x + \frac{3}{x} = x^2$

(v) $x^2 - \frac{1}{x^2} = 5$

2. Is $x = -2$ a solution of the equation $x^2 - 2x + 8 = 0$?

3. Show that $x = -3$ is a solution of equation $x^2 + 6x + 9 = 0$.

4. Which of the following are the solutions of $2x^2 - 5x - 3 = 0$?

(i) $x = 3$ (ii) $x = 4$

(iii) $x = \frac{-1}{2}$ (iv) $x = -1$

(v) $x = 2/3$ (vi) $x = 2$

5. In each of the following equations, determine whether the given values are a solution of the given equations or not.

(i) $x^2 - x + 1 = 0; x = 1, x = -1$

(ii) $x^2 + \sqrt{2}x - 4 = 0; x = \sqrt{2}, x = -2\sqrt{2}$

(iii) $2x^2 - x + 9 = x^2 + 4x + 3; x = 2, x = 3$

(iv) $\sqrt{5}x^2 - 8x + 3\sqrt{5} = 0; x = \sqrt{5}, x = \frac{3}{\sqrt{5}}$

Directions (Q. Nos. 6-12) *Solve each of the following equations by factorisation.*

6. $t^2 + 3t - 10 = 0$

7. $x^2 - 10x + 21 = 0$

8. $8x^2 - 22x - 21 = 0$

9. $15y^2 = 41y - 14$

10. $6x^2 + 40 = 31x$

11. $21x^2 - 2x + \frac{1}{21} = 0$

12. $3x^2 + 2\sqrt{5}x - 5 = 0$

Directions (Q. Nos. 13-15) *Solve for x*.

13. $x^2 + 5x - (a^2 + a - 6) = 0$

14. $6a^2x^2 - 7abx - 3b^2 = 0, a \neq 0$

15. $4x^2 - 4a^2x + (a^4 - b^4) = 0$

16. Solve $x^2 + 7x = 7$ and give your answer correct to two decimal places. *[2018]*

Directions (Q. Nos. 17-19) *Find the roots of the following quadratic equations by applying the quadratic formula.*

17. $\dfrac{14}{x+3} - 1 = \dfrac{5}{x+1}$; $x \neq -3, -1$

18. $abx^2 + (b^2 - ac)x - bc = 0, a, b \neq 0$

19. $3a^2x^2 + 8abx + 4b^2 = 0, a \neq 0$

20. In each of the following equations, find the value of unknown constant(s) for which the given value(s) is (are) solution of the equations.

(i) $x^2 - k^2 = 0, x = 0.3$

(ii) $3x^2 + 2ax - 3 = 0$; $x = \dfrac{-1}{2}$

(iii) $7x^2 + kx - 3 = 0$; $x = \dfrac{2}{3}$

(iv) $kx^2 + \sqrt{2}x - 4 = 0$; $x = \sqrt{2}$

(v) $ax^2 + bx + 1 = 0$; $x = 1, x = 2$

Hints and Answers

1. Do same as Example 1.
Ans. (i) Yes (ii) No (iii) Yes (iv) No (v) No

2. Do same as Example 2. **Ans.** No

3. Do same as Example 2.

4. Do same as Example 2. **Ans.** (i) and (iii)

5. Do same as Example 2.
Ans.
(i) Both 1 and -1 are not the solution of the given equation.
(ii) Both $\sqrt{2}$ and $-2\sqrt{2}$ are the solution of the given equation.
(iii) Both 2 and 3 are the solution of the given equation.
(iv) Both $\sqrt{5}$ and $3/\sqrt{5}$ are the solution of the given equation.

6. **Hint** $ac = -10 = 5 \times (-2)$ and $5 + (-2) = 3$
Ans. $-5, 2$

7. **Hint** $ac = 21 = (-7)(-3)$ and $(-7) + (-3) = -10$
Ans. 3, 7

8. **Hint** $ac = -168 = (-28)(6)$ and $(-28) + 6 = -22$
Ans. $\dfrac{7}{2}, \dfrac{-3}{4}$

9. **Hint** $ac = 210 = (-35) \times (-6)$ and $(-35) + (-6) = -41$
Ans. $\dfrac{7}{3}, \dfrac{2}{5}$

10. **Hint** $ac = 240 = (-16)(-15)$ and $(-16) + (-15) = -31$
Ans. $\dfrac{8}{3}, \dfrac{5}{2}$

11. **Hint** First, multiply the equation with 21, to eliminate the denominator. Then, write $ac = 441 = (-21)(-21)$.
Ans. $\dfrac{1}{21}, \dfrac{1}{21}$

12. **Hint** $ac = -15$, take factors as $3\sqrt{5}$ and $-\sqrt{5}$.
Ans. $-\sqrt{5}, \dfrac{\sqrt{5}}{3}$

13. **Hint** First, factorise $a^2 + a - 6 = (a-2)(a+3)$, then factorise $x^2 + 5x - (a-2)(a+3) = 0$.
Ans. $-(a+3), (a-2)$

14. Do same as Example 3. **Ans.** $-\dfrac{b}{3a}, \dfrac{3b}{2a}$

15. **Hint** First, factorise $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$, then factorise $4x^2 - 4a^2x + (a^2 - b^2)(a^2 + b^2) = 0$.
Ans. $\dfrac{a^2 - b^2}{2}, \dfrac{a^2 + b^2}{2}$

16. **Hint** Use the quadratic formula
Ans. $x = 0.88$ or -7.88

17. **Hint** First, simplify the given equation, then do same as Example 8. **Ans.** 1, 4

18. **Hint** $x = \dfrac{-(b^2 - ac) \pm \sqrt{(b^2 - ac)^2 + 4ab^2c}}{2ab}$
Ans. $\dfrac{c}{b}, \dfrac{-b}{a}$

19. **Hint** $x = \dfrac{-8ab \pm \sqrt{64a^2b^2 - 4 \cdot 3a^2 \cdot 4b^2}}{6a^2}$
Ans. $\dfrac{-2b}{a}, \dfrac{-2b}{3a}$

20. (i) Do same as Example 12. **Ans.** $k = -0.3$ or 0.3
(ii) Do same as Example 12. **Ans.** $a = -\dfrac{9}{4}$
(iii) Do same as Example 12. **Ans.** $k = -\dfrac{1}{6}$
(iv) Do same as Example 12. **Ans.** $k = 1$
(v) Do same as Example 13. **Ans.** $a = \dfrac{1}{2}, b = -\dfrac{3}{2}$

Topic 2
Nature of Roots of a Quadratic Equation

By quadratic formula, the roots of the quadratic equation $ax^2+bx+c=0,\ a\neq 0$ are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{D}}{2a}$$

where, $D=b^2-4ac$ is called **discriminant**. The nature of roots depends upon the nature of the discriminant D. Since, D can be zero, positive or negative, so three cases arise.

***Case* I** When $D=0$, i.e. $b^2-4ac=0$.

If $D=b^2-4ac=0$, then $x=\dfrac{-b\pm 0}{2a}$

$\Rightarrow\ x=-\dfrac{b}{2a},-\dfrac{b}{2a}$

When $D=0$, the quadratic equation has two equal real roots.

***Case* II** When $D>0$, i.e. $b^2-4ac>0$.

If $D=b^2-4ac>0$, then

$$x=\frac{-b+\sqrt{D}}{2a},\frac{-b-\sqrt{D}}{2a}$$

When $D>0$, the quadratic equation has two distinct real roots.

***Case* III** When $D<0$, i.e. $b^2-4ac<0$.

If $D=b^2-4ac<0$, then there is no real number, whose square is b^2-4ac. So, the quadratic equation has no real roots. **When $D<0$, the quadratic equation has imaginary roots.**

Points to be Remembered

For quadratic equation $ax^2+bx+c=0$,

Discriminant	Nature of roots	Roots
$b^2-4ac=0$	Two equal real roots	$-\dfrac{b}{2a},-\dfrac{b}{2a}$
$b^2-4ac>0$	Two distinct real roots	$\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$
$b^2-4ac<0$	Non-real roots imaginary	Cannot determined in real number system.

Example 1. Find the discriminant in the following quadratic equations and state the nature of the roots.

(i) $3x^2-5x+2=0$ (ii) $x^2-\dfrac{1}{3}x+\dfrac{3}{2}=0$

Sol.

(i) Given quadratic equation is $3x^2-5x+2=0$.

On comparing it with $ax^2+bx+c=0$, we get

$$a=3,\ b=-5 \text{ and } c=2$$

Now, discriminant, $D=b^2-4ac=(-5)^2-4\times 3\times 2$
$=25-24=1$

Since, $D>0$, so the roots of the quadratic equation are real and distinct.

(ii) Given quadratic equation is $x^2-\dfrac{1}{3}x+\dfrac{3}{2}=0$.

On comparing it with $ax^2+bx+c=0$, we get

$$a=1,\ b=-\frac{1}{3} \text{ and } c=\frac{3}{2}$$

Now, discriminant, $D=b^2-4ac$

$$=\left(\frac{-1}{3}\right)^2-4\times 1\times\left(\frac{3}{2}\right)=\frac{1}{9}-6=\frac{-53}{9}$$

Since, $D<0$, therefore the roots of the quadratic equation are imaginary.

Example 2. Discuss the nature of the roots of the following equations.

(i) $2x^2-4x+3=0$ (ii) $5x^2-2x-3=0$

Sol.

(i) Given quadratic equation is $2x^2-4x+3=0$.

On comparing it with $ax^2+bx+c=0$, we get

$$a=2,\ b=-4 \text{ and } c=3$$

Now, discriminant, $D=b^2-4ac$
$=(-4)^2-4\times 2\times 3=16-24=-8<0$

$\therefore$ The roots of given quadratic equation are not real.

(ii) Given quadratic equation is $5x^2-2x-3=0$.

On comparing it with $ax^2+bx+c=0$, we get

$$a=5,\ b=-2 \text{ and } c=-3$$

Now, discriminant, $D=b^2-4ac=(-2)^2-4(5)(-3)$
$=4+60=64>0$

$\therefore$ The roots of given quadratic equation are real and distinct.

Example 3. Find the nature of roots of the following quadratic equations. If the real roots exist, then also find them.

(i) $2x^2-3x+5=0$

(ii) $3x^2-4\sqrt{3}x+4=0$

Sol.

(i) Given equation is $2x^2-3x+5=0$.

On comparing it with $ax^2+bx+c=0$, we get

$$a=2,\ b=-3 \text{ and } c=5$$

Now, discriminant, $D=b^2-4ac$
$=(-3)^2-4\,(2)\,(5)=9-40=-31<0$

$\therefore$ The given equation has no real roots.

(ii) Given equation is $3x^2 - 4\sqrt{3}x + 4 = 0$. ...(i)

On comparing it with $ax^2 + bx + c = 0$, we get

$$a = 3,\ b = -4\sqrt{3} \text{ and } c = 4$$

Now, discriminant, $D = b^2 - 4ac = (-4\sqrt{3})^2 - 4(3)(4)$

$$= 48 - 48 = 0$$

$\therefore$ The given equation has two equal real roots.

Now, Eq. (i) can be written as

$$(\sqrt{3}x)^2 - 2(\sqrt{3}x)(2) + (2)^2 = 0$$

$\Rightarrow \quad (\sqrt{3}x - 2)^2 = 0 \quad [\because a^2 - 2ab + b^2 = (a-b)^2]$

$\Rightarrow \quad (\sqrt{3}x - 2)(\sqrt{3}x - 2) = 0$

$\Rightarrow \quad \sqrt{3}x - 2 = 0$ and $\sqrt{3}x - 2 = 0$

$\therefore \quad x = \dfrac{2}{\sqrt{3}}, \dfrac{2}{\sqrt{3}}$

Hence, the equal roots are $\dfrac{2}{\sqrt{3}}$ and $\dfrac{2}{\sqrt{3}}$.

Method to Determine the Value of Unknown, When Nature of Roots is Given

If nature of roots of a quadratic equation is given and quadratic equation involves an unknown, then to find the value of unknown, firstly we find the value of discriminant in terms of unknown and then take suitable condition $D > 0$ or $D = 0$ or $D < 0$, according to the given nature and simplify it.

Example 4. Find the values of k, for which the equation $x^2 + 5kx + 16 = 0$ has no real roots.

Sol. Given equation is $x^2 + 5kx + 16 = 0$.

On comparing it with $ax^2 + bx + c = 0$, we get

$$a = 1,\ b = 5k \text{ and } c = 16$$

Now, discriminant, $D = b^2 - 4ac$

$$= (5k)^2 - 4 \times 1 \times 16 = 25k^2 - 64$$

Since, the given equation has no real roots.

$\therefore \quad D < 0$

$\Rightarrow \quad 25k^2 - 64 < 0$

$\Rightarrow \quad 25\left(k^2 - \dfrac{64}{25}\right) < 0 \Rightarrow k^2 - \dfrac{64}{25} < 0$

$\Rightarrow \quad k^2 < \dfrac{64}{25} \Rightarrow -\dfrac{8}{5} < k < \dfrac{8}{5}$

Example 5. Without solving the following quadratic equation, find the value of p, for which the roots are equal.

$$px^2 - 4x + 3 = 0 \qquad \textit{[2010]}$$

Sol. Given quadratic equation is $px^2 - 4x + 3 = 0$.

On comparing it with $ax^2 + bx + c = 0$, we get

$$a = p,\ b = -4 \text{ and } c = 3$$

Since, the given equation has equal roots.

So, the discriminant D will be zero.

i.e. $\quad D = b^2 - 4ac = 0$

$\therefore \quad (-4)^2 - 4 \times p \times 3 = 0$

$\Rightarrow \quad 16 - 12p = 0 \Rightarrow 12p = 16$

$\therefore \quad p = \dfrac{16}{12} = \dfrac{4}{3}$

Hence, the value of p is $\dfrac{4}{3}$.

Example 6. Without solving the following quadratic equation, find the value of m for which the given equation has real and equal roots.

$$x^2 + 2(m-1)x + (m+5) = 0 \qquad \textit{[2012]}$$

Sol. Given quadratic equation is

$$x^2 + 2(m-1)x + (m+5) = 0$$

On comparing it with $ax^2 + bx + c = 0$, we get

$$a = 1,\ b = 2(m-1) \text{ and } c = m + 5$$

Since, the given equation has real and equal roots.

So, the discriminant will be zero.

i.e. $\quad b^2 - 4ac = 0$

$\Rightarrow \quad [2(m-1)]^2 - 4 \times 1 \times (m+5) = 0$

$\Rightarrow \quad 4(m-1)^2 - 4(m+5) = 0$

$\Rightarrow \quad 4[m^2 + 1^2 - 2m - m - 5] = 0 \quad [\because (a-b)^2 = a^2 + b^2 - 2ab]$

$\Rightarrow \quad m^2 - 3m - 4 = 0 \quad$ [dividing by 4]

$\Rightarrow \quad m^2 - 4m + m - 4 = 0 \quad$ [splitting the middle term]

$\Rightarrow \quad m(m-4) + 1(m-4) = 0$

$\Rightarrow \quad (m+1)(m-4) = 0$

$\Rightarrow \quad m + 1 = 0$ or $m - 4 = 0$

$\therefore \quad m = -1$ or $m = 4$

Hence, the values of m are -1 and 4.

Example 7. For what value of k, $(4-k)x^2 + (2k+4)x + (8k+1) = 0$ is a perfect square?

Sol. Given equation is $(4-k)x^2 + (2k+4)x + (8k+1) = 0$

On comparing it with $ax^2 + bx + c = 0$, we get

$$a = 4 - k,\ b = 2k + 4 \text{ and } c = 8k + 1$$

For perfect square, discriminant, $D = 0$.

$\therefore \quad b^2 - 4ac = 0$

$\Rightarrow \quad (2k+4)^2 - 4 \times (4-k)(8k+1) = 0$

$\Rightarrow \quad 4k^2 + 16 + 16k - 4(32k + 4 - 8k^2 - k) = 0$

$\Rightarrow \quad 4k^2 + 16 + 16k - 124k - 16 + 32k^2 = 0$

$\Rightarrow \quad 36k^2 - 108k = 0 \Rightarrow 36k(k-3) = 0$

$\therefore \quad k = 0, 3$

Example 8. Find the values of k, so that the equation $3x^2 + kx + 2 = 0$ has equal roots. Also, find the roots in each case.

Sol. Given equation is $3x^2 + kx + 2 = 0$. ...(i)

On comparing it with $ax^2 + bx + c = 0$, we get

$$a = 3,\ b = k \text{ and } c = 2$$

Since, Eq. (i) will have equal roots.

$\therefore$ Discriminant, $D = 0 \Rightarrow b^2 - 4ac = 0$

$\Rightarrow \quad (k)^2 - 4 \times 3 \times 2 = 0$

$\Rightarrow \quad k^2 = 24$

$\Rightarrow \quad k = \pm 2\sqrt{6}$

$\therefore$ The given equation has equal roots, if $k = 2\sqrt{6}$ or $-2\sqrt{6}$.

When $k = 2\sqrt{6}$, then Eq. (i) becomes $3x^2 + 2\sqrt{6}x + 2 = 0$ and its roots are given by

$$x = \frac{-2\sqrt{6} \pm \sqrt{(2\sqrt{6})^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$= \frac{-2\sqrt{6} \pm 0}{6} = -\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3}$$

Thus, the required roots are $\frac{-\sqrt{6}}{3}$ and $-\frac{\sqrt{6}}{3}$.

When $k = -2\sqrt{6}$, then Eq. (i) becomes $3x^2 - 2\sqrt{6}x + 2 = 0$ and its roots are given by

$$x = \frac{-(-2\sqrt{6}) \pm \sqrt{(-2\sqrt{6})^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$= \frac{2\sqrt{6} \pm 0}{6} = \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3}$$

Thus, the required roots are $\frac{\sqrt{6}}{3}$ and $\frac{\sqrt{6}}{3}$.

Example 9. If -4 is a root of the quadratic equation $x^2 + px - 4 = 0$ and the quadratic equation $x^2 + px + k = 0$ has equal roots, then find the value of k.

Sol. Given, -4 is a root of the equation $x^2 + px - 4 = 0$.

So, -4 will satisfy the given equation.

$\therefore \quad (-4)^2 + p \times (-4) - 4 = 0$

$\Rightarrow \quad 16 - 4p - 4 = 0$

$\Rightarrow \quad 4p = 12$

$\Rightarrow \quad p = 3$

Now, the equation $x^2 + px + k = 0$ can be rewritten as $x^2 + 3x + k = 0$. Since, it has equal roots.

$\therefore$ Discriminant, $D = 0$

$\Rightarrow \quad b^2 - 4ac = 0$

$\Rightarrow \quad 3^2 - 4k = 0 \quad$ [here $a = 1$, $b = 3$ and $c = k$]

$\Rightarrow \quad 9 - 4k = 0$

$\therefore \quad k = \frac{9}{4}$

Example 10. Determine the positive values of k, for which the equations $x^2 + kx + 64 = 0$ and $x^2 - 8x + k = 0$ will both have real roots.

Sol. Given quadratic equations are

$$x^2 + kx + 64 = 0 \quad \text{...(i)}$$

and

$$x^2 - 8x + k = 0 \quad \text{...(ii)}$$

Let D_1 and D_2 be the discriminants of Eqs. (i) and (ii), respectively.

Then, $D_1 = k^2 - 4 \times 64 = k^2 - 256$

and $D_2 = (-8)^2 - 4k = 64 - 4k$

Since, both the equations have real roots.

$\therefore \quad D_1 \geq 0$ and $D_2 \geq 0$

$\Rightarrow \quad k^2 - 256 \geq 0$ and $64 - 4k \geq 0$

$\Rightarrow \quad k^2 \geq 256$ and $4k \leq 64$

$\Rightarrow \quad k \geq 16$ and $k \leq 16 \quad [\because k > 0]$

$\therefore \quad k = 16$

Hence, for $k = 16$, both the equations will have real roots.

Topic Exercise 2

1. Find the value of the discriminant in the following equations.
   (i) $x^2 + 2x + 4 = 0$
   (ii) $10x^2 - 1 = 3x$
   (iii) $x^2 - 5x + 1 = 0$
   (iv) $4x^2 - x - 3 = 0$
   (v) $4\sqrt{3}x^2 + 5x - 2\sqrt{3} = 0$

2. Determine the nature of the roots of the following quadratic equations.
   (i) $2x^2 + x - 1 = 0$ (ii) $x^2 - 4x + 4 = 0$
   (iii) $4x^2 - 8x + 5 = 0$ (iv) $7x^2 + 2x - 1 = 0$
   (v) $m^2 - 4m + 11 = 0$

3. Determine the nature of the roots of the following quadratic equations.
   (i) $2x^2 - 3x + 4 = 0$ (ii) $x^2 + 2\sqrt{3}x - 1 = 0$
   (iii) $(b + c)x^2 - (a + b + c)x + a = 0$
   (iv) $2x^2 - x - 3 = 0$ (v) $12x^2 = x + 1$

4. Find the value of k, for which the roots are real and equal, in each of the following equations.
   (i) $kx^2 - 2\sqrt{5}x + 4 = 0$
   (ii) $2kx^2 - 40x + 25 = 0$
   (iii) $(3k + 1)x^2 + 2(k + 1)x + k = 0$
   (iv) $x^2 - kx + 27 = 0$
   (v) $x^2 + 4kx + (k^2 - k + 2) = 0$ ***[2018]***

5. For the following equations determine the value of k, for which the given quadratic equation has real roots.
   (i) $2x^2 + kx + 3 = 0$ (ii) $kx^2 + 6x + 1 = 0$
   (iii) $3x^2 + 2x + k = 0$ (iv) $kx^2 + 2x - 7 = 0$
   (v) $3x^2 - kx - 4 = 0$

6. Find the value of k, for which the equation $5x^2 - kx + 1 = 0$ has real roots.

7. Find the values of k, for which the quadratic equation $9x^2 + kx + 1 = 0$ has equal roots. Also, find the roots in each case.

8. If the roots of the equation $(b-c)x^2 + (c-a)x + (a-b) = 0$ are equal, then prove that $2b = a + c$.

9. If the roots of the equations $ax^2 + 2bx + c = 0$ and $bx^2 - 2\sqrt{ac}\,x + b = 0$ are simultaneously real, then show that $b^2 = ac$.

10. If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal, prove that either $a = 0$ or $a^3 + b^3 + c^3 = 3abc$.

11. If p, q, r and s are real numbers such that $pr = 2(q + s)$, then show that atleast one of the equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ has real roots.

12. Prove that both the roots of the equation $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$ are real but they are equal only which $a = b = c$.

Hints and Answers

1. Do same as Example 1.
Ans. (i) -12 (ii) 49 (iii) 21 (iv) 49 (v) 121

2. Do same as Example 2.
Ans. (i) Real and distinct
(ii) Real and equal (iii) Imaginary
(iv) Real and distinct (v) Imaginary

3. Do same as Example 2.
Ans. (i) Imaginary (ii) Real and distinct
(iii) Real and distinct (iv) Real and distinct
(v) Real and distinct

4. Do same as Example 6.
Ans. (i) $k = \frac{5}{4}$ (ii) $k = 8$
(iii) $k = \frac{-1}{2}, 1$ (iv) $k = \pm 6\sqrt{3}$ (v) $k = \frac{2}{3}, -1$

5. **Hint** First, find $D = b^2 - 4ac$ and then put $D \geq 0$, because for real roots, discriminant should be greater than or equal to zero.
Ans. (i) $k \leq -2\sqrt{6}$ or $k \geq 2\sqrt{6}$ (ii) $k \leq 9$
(iii) $k \leq \frac{1}{3}$ (iv) $k \geq \frac{-1}{7}$ (v) $-\infty < k < \infty$

6. **Hint** Put $D = b^2 - 4ac \geq 0$ **Ans.** $k \leq -\sqrt{20}$ or $k \geq \sqrt{20}$

7. Do same as Example 8.
Ans. $k = 6, -6$, when $k = 6$, then roots are $\frac{-1}{3}, \frac{-1}{3}$ and when $k = -6$, then roots are $\frac{1}{3}, \frac{1}{3}$.

8. **Hint** For equal roots, $D = b^2 - 4ac = 0$.

9. **Hint** $D_1 = 4b^2 - 4ac \geq 0$ and $D_2 = 4ac - 4b^2 \geq 0$

10. **Hint** $D = 4a(a^3 + b^3 + c^3 - 3abc)$
Put $D = 0$ as roots are equal.

11. **Hint** $D_1 + D_2 = p^2 - 4q + r^2 - 4s$
$$= (p^2 + r^2) - 4\left(\frac{pr}{2}\right) \quad [\because pr = 2(q+s)]$$
$$= (p-q)^2$$
$\Rightarrow (D_1 + D_2) \geq 0$
$\therefore$ Atleast one of D_1 and D_2 is greater than equal to zero.

12. **Hint** Given equation can be rewritten as
$$3x^2 - 2x(a+b+c) + (ab + bc + ca) = 0$$
Now, find D and show it ≥ 0 and then put it equal zero for equal roots.

Topic 3
Applications of Quadratic Equation

Many real life situations can be expressed as quadratic equations. The solution of such quadratic equations gives us the answer(s) to the problem. In this topic, we will discuss some simple applications of quadratic equations.

Working Rule to Solve the Word Problems

Generally, we use these steps to solve such problems

Step I First, consider the first quantity as a variable (say x) and then write the other quantities in terms of x by using the given conditions.

Step II According to the given conditions, write the equation in the quantities, which is obtained from Step I.

Step III Simplify the equation obtained in Step II to get the quadratic equation.

Step IV Now, simplify the quadratic equation by any one of the methods, i.e. factorisation or quadratic formula, to obtain the value(s) of variable x.

Step V Further, check whether the value(s) of variable x, satisfies the given condition or not.

Step VI Find the values of the quantities. Also, find the values of other information, if asked in the question.

Different Types of Word Problems

Here, we will discuss various types of word problems and analyse, how to represent them as a quadratic equation.

1. Word Problems Based on Numbers

Some concepts that will be helpful in solving these types of problems are given below

(i) Two consecutive numbers can be chosen as x and $x+1$.

(ii) Two consecutive odd/even numbers can be chosen as x and $x+2$.

(iii) If sum of two numbers is given to us, say k, then the numbers can be chosen as x and $k - x$.

(iv) If difference of two numbers is given to us, say k, then the number can be chosen as x and $x+k$.

Example 1. The sum of the squares of two consecutive odd integers is 394. Find the integers.

Sol. Let two consecutive odd integers be x and $x + 2$.

Then, according to the question, $x^2 + (x+2)^2 = 394$

$\Rightarrow \quad x^2 + (x^2 + 4 + 4x) = 394$

$\Rightarrow \quad 2x^2 + 4x = 390$

$\Rightarrow \quad x^2 + 2x = 195$

$\Rightarrow \quad x^2 + 2x - 195 = 0$

$\Rightarrow \quad x^2 + 15x - 13x - 195 = 0$

$\Rightarrow \quad x(x+15) - 13(x+15) = 0$

$\Rightarrow \quad (x+15)(x-13) = 0$

$\Rightarrow \quad x = 13 \text{ or } -15$

When $x = 13$, then other integer is 15 and when $x = -15$, then other integer is -13.

Thus, the integers are 13 and 15 or -13 and -15.

Example 2. The difference of two natural numbers is 4 and the difference of their reciprocals is 1/8. Find the numbers.

Sol. As the difference of two natural numbers is 4, let the numbers be x and $x + 4$, where $x \in N$.

We know that, $x + 4 > x$

$\therefore \quad \frac{1}{x+4} < \frac{1}{x} \Rightarrow \frac{1}{x} - \frac{1}{x+4} > 0$

According to the question,

$$\frac{1}{x} - \frac{1}{x+4} = \frac{1}{8}$$

$\Rightarrow \quad \frac{(x+4)-x}{x(x+4)} = \frac{1}{8} \Rightarrow \frac{4}{x(x+4)} = \frac{1}{8}$

$\Rightarrow \quad x(x+4) = 32 \Rightarrow x^2 + 4x - 32 = 0$

$\Rightarrow \quad x^2 + 8x - 4x - 32 = 0$ [splitting the middle term]

$\Rightarrow \quad x(x+8) - 4(x+8) = 0$

$\Rightarrow \quad (x-4)(x+8) = 0$

$\Rightarrow \quad x - 4 = 0 \text{ or } x + 8 = 0$

$\Rightarrow \quad x = 4 \text{ or } x = -8$

But $\quad x \in N$

$\therefore \quad x = 4$

When $x = 4$, then $x + 4 = 4 + 4 = 8$

Hence, the required numbers are 4 and 8.

Example 3. The sum of a number and its positive square root is $\frac{6}{25}$. Find the number.

Sol. Let the number be x. Then, its positive square root be $\sqrt{x}$.

$\therefore \quad x + \sqrt{x} = \frac{6}{25}$ [by given condition]

$\Rightarrow \quad 25x + 25\sqrt{x} = 6$ [multiplying both sides by 25]

$\Rightarrow \quad 25x + 25\sqrt{x} - 6 = 0$

Put $\sqrt{x} = y \Rightarrow x = y^2$, then above equation becomes

$25y^2 + 25y - 6 = 0$

$\Rightarrow \quad 25y^2 + 30y - 5y - 6 = 0$ [splitting the middle term]

$\Rightarrow \quad 5y(5y+6) - 1(5y+6) = 0$

$\Rightarrow \quad (5y-1)(5y+6) = 0$

$\Rightarrow \quad y = \frac{1}{5} \text{ or } y = -\frac{6}{5}$

Again, substituting the values of y in $\sqrt{x} = y$, we get

$$\sqrt{x} = \frac{1}{5} \text{ or } \sqrt{x} = -\frac{6}{5}$$

But $\sqrt{x} = \frac{-6}{5}$ is not possible, because $\sqrt{x}$ is positive.

$\therefore \quad \sqrt{x} = \frac{1}{5} \Rightarrow x = \frac{1}{25}$

Hence, the required number is $\frac{1}{25}$.

2. Word Problems Based on Digits

Some concepts that will be helpful in solving these types of problems, are given below

If the digit in unit's place is x and that in ten's place is y. Then, two-digit number is given by $10y + x$.

On interchanging the positions of the digits, the digit in unit's place becomes y and in ten's place becomes x and then two-digit number becomes $10x + y$.

Example 4. A two-digit positive number is such that the product of its digits is 6. If 9 is added to the number, then the digits interchange their places. Find the number. ***[2014]***

Sol. Let the required two-digit number be $10x + y$, where y is unit's digit and x is ten's digit.

According to the question,

$$xy = 6 \text{ and } 10x + y + 9 = 10y + x$$

$$\Rightarrow \quad y = \frac{6}{x} \text{ and } 9x - 9y + 9 = 0$$

Now, $\quad 9(x - y) = -9$

$$\Rightarrow \quad x - y = -1 \quad \text{[dividing both sides by 9]} \quad \ldots\text{(i)}$$

$$\Rightarrow \quad x - \frac{6}{x} = -1 \quad \left[\because y = \frac{6}{x}\right]$$

$$\Rightarrow \quad \frac{x^2 - 6}{x} = -1$$

$$\Rightarrow \quad x^2 - 6 = -x$$

$$\Rightarrow \quad x^2 + x - 6 = 0$$

$$\Rightarrow \quad x^2 + 3x - 2x - 6 = 0 \quad \text{[splitting the middle term]}$$

$$\Rightarrow \quad x(x + 3) - 2(x + 3) = 0$$

$$\Rightarrow \quad (x + 3)(x - 2) = 0 \quad \text{[factorising left side]}$$

$$\Rightarrow \quad x = -3 \text{ or } x = 2$$

$$\therefore \quad x = 2 \quad [\because \text{digit cannot be negative}]$$

On putting $x = 2$ in Eq. (i), we get

$$2 - y = -1 \Rightarrow -y = -1 - 2 = -3 \Rightarrow y = 3$$

$\therefore$ The required two-digit number

$$= 10x + y = 10 \times 2 + 3 = 23$$

Hence, the required number is 23.

3. Word Problems Based on Fractions

A concept that will be helpful in solving these types of problems is given below

If the numerator of the fraction is x and denominator is y, then the fraction is given by x / y.

Example 5. **The denominator of a fraction is one more than twice the numerator. If the sum of fraction and its reciprocal is $2\frac{16}{21}$, then find the fraction.**

Sol. Let the numerator of the fraction be $x (x \in I)$, then its denominator $= 2x + 1$ and so the fraction is $\frac{x}{2x+1}$.

According to the question,

$$\frac{x}{2x + 1} + \frac{2x + 1}{x} = 2\frac{16}{21}$$

$$\Rightarrow \quad \frac{x^2 + (2x + 1)^2}{x(2x + 1)} = \frac{58}{21}$$

$$\Rightarrow \quad 58x(2x + 1) = 21(x^2 + 4x^2 + 4x + 1)$$

$$\Rightarrow \quad 116x^2 + 58x = 105x^2 + 84x + 21$$

$$\Rightarrow \quad 11x^2 - 26x - 21 = 0$$

$$\Rightarrow \quad 11x^2 - 33x + 7x - 21 = 0 \quad \text{[splitting the middle term]}$$

$$\Rightarrow \quad 11x(x - 3) + 7(x - 3) = 0$$

$$\Rightarrow \quad (x - 3)(11x + 7) = 0$$

$$\Rightarrow \quad x - 3 = 0 \text{ or } 11x + 7 = 0$$

$$\Rightarrow \quad x = 3 \text{ or } x = -\frac{7}{11} \quad [\because x \text{ is an integer}]$$

$$\Rightarrow \quad x = 3$$

$$\therefore \text{ Required fraction} = \frac{3}{2 \times 3 + 1} = \frac{3}{7}$$

Example 6. **The sum of the numerator and denominator of a certain fraction is 8. If 2 is added to both the numerator and denominator, the fraction is increased by $\frac{4}{35}$. Find the fraction.**

Sol. Let the fraction be x/y. Then, according to the question,

$$x + y = 8 \quad \ldots\text{(i)}$$

and $$\frac{x + 2}{y + 2} = \frac{x}{y} + \frac{4}{35} \quad \ldots\text{(ii)}$$

From Eq. (ii), we get

$$\frac{x + 2}{y + 2} - \frac{x}{y} = \frac{4}{35} \Rightarrow \frac{(x + 2)y - x(y + 2)}{y(y + 2)} = \frac{4}{35}$$

$$\Rightarrow \quad 35(xy + 2y - xy - 2x) = 4y(y + 2)$$

$$\Rightarrow \quad 35(2y - 2x) = 4y(y + 2)$$

$$\Rightarrow \quad 35(y - x) = 2y(y + 2) \quad \text{[dividing both sides by 2]}$$

$$\Rightarrow \quad 35(y - (8 - y)) = 2y(y + 2) \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 35(2y - 8) = 2y(y + 2)$$

$$\Rightarrow \quad 35(2(y - 4)) = 2y(y + 2)$$

$$\Rightarrow \quad 35(y - 4) = y(y + 2) \quad \text{[dividing both sides by 2]}$$

$$\Rightarrow \quad 35y - 140 = y^2 + 2y$$

$$\Rightarrow \quad y^2 + 2y - 35y + 140 = 0$$

$$\Rightarrow \quad y^2 - 33y + 140 = 0$$

$$\Rightarrow \quad y^2 - 28y - 5y + 140 = 0 \quad \text{[splitting the middle term]}$$

$$\Rightarrow \quad y(y - 28) - 5(y - 28) = 0$$

$$\Rightarrow \quad (y - 28)(y - 5) = 0$$

$$\Rightarrow \quad y = 28 \text{ or } y = 5$$

$$\Rightarrow \quad y = 5 \quad [\because y \neq 28, \text{ as sum of fraction is } 8]$$

On substituting $y = 5$ in Eq. (i), we get

$$x + 5 = 8 \Rightarrow x = 3$$

Hence, the fraction is $\frac{3}{5}$.

4. Word Problems Based on Age

Some concepts that will be helpful in solving these types of problems, are given below

If the present age of a person is x yr, then

(i) 'a' yr ago, age of person was $(x - a)$ yr.

(ii) after 'b' yr, age of person will be $(x + b)$ yr.

Example 7. **Seven years ago, Varun's age was five times the square of Swati's age. Three years hence, Swati's age will be two-fifth of Varun's age. Find their present ages.**

Sol. Let seven years ago, Swati's age be x yr.

Then, Varun's age $= 5x^2$ yr

$\therefore$ Swati's present age $= (x + 7)$ yr

and Varun's present age $= (5x^2 + 7)$ yr

Three years hence, i.e. after 3 years from now,

Swati's age $= (x + 7 + 3) = (x + 10)$ yr

Varun's age $= (5x^2 + 7 + 3) = (5x^2 + 10)$ yr

According to the question,

$$x + 10 = \frac{2}{5}(5x^2 + 10)$$

$\Rightarrow \quad x + 10 = 2x^2 + 4$

$\Rightarrow \quad 2x^2 - x - 6 = 0$

$\Rightarrow \quad 2x^2 - 4x + 3x - 6 = 0$ [splitting the middle term]

$\Rightarrow \quad 2x(x - 2) + 3(x - 2) = 0 \Rightarrow (2x + 3)(x - 2) = 0$

$\Rightarrow \quad x - 2 = 0$ or $2x + 3 = 0 \Rightarrow x = 2$ or $x = \frac{-3}{2}$

But $\quad x \neq \frac{-3}{2}$ [since, age cannot be negative]

$\therefore \quad x = 2$

Hence, Swati's present age $= 2 + 7 = 9$ yr

and Varun's present age $= 5 \times 2^2 + 7 = 27$ yr

Example 8. Five years ago, a woman's age was the square of her son's age. Ten years hence, her age will be twice that of her son's age. Find

(i) the age of the son five years ago.

(ii) the present age of the woman. ***[2007]***

Sol. Let the age of the son 5 yr ago be x yr, then the age of the woman 5 yr ago $= x^2$ yr.

Now, the present age of the son $= (x + 5)$ yr

and the present age of the woman $= (x^2 + 5)$ yr

Ten years hence, i.e. after 10 years from now,

the age of the son $= ((x + 5) + 10)$ yr $= (x + 15)$ yr

and the age of the woman $= ((x^2 + 5) + 10)$ yr

$= (x^2 + 15)$ yr

According to the question, $x^2 + 15 = 2(x + 15)$

$\Rightarrow \quad x^2 + 15 = 2x + 30 \Rightarrow x^2 - 2x - 15 = 0$

$\Rightarrow \quad x^2 - 5x + 3x - 15 = 0$ [splitting the middle term]

$\Rightarrow \quad x(x - 5) + 3(x - 5) = 0$

$\Rightarrow \quad (x - 5)(x + 3) = 0 \Rightarrow x - 5 = 0$ or $x + 3 = 0$

$\Rightarrow \quad x = 5$ or $x = -3$ but x being age cannot be negative

$\Rightarrow \quad x = 5$

(i) The age of the son 5 yr ago = 5 yr.

(ii) The present age of the woman $= (x^2 + 5)$ yr

$= (5^2 + 5)$ yr $= 30$

yr

Example 9. The sum of the ages of Vivek and his younger brother Amit is 47 yr. The product of their ages in years is 550. Find their ages. ***[2017]***

Sol. Let the age of Vivek $= x$ yr and age of younger brother $= y$ yr

According to the question,

$x + y = 47 \quad \ldots$(i)

and $\quad xy = 550 \quad \ldots$(ii)

Now, $\quad x - y = \sqrt{(x + y)^2 - 4xy} = \sqrt{(47)^2 - 4 \times 550}$

$= \sqrt{2209 - 2200} = \sqrt{9}$

$\Rightarrow \quad x - y = 3 \quad \ldots$(iii)

On subtracting Eq. (iii) from Eq. (i), we get

$$2y = 44 \Rightarrow y = \frac{44}{2} = 22$$

On putting the value of y in Eq. (i), we get

$x + 22 = 47 \Rightarrow x = 47 - 22 = 25$

$\therefore$ Age of Vivek = 25 yr

and age of his younger brother = 22 yr

5. Word Problems Based on Time and Work

The concept that will be used for these types of problems, is given below

If a person can do a piece of work in n days, then he will do $1/n$ of the work in one day and *vice-versa*.

Example 10. B takes 16 days less than A to do a piece of work. If both, working together can do it in 15 days, then in how many days, will B alone complete the work?

Sol. Let A complete the work in x days. Then, B will complete it in $(x - 16)$ days.

$\therefore$ A's one day work $= \frac{1}{x}$ and B's one day work $= \frac{1}{x - 16}$

Since, A and B together can complete the work in 15 days.

$\therefore$ A's one day work + B's one day work $= \frac{1}{15}$

$\Rightarrow \quad \frac{1}{x} + \frac{1}{x - 16} = \frac{1}{15}$

$\Rightarrow \quad \frac{x - 16 + x}{x(x - 16)} = \frac{1}{15}$

$\Rightarrow \quad x(x - 16) = 15(2x - 16)$

$\Rightarrow \quad x^2 - 16x = 30x - 240$

$\Rightarrow \quad x^2 - 46x + 240 = 0$

$\Rightarrow \quad x^2 - 40x - 6x + 240 = 0$ [splitting the middle term]

$\Rightarrow \quad x(x - 40) - 6(x - 40) = 0$

$\Rightarrow \quad (x - 6)(x - 40) = 0$

$\Rightarrow \quad x - 6 = 0$ or $x - 40 = 0$

$\Rightarrow \quad x = 6$ or $x = 40$

If $x = 6$, then $x - 16 = 6 - 16 = -10$, which is not possible.

$\therefore \quad x = 40$

Hence, B alone will complete the work in $(40 - 16)$ days, i.e. 24 days.

Example 11. To fill a swimming pool, two pipes are to be used. If the pipe of the larger diameter is used for 4 h and the pipe of the smaller diameter is used for 9 h, only half of the pool can be filled. Find how long it would take for each pipe to fill the pool separately, if the pipe of the smaller diameter takes 10 h more than the pipe of the larger diameter to fill the pool?

Sol. Let the time taken by the pipe of the larger diameter to fill the pool separately be x h, then the time taken by the pipe of the smaller diameter to fill the pool separately $= (x + 10)$ h.

Then, the part of the pool filled by the pipe of larger diameter in 4 h $= \frac{4}{x}$ and the part of the pool filled by the pipe of the smaller diameter in 9 h $= \frac{9}{x + 10}$.

According to the question, $\frac{4}{x} + \frac{9}{x + 10} = \frac{1}{2}$

$\Rightarrow \quad \frac{4(x + 10) + 9x}{x(x + 10)} = \frac{1}{2} \Rightarrow \frac{13x + 40}{x^2 + 10x} = \frac{1}{2}$

$\Rightarrow \quad x^2 + 10x = 26x + 80 \Rightarrow x^2 - 16x - 80 = 0$

$\Rightarrow \quad x^2 - 20x + 4x - 80 = 0$ [splitting the middle term]

$\Rightarrow \quad x(x-20) + 4(x-20) = 0$

$\Rightarrow \quad (x-20)(x+4) = 0 \Rightarrow x = 20$ or $x = -4$

But x being the time taken cannot be negative, so $x = 20$.

Hence, the time taken to fill the pool by the two pipes separately is 20 h and $(x + 10)$ h, i.e. 30 h.

6. Word Problems Based on Distance, Speed and Time

Some formulae that will be used for these types of problems, are given below

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}, \text{Distance} = \text{Speed} \times \text{Time}$$

or $$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

If the problem is related to speed of boat going downstream and upstream and if the speed of boat in still water is x km/h and speed of stream is y km/h, then speed of boat downstream $= (x + y)$ km/h and speed of boat upstream $= (x - y)$ km/h.

Example 12. A car covers a distance of 400 km at a certain speed. Had the speed been 12 km/h more, the time taken for the journey would have been 1 h 40 min less. Find the original speed of the car. ***[2012]***

Sol. Let the original speed of the car be x km/h.

$\because$ Distance covered by a car = 400 km

$\therefore$ Original time taken, $T = \frac{400}{x}$ h $\quad \left[\because \text{time} = \frac{\text{distance}}{\text{speed}}\right]$

Further, speed is increased by 12 km/h.

$\therefore \quad$ Speed $= (x + 12)$ km/h

Now, new time taken $= \frac{400}{x+12}$ h

According to the question, $\frac{400}{x} - \frac{400}{x+12} = \frac{5}{3}$

$$\left[\begin{aligned}\because 1 \text{ h } 40 \text{ min} &= 1 \text{ h} + 40 \text{ min} = 1 \text{ h} + \frac{40}{60}\text{h} \\ &= 1 \text{ h} + \frac{2}{3}\text{h} = \left(1 + \frac{2}{3}\right)\text{h} = \frac{5}{3}\text{h}\end{aligned}\right]$$

$\Rightarrow \quad 400\left(\frac{1}{x} - \frac{1}{x+12}\right) = \frac{5}{3} \Rightarrow 400\left[\frac{x+12-x}{x(x+12)}\right] = \frac{5}{3}$

$\Rightarrow \quad 80(12) \times 3 = x(x+12)$ [dividing both sides by 5]

$\Rightarrow \quad 2880 = x^2 + 12x$

$\Rightarrow \quad x^2 + 12x - 2880 = 0$

$\Rightarrow \quad x^2 + 60x - 48x - 2880 = 0$ [splitting the middle term]

$\Rightarrow \quad x(x+60) - 48(x+60) = 0$

$\Rightarrow \quad (x+60)(x-48) = 0$

$\Rightarrow \quad x + 60 = 0$ or $x - 48 = 0$

$\Rightarrow \quad x = 48$ or $x = -60$

$\therefore \quad x = 48$ km/h [$\because$ speed cannot be negative]

Hence, the original speed of the car is 48 km/h.

Example 13. Sonal can row a boat at a speed of 5 km/h. If it takes her 1 h more to row the boat 5.25 km upstream than to return downstream, then find the speed of the stream.

Sol. Let the speed of the stream be x km/h, where $0 < x < 5$.

Then, speed of the boat upstream $= (5 - x)$ km/h

and speed of the boat downstream $= (5 + x)$ km/h

$\therefore$ Time taken for going 5.25 km upstream $= \frac{5.25}{5-x}$ h

and time taken for going 5.25 km downstream $= \frac{5.25}{5+x}$ h

According to the question,

$$\frac{5.25}{5-x} - \frac{5.25}{5+x} = 1$$

$\Rightarrow \quad \frac{21}{4}\left(\frac{1}{5-x} - \frac{1}{5+x}\right) = 1$

$\Rightarrow \quad 21[(5+x) - (5-x)] = 4(5-x)(5+x)$

$\Rightarrow \quad 42x = 4(25 - x^2) \quad [\because (a-b)(a+b) = a^2 - b^2]$

$\Rightarrow \quad 4x^2 + 42x - 100 = 0$

$\Rightarrow \quad 2x^2 + 21x - 50 = 0$ [dividing both sides by 2]

$\Rightarrow \quad 2x^2 + 25x - 4x - 50 = 0$ [splitting the middle term]

$\Rightarrow \quad x(2x+25) - 2(2x+25) = 0$

$\Rightarrow \quad (2x+25)(x-2) = 0$

$\Rightarrow \quad 2x + 25 = 0$ or $x - 2 = 0$

$\Rightarrow \quad x = \frac{-25}{2}$ or $x = 2$

$\therefore \quad x = 2$ [$\because$ speed cannot be negative]

Hence, the required speed is 2 km/h.

Example 14. In a flight of 600 km, an aircraft was slowed down due to bad weather. Its average speed for the trip was reduced by 200 km/h and the time of flight increased by 30 min. Find the duration of flight.

Sol. Let the original speed of the aircraft be x km/h. Then,

New speed $= (x - 200)$ km/h

Duration of flight at original speed $= \left(\frac{600}{x}\right)$ h

Duration of flight at reduced speed $= \left(\frac{600}{x-200}\right)$ h

According to the question,

$$\frac{600}{x-200} - \frac{600}{x} = 30 \text{ min} = \frac{1}{2} \text{h}$$

$\Rightarrow \quad \frac{600}{x-200} - \frac{600}{x} = \frac{1}{2} \quad \left[\because 30 \text{ min} = \frac{30}{60}\text{h} = \frac{1}{2}\text{h}\right]$

$\Rightarrow \quad \frac{600x - 600(x-200)}{x(x-200)} = \frac{1}{2} \Rightarrow \frac{120000}{x^2 - 200x} = \frac{1}{2}$

$\Rightarrow \quad x^2 - 200x - 240000 = 0$

$\Rightarrow \quad x^2 - 600x + 400x - 240000 = 0$

[splitting the middle term]

$\Rightarrow \quad x(x-600) + 400(x-600) = 0$

$\Rightarrow \quad (x-600)(x+400)=0$

$\Rightarrow \quad x-600=0$ or $x+400=0$

$\Rightarrow \quad x=600$ or $x=-400$

$\therefore \quad x=600$ [$\because$ speed cannot be negative]

So, the original speed of the aircraft was 600 km/h.

Hence, duration of flight $= \frac{600}{x}\text{h} = \frac{600}{600}\text{h} = 1\text{h}$

Example 15. A bus covers a distance of 240 km at a uniform speed. Due to heavy rain, its speed gets reduced by 10 km/h and as such it takes two hours longer to cover the total distance. Assuming the uniform speed to be 'x' km/h, form an equation and solve it to evaluate 'x'. ***[2016]***

Sol. Let the uniform speed of bus be x km/h.

$\therefore$ Time taken by bus to cover a distance of 240 km

$$T_1 = \frac{240}{x}\text{h}$$

$\because$ Due to heavy rain, speed of bus is reduced by 10 km/h.

$\therefore$ Reduced speed $=(x-10)$ km/h

Now, time taken by bus to cover a distance of 240 km with speed $(x-10)$ km/h

$$T_2 = \left(\frac{240}{x-10}\right)\text{h}$$

According to the question,

$$T_2 - T_1 = 2$$

$\Rightarrow \quad \frac{240}{x-10} - \frac{240}{x} = 2$

$\Rightarrow \quad 240\left[\frac{x-x+10}{x(x-10)}\right] = 2 \Rightarrow \frac{2400}{x(x-10)} = 2$

$\Rightarrow \quad 2x(x-10) = 2400$

$\Rightarrow \quad x(x-10)=1200$

$\Rightarrow \quad x^2-10x-1200=0$

which is the required equation.

$\therefore \quad x = \frac{10 \pm \sqrt{100+4800}}{2}$

[by Sridharacharya formula]

$= \frac{10 \pm \sqrt{4900}}{2} = \frac{10 \pm 70}{2} = 40, -30$

$\therefore \quad x = 40$ [$\because$ speed cannot be negative]

Hence, the speed of bus is 40 km/h.

7. Word Problems Based on Monetary Transaction

Some concepts that will be helpful in solving these types of problems, are given below.

(i) If the cost of x articles is ₹ y, then the cost of each article is ₹ $\left(\frac{y}{x}\right)$.

(ii) Selling price of an article = Cost of an article + Profit
or Cost of an article − Loss

Example 16. A dealer sells a toy for ₹ 24 and gains as much per cent as the cost price of the toy. Find the cost price of the toy.

Sol. Let the cost price of the toy be ₹ x.

Then, gain per cent $= x\%$

$\therefore \quad$ Gain $= x \times \frac{x}{100} =$ ₹ $\frac{x^2}{100}$

Now, SP = CP + Gain $= x + \frac{x^2}{100}$

But SP = ₹ 24 [given]

$\therefore \quad x + \frac{x^2}{100} = 24 \Rightarrow 100x + x^2 = 2400$

$\Rightarrow \quad x^2+100x-2400=0$

$\Rightarrow \quad x^2+120x-20x-2400=0$

[splitting the middle term]

$\Rightarrow \quad x(x+120)-20(x+120)=0$

$\Rightarrow \quad (x+120)(x-20)=0$

$\Rightarrow \quad x+120=0$ or $x-20=0$

$\Rightarrow \quad x=20, -120$

$\therefore \quad x=20$ [$\because$ cost cannot be negative]

Hence, the cost price of a toy is ₹ 20.

Example 17. ₹ 480 is divided equally among x children. If the number of children was 20 more than each would have got ₹ 12 less. Find the value of x. ***[2011]***

Sol. Given, total money = ₹ 480

and original number of children $= x$

Then, money received by each child = ₹ $\frac{480}{x}$

Since, number of children was 20 more.

$\therefore$ Total number of children $= x+20$

and the money received by each child = ₹ $\frac{480}{x+20}$

According to the question, $\frac{480}{x} - \frac{480}{x+20} = 12$

$\Rightarrow \quad \frac{480(x+20)-480x}{x(x+20)} = 12$

$\Rightarrow \quad \frac{480(x+20-x)}{x(x+20)} = 12$

$\Rightarrow \quad \frac{40(20)}{x(x+20)} = 1$ [dividing both sides by 12]

$\Rightarrow \quad x(x+20)=800$

$\Rightarrow \quad x^2+20x-800=0$

$\Rightarrow \quad x^2+40x-20x-800=0$ [splitting the middle term]

$\Rightarrow \quad x(x+40)-20(x+40)=0$

$\Rightarrow \quad (x-20)(x+40)=0$

$\Rightarrow \quad x=20$ or $x=-40$

$\therefore \quad x=20$ [$\because x$ cannot be negative]

Hence, the number of children is 20.

Example 18. A piece of cloth costs ₹ 200. If the piece was 5 m longer and each metre of cloth costs ₹ 2 less, the cost of the piece would have remained unchanged. How long is the piece and what is the original rate per metre?

Sol. Let the original length of piece be x m.

Then, rate = ₹ $\frac{200}{x}$ per m

$\therefore$ New length = $(x + 5)$ m

Since, the cost remains same.

$\therefore$ New rate = ₹ $\frac{200}{x+5}$ per m

According to the question,

$$\frac{200}{x+5} = \frac{200}{x} - 2 \Rightarrow \frac{200}{x} - \frac{200}{x+5} = 2$$

$$\Rightarrow \quad 200\left(\frac{1}{x} - \frac{1}{x+5}\right) = 2 \Rightarrow \frac{x+5-x}{x(x+5)} = \frac{2}{200}$$

$$\Rightarrow \quad \frac{5}{x^2+5x} = \frac{1}{100} \Rightarrow x^2 + 5x = 500$$

$$\Rightarrow \quad x^2 + 5x - 500 = 0$$

$$\Rightarrow \quad x^2 + 25x - 20x - 500 = 0 \text{ [splitting the middle term]}$$

$$\Rightarrow \quad x(x+25) - 20(x+25) = 0 \Rightarrow (x+25)(x-20) = 0$$

$$\Rightarrow \quad x + 25 = 0 \text{ or } x - 20 = 0 \Rightarrow x = -25 \text{ or } x = 20$$

$$\Rightarrow \quad x = 20 \text{ m} \quad [\because \text{length cannot be negative}]$$

$$\therefore \text{ Rate } = \frac{200}{x} = \frac{200}{20} = ₹\, 10 \text{ per m}$$

Hence, the original length of the piece is 20 m and its original rate per metre is ₹ 10.

Example 19. A shopkeeper purchases a certain number of books for ₹ 960. If the cost per book was ₹ 8 less the number of books that could be purchased for ₹ 960, would be 4 more. Write an equation, taking the original cost of each book to be ₹ x and solve it to find the original cost of the books. ***[2013]***

Sol. Given, total cost of books purchased = ₹ 960

Let the original cost of one book purchased be ₹ x.

Then, the total number of books purchased = $\frac{960}{x}$

If the cost of per book is ₹ 8 less, then

Cost of one book purchased = ₹ $(x - 8)$

and total number of books purchased = $\frac{960}{x-8}$

According to the question, $\frac{960}{x-8} = \frac{960}{x} + 4$

$$\Rightarrow \quad \frac{960}{x-8} - \frac{960}{x} = 4 \Rightarrow 960\left(\frac{1}{x-8} - \frac{1}{x}\right) = 4$$

$$\Rightarrow \quad \frac{960\,(x - x + 8)}{x(x-8)} = 4 \Rightarrow \frac{240 \times 8}{x\,(x-8)} = 1$$

[dividing both sides by 4]

$$\Rightarrow \quad x^2 - 8x = 1920$$

$$\Rightarrow \quad x^2 - 8x - 1920 = 0$$

$$\Rightarrow \quad x^2 - 48x + 40x - 1920 = 0 \text{ [splitting the middle term]}$$

$$\Rightarrow \quad x\,(x-48) + 40\,(x-48) = 0$$

$$\Rightarrow \quad (x-48)\,(x+40) = 0$$

$$\Rightarrow \quad x - 48 = 0 \text{ or } x + 40 = 0$$

$$\Rightarrow \quad x = 48 \text{ or } x = -40$$

$$\therefore \quad x = 48 \quad [\because \text{cost cannot be negative}]$$

Hence, the original cost of each book is ₹ 48.

Topic Exercise 3

1. The sum of the squares of two consecutive natural numbers is 481. Find the numbers.
2. The sum of the squares of two consecutive odd positive integers is 290. Find them.
3. Divide 25 into two parts, such that twice the square of the larger part exceeds thrice the square of the smaller part by 29.
4. The sum of two numbers is 18. If the sum of their reciprocals is $\frac{1}{4}$, then find the numbers.
5. A two-digit number is such that the product of its digits is 18. When 63 is subtracted from the number, the digits interchange their places. Find the number.
6. A two-digit number contains the smaller of two digits in the unit's place .The product of the digits is 24 and the difference between the digits is 5. Find the number.
7. In a certain positive fraction, the denominator is greater than the numerator by 3. If 1 is subtracted from both the numerator and denominator, the fraction is decreased by $\frac{1}{14}$. Find the fraction.
8. The product of Shikha's age five years ago and her age 8 yr later is 30, find her percentage.
9. The sum of the reciprocals of Rehman's ages (in years) 3 yr ago and 5 yr from now is 1/3. Find his percentage.
10. A take 10 days less than the time taken by B to finish a piece of work. If both A and B together can finish the work in 12 days, find the time taken by B to finish the work.
11. Two water pipes together can fill a tank in $9\frac{3}{8}$ h. The pipe of larger diameter takes 10 h less than the smaller one to fill the tank separately. Find the time in which each pipe can fill the tank separately.
12. An aeroplane travelled a distance of 400 km at an average speed of x km/h. On the return journey, the speed was increased by 40 km/h. Write down an expression for the time taken for

(i) the onward journey. (ii) the return journey.

If the return journey took 30 min less than the onward journey, then write down an equation in x and find its value. **[2002]**

13. The speed of an express train is x km/h and the speed of an ordinary train is 12 km/h less than that of the express train. If the ordinary train takes one hour more than the express train to cover a distance of 240 km, then find the speed of the express train. **[2009]**

14. An aeroplane flying with a wind of 30 km/h takes 40 min less to fly 3600 km than what it would to have taken to fly against the same wind. Find the aeroplane's speed of flying in still air.

15. A piece of cloth costs ₹ 300. If the piece was 5 cm longer and each metre of cloth costs ₹ 2 less, the cost of the piece would have remained unchanged. How long is the original piece of cloth and what is the rate per metre?

16. Some students planned a picnic. The budget for the food was ₹ 480. As eight of them failed to join the party, the cost of the food for each member is increased by ₹ 10. Find how many students went for the picnic? **[2008]**

17. The hotel bill for a number of persons for overnight stay is ₹ 4800. If there were 4 more persons, the bill each person had to pay, would have reduced by ₹ 200. Find the number of persons staying overnight. **[2000]**

18. The hypotenuse of a right triangle is 25 cm. The difference between the lengths of the other two sides of the triangle is 5 cm. Find the lengths of these sides.

19. If the perimeter of a rectangular plot is 68 m and length of its diagonal is 26 m, then find its area.

20. A grassy land is in the shape of a right triangle. The hypotenuse of the land is 1 m more than twice the shortest side. If the third side is 7 m more than the shortest side, find the sides of the grassy land.

21. ₹7500 were divided equally among a certain number of children. Had there been 20 less children, each would have received ₹100 more. Find the original number of children. **[2018]**

Hints and Answers

1. Hint Let two consecutive natural numbers be p and $p+1$, respectively.

Then, $p^2+(p+1)^2=481$.

Ans. 15, 16

2. Do same as Example 1.

Ans. 11, 13

3. Hint Let the smaller part be x and the larger part be $(25-x)$.

According to the question, $2(25-x)^2=3x^2+29$

Ans. 14, 11

4. Hint Let the numbers be x and $(18-x)$, respectively.

According to the question, $\frac{1}{x}+\frac{1}{(18-x)}=\frac{1}{4}$

$\Rightarrow x^2-18x+72=0$

Ans. 6, 12

5. Do same as Example 4.

Ans. 92

6. Hint Let the larger digit be x and smaller digit be y.

Then, the number is $10x+y$.

Also, $x-\frac{24}{x}=5$

Ans. 83

7. Hint $\frac{x-1}{(x+3)-1}=\frac{x}{x+3}-\frac{1}{14}$ **Ans.** $\frac{4}{7}$

8. Hint Let Shikha's present age be x yr.

According to the question, $(x-5)(x+8)=30$

Ans. 7 yr

9. Hint Let Rehman's age be x yr.

Then, we have $\frac{1}{x-3}+\frac{1}{x+5}=\frac{1}{3}$

Ans. 7 yr

10. Do same as Example 10.

Ans. 30 days

11. Hint Let the time taken by the pipe of larger diameter to fill the tank separately be x h. Then, the time taken by the pipe of smaller diameter to fill the tank separately $=(x+10)$ h.

According to the question,

$$\frac{1}{x}+\frac{1}{x+10}=\frac{8}{75}$$

Ans. 15 h, 25 h

12. **Hint** Let the time taken by the aeroplane at a speed of x km/h $= t_1$ h and the time taken by the aeroplane at a speed of $(x + 40)$ km/h $= t_2$ h

Then, according to the question,

$$t_1 = \frac{400}{x} \text{ and } t_2 = \frac{400}{x+40} \qquad \left[\because \text{time} = \frac{\text{distance}}{\text{speed}}\right]$$

Now, as $t_1 - t_2 = \frac{1}{2}$ h

$$\therefore \frac{400}{x} - \frac{400}{x+40} = \frac{1}{2}$$

Ans. $x^2 + 40x - 32000 = 0$ and $x = 160$

13. **Hint** Time taken by express train, $t_1 = \frac{240}{x}$

$$\left[\because \text{time} = \frac{\text{distance}}{\text{speed}}\right]$$

and time taken by ordinary train, $t_2 = \frac{240}{x-12}$

According to the question, $\frac{240}{x-12} - \frac{240}{x} = 1$

Ans. 60 km/h

14. **Hint** Let the speed of aeroplane in still air be x km/h.

Then, $\frac{3600}{x-30} - \frac{3600}{x+30} = \frac{2}{3}$

Ans. 570 km/h

15. Do same as Example 18.

Ans. 25 m, ₹ 12 per m

16. **Hint** Let x be the number of students planned for the picnic.

According to the question, $\frac{480}{x-8} - \frac{480}{x} = 10$

Ans. 16

17. Do same as Q. 16. **Ans.** 8

18. **Hint** Let the other two sides of triangle be x and $(x+5)$ cm.

Using Pythagoras theorem, we get

$$(x)^2 + (x+5)^2 = (25)^2$$

Ans. 15 cm, 20 cm

19. **Hint** Perimeter of rectangle $= 2(l + b)$

and $(\text{diagonal})^2 = (\text{length})^2 + (\text{breadth})^2$

Let the length of rectangular plot be x m and breadth be y m.

Then, we have $\quad x + y = 34 \quad$ ...(i)

and $\quad (26)^2 = x^2 + y^2 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get the result.

Ans. 240 m^2

20. **Hint** Let the length of shortest side be x m.

According to the question,

$$(2x+1)^2 = x^2 + (x+7)^2$$

Ans. 8 m, 17 m and 15 m

21. **Hint** Let the number of children $= x$

According to question,

$$\frac{7500}{(x-20)} - \frac{7500}{x} = 100$$

Ans. Original number of children $= 50$

CHAPTER EXERCISE

a 3 Marks Questions

1. Solve the following quadratic equations by factorisation method.

$$(2x+1)(x+3)+3=0$$

2. Solve the following and calculate the answer correct to two decimal places.

$$x^2-5x-10=0$$ **[2013]**

3. Solve the quadratic equation $x^2-3x-9=0$ for x and give your answer correct to two decimal places. **[2007]**

4. Solve the quadratic equation $x^2-3(x+3)=0$. Give your answer correct to two significant figures. **[2016]**

5. Solve the equation $4x^2-5x-3=0$ and give your answer correct to two decimal places. **[2017]**

6. Solve the quadratic equation $2x^2+5\sqrt{3}x+6=0$ upto three decimal places. [take $\sqrt{3}=1.732$]

7. Solve the quadratic equation $5x(x+2)=3$ and give your answer correct to two decimal places. **[2008]**

8. Solve the equation $x-\dfrac{18}{x}=6$. Give your answer correct to two significant figures. **[2011]**

9. Find the value of 'k' for which $x=3$ is a solution of the quadratic equation $(k+2)x^2-kx+6=0$. Also, find the other root of the equation. **[2015]**

10. Solve $2x-3=\sqrt{2x^2-2x+21}$ by factorisation method.

11. Find the roots of equation

$$\sqrt{2x-3}+\sqrt{3x-5}-\sqrt{5x-6}=0.$$

12. Solve for x

$$\frac{2x}{x-3}+\frac{1}{2x+3}+\frac{3x+9}{(x-3)(2x+3)}=0,\ x\neq 3,-\frac{3}{2}.$$

13. Solve the following quadratic equation for x.

$$x^2+\frac{a}{a+b}x+\frac{a+b}{a}x+1=0$$

14. Solve the following quadratic equation for x.

$$9x^2-9(a+b)x+2a^2+5ab+2b^2=0$$

15. Solve $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x},\ a+b\neq 0.$

16. Find the nature of roots of each of the following equations.

(i) $x^2+2\sqrt{3}x-9=0$

(ii) $2x^2+8x+9=0$

(iii) $3x^2-2\sqrt{6}x+2=0$

17. Without solving the following quadratic equation, find the value of p for which the given equation has real and equal roots.

$$x^2+(p-3)x+p=0$$ **[2013]**

18. If the equation $(1+m^2)x^2+2mc\,x+(c^2-a^2)=0$ has equal roots, then prove that $c^2=a^2(1+m^2)$.

19. Find the values of k for which the equation $(3k+1)x^2+2(k+1)x+1=0$ has equal roots. Also, find the roots.

20. Find the values of k for which the quadratic equations $(k+4)x^2+(k+1)x+1=0$ has equal roots. Also, find the roots.

21. If -4 is a root of the equation $x^2+2x+4p=0$, then find the values of k for which the equation $x^2+p(1+3k)\,x+7\,(3+2k)=0$ has equal roots.

22. If -5 is a root of the quadratic equation $2x^2+px-15=0$ and the quadratic equation $p\,(x^2+x)+k=0$ has equal roots, then find the value of k.

23. If $x=-2$ is a root of the equation $3x^2+7x+p=0$, then find the value of k, so that the roots of the equation $x^2+k\,(4x+k-1)+p=0$ are equal.

24. If the product of two positive consecutive even integers is 288, then find the integers.

25. The product of two successive multiples of 4 is 28 more than the first multiple. Find them.

26. Divide 29 into two parts, such that the sum of the squares of the parts is 425.

27. The difference of squares of two numbers is 180. The square of the smaller number is 8 times the larger number. Find the two numbers.

28. Sum of two natural numbers is 8 and the difference of their reciprocals is $\dfrac{2}{15}$. Find the numbers. **[2015]**

29. A number consists of two digits, whose product is 18. When 27 is subtracted from the number, the digits interchange their places. Find the number.

30. A two-digit number is such that, the product of its digits is 12. When 36 is added to this number, the digits interchange their places. Find the number.

31. The denominator of a fraction is 1 more than its numerator. The sum of the fraction and its reciprocal is 5/2. Find the fraction.

32. Two years ago, a man's age was three times the square of his son's age. In three years time, his age will be four times his son's age. Find their present ages.

33. A train travels a distance of 300 km at a constant speed. If the speed of the train is increased by 10 km/h, the journey would have taken 1 h less. Find the original speed of the train.

34. A train covers a distance of 600 km at x km/h. Had the speed been $(x + 20)$ km/h, the time taken to cover the distance would have been reduced by 5 h. Write down an equation in x and solve it to evaluate x.

35. The speed of a boat in still water is 15 km/h. It can go 30 km upstream and return downstream to the original point in 4 h 30 min. Find the speed of the stream.

b 4 Marks Questions

36. A positive number is divided into two parts, such that the sum of the squares of the two parts is 208. The square of the larger part is 18 times the smaller part. Taking x as the smaller part of the two parts, find the number. *[2010]*

37. Three consecutive natural numbers are such that the square of the middle number exceeds the difference of the squares of the other two by 60. Assume the middle number to be x and form a quadratic equation satisfying the above statement. Hence, find the three numbers.

38. A takes 6 days less than the time taken by B to finish a work. If both can finish the work in 4 days, then find the time taken by B to finish the work.

39. Two pipes running together can fill a tank in $11\frac{1}{9}$ min. If one pipe takes 5 min more than the other to fill the tank separately, find the time in which each pipe would fill the tank separately.

40. Two pipes running together can fill a cistern in $2\frac{8}{11}$ min. If one pipe takes 1 min more than the other to fill the cistern, then find the time in which each pipe would fill the cistern.

41. A boat can cover 10 km up the stream and 5 km down the stream in 6 h. If the speed of the stream is 1.5 km/h, find the speed of the boat in still water.

42. The speed of a boat in still water is 11 km/h. It can go 12 km upstream and return downstream to the original point in 2 h 45 min. Find the speed of the stream.

43. By increasing the speed of a car by 10 km/h, the time of journey for a distance of 72 km is reduced by 36 min. Find the original speed of the car. *[2005]*

44. A scholarship amount of ₹ 75000 was distributed equally among a certain number of students. Had there been 10 students more, each would have got ₹ 250 less. Find the original number of students.

45. A shopkeeper buys a number of books of ₹ 80. If he had bought 4 more books for the same amount, each book would have cost ₹ 1 less. Taking the original number of books as x, form an equation in x and solve it.

46. A trader buys x articles for a total cost of ₹ 600.

(i) Write down the cost of one article in terms of x. If the cost per article were ₹ 5 more, the number of articles that can be bought for ₹ 600, would be four less.

(ii) Write down the equation in x for the above situation and solve it to find x.

47. A man on tour has ₹ 18000 for his daily expenses. If he extends his tour for 4 days, he has to cut down his daily expenses by ₹ 750. Find the original duration of the tour.

48. ₹ 6500 were divided equally among a certain number of persons. There had been 15 more persons, each would have got ₹ 30 less. Find the original number of persons.

49. In a class test, the sum of Shefali's marks in Mathematics and English is 30. Had she got 2 marks more in Mathematics and 3 marks less in English, the product of their marks would have been 210. Find her marks in the two subjects.

50. In an auditorium, seats are arranged in rows and columns. The number of rows was equal to the number of seats in each row. When the number of rows was doubled and the number of seats in each row was reduced by 10, the total number of seats increased by 300. Find

(i) the number of rows in the original arrangement.

(ii) the number of seats in the auditorium after re-arrangement. *[2003]*

51. The sum of the areas of two squares is 468 m. If the difference of their perimeters is 24 m, then find the sides of the two squares.

52. The length of the sides forming right angles of a right angled triangle are $5x$ cm and $(3x-1)$ cm. If the area of the triangle is $60\,\text{cm}^2$, then find its hypotenuse.

53. The perimeter of a rectangular field is 82 m and its area is 400 sq m. Find the length and breadth of the rectangle.

Hints and Answers

1. Hint $2x^2+7x+6=0 \Rightarrow 2x^2+(3+4)x+6=0$

Ans. $-2, \dfrac{-3}{2}$

2. Hint Use quadratic formula. **Ans.** 6.53, – 1.53

3. Hint Use quadratic formula. **Ans.** 4.85, – 1.85

4. Do same as Q. 3.

Ans. – 4.85, – 1.85

5. Hint Use quadratic formula.

Ans. –0.44, 1.69

6. Hint Use quadratic formula. **Ans.** – 0.866, – 3.464

7. Hint Use quadratic formula **Ans.** – 2.26, 0.26

8. Hint $\dfrac{x^2-18}{x}=6 \Rightarrow x^2-6x-18=0$

Ans. 8.20, – 2.20

9. Hint Put the value $x=3$ in $(k+2)x^2-kx+6=0$

Ans. $k=-4$, $x=-1$

10. Hint Squaring on both sides and solve.

Ans. 6

11. Hint $\sqrt{2x-3}+\sqrt{3x-5}=\sqrt{5x-6}$

On squaring both sides, we get

$$\sqrt{6x^2-19x+15}=1$$

Again, squaring both sides, we get

$$x=2 \text{ or } \frac{7}{6}.$$

The value $x=\dfrac{7}{6}$ is not possible as

$\sqrt{2x-3}=\sqrt{2\times\dfrac{7}{6}-3}=\sqrt{\dfrac{-2}{3}}$ at $x=\dfrac{7}{6}$, which is not defined.

Ans. 2

12. Hint Do same as Example 7 of Topic 1. **Ans.** –1

13. Hint $x\left(x+\dfrac{a}{a+b}\right)+\dfrac{a+b}{a}\left(x+\dfrac{a}{a+b}\right)=0$

$$\Rightarrow \quad \left(x+\frac{a}{a+b}\right)\left(x+\frac{a+b}{a}\right)=0$$

Ans. $\dfrac{-a}{a+b}, \dfrac{-(a+b)}{a}$

14. Hint Consider $2a^2+5ab+2b^2=(a+2b)(2a+b)$

Hence, the equation will become

$$9x^2-9(a+b)x+(a+2b)(2a+b)=0$$

$$\Rightarrow \quad [3x-(a+2b)][3x-(2a+b)]=0$$

Ans. $x=\dfrac{a+2b}{3}, \dfrac{2a+b}{3}$

15. Hint $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}$

$\Rightarrow \quad x^2+(a+b)x+ab=0 \qquad [\because (a+b)\neq 0]$

Ans. $-a, -b$

16. Do same as Example 2 of Topic 2.

Ans. (i) Real and distinct

(ii) Imaginary (iii) Real and equal

17. Do same as Example 5 of Topic 2. **Ans.** 1 or 9

18. Hint Since, the roots are equal.

$\therefore \quad D=b^2-4ac=0$

19. Do same as Example 8 of Topic 2.

Ans. $k=0, 1$ and $x=-1, -1$ or $\dfrac{-1}{2}, \dfrac{-1}{2}$

20. Hint We know that, for equal roots, $D=b^2-4ac=0$.

Ans. $k=5$ or -3; $x=1, 1$ or $\dfrac{-1}{3}, \dfrac{-1}{3}$

21. Do same as Example 9 of Topic 2.

Ans. $k = 2, \frac{-10}{9}$

22. Do same as Example 9 of Topic 2. **Ans.** $\frac{7}{4}$

23. Do same as Example 9 of Topic 2. **Ans.** $\frac{2}{3}, -1$

24. Hint Let two consecutive even positive integers be x and $x+2$.

According to the question, $x(x+2) = 288$

Ans. 16, 18

25. Hint Let two successive multiples of 4 be $4k$ and $(4k+4)$.

According to the question, $4k(4k+4) = 4k+28$

Ans. 4, 8

26. Hint Let two parts be x and $(29-x)$. Then, we have $x^2 + (29-x)^2 = 425$

Ans. 16, 13

27. Hint Let the larger number be x and smaller number be y.

Then, $x^2 - y^2 = 180$ ...(i)

and $y^2 = 8x$ ...(ii)

Ans. 18 and 12 or 18 and -12

28. Hint Let two natural numbers be x and y and also let $x < y$.

According to the question,

$$x + y = 8 \text{ and } \frac{1}{x} - \frac{1}{y} = \frac{2}{15}$$

Ans. 3 and 5

29. Hint Let ten's place digit be x and unit's place digit be y.

Then, $xy = 18$ ...(i)

and $(10x + y) - 27 = 10y + x$ ...(ii)

Ans. 63

30. Do same as Q. 29. **Ans.** 26

31. Hint Let the numerator of the fraction be x.

Then, denominator $= x+1$

According to the question, $\frac{x}{x+1} + \frac{x+1}{x} = \frac{5}{2}$ **Ans.** $\frac{1}{2}$

32. Do same as Example 7 of Topic 3.

Ans. Father's age = 29 yr, Son's age = 5 yr

33. Hint Let the original speed be x km/h.

Then, $\frac{300}{x} - \frac{300}{x+10} = 1$

Ans. 50 km/h

34. Do same as Q. 33. **Ans.** $x = 40$ km/h

35. Do same as Example 13 of Topic 3.

Ans. 5 km/h

36. Hint Let x and y be two numbers, in which x is smaller number.

Now, according to the question,

$$x^2 + y^2 = 208 \text{ and } y^2 = 18x$$

Ans. 20

37. Hint Let the middle number of the three consecutive numbers be x, therefore the other two numbers are $(x-1)$ and $(x+1)$.

Ans. 9, 10 and 11

38. Hint Let the time taken by $B = x$ days

Then, the time taken by $A = (x-6)$ days

According to the question,

$$\left(\frac{1}{x} + \frac{1}{x-6}\right) = \frac{1}{4}$$

Ans. 12 days

39. Hint Let time taken by pipe A be x min and time taken by pipe B be $(x+5)$ min.

In one minute, pipe A will fill $\frac{1}{x}$ tank.

In one minute, pipe B will fill $\frac{1}{x+5}$ tank.

According to the question, $\frac{1}{x} + \frac{1}{x+5} = \frac{9}{100}$

Ans. Pipe $A = 20$ min and pipe $B = 25$ min

40. Do same as Q. 39.

Ans. 5 min, 6 min

41. Hint Let speed of the boat in still water $= x$ km/h

Then, speed of the boat upstream $= (x - 1.5)$ km/h

and speed of the boat downstream $= (x + 1.5)$ km/h

Now, according to the question,

$$\frac{10}{x-1.5} + \frac{5}{x+1.5} = 6$$

Ans. 3.5 km/h

42. Hint Let speed of the stream be x km/h.

Relative speed of the boat when going upstream

$= (11 - x)$ km/h

Relative speed of the boat when going downstream

$= (11 + x)$ km/h

According to the question,

$$\frac{12}{11-x} + \frac{12}{11+x} = 2\frac{45}{60}$$

Ans. 5 km/h

43. Hint Let the original speed of car $= x$ km/h

According to the question,

$$\frac{72}{x} - \frac{72}{x+10} = \frac{36}{60}$$

Ans. 30 km/h

44. Hint Let the number of students be x.

According to the question,

$$\frac{75000}{x+10} = \frac{75000}{x} - 250 \quad \textbf{Ans. } 50$$

45. Hint Do same as Example 19 of Topic 3.

Ans. 16

46. Hint $\because$ Cost of x articles = ₹600

$\therefore$ Cost of 1 article = ₹$\frac{600}{x}$

Ans. (i) ₹$\frac{600}{x}$ (ii) $\frac{600}{x-4} - \frac{600}{x} = 5$; $x =$ ₹24

47. Hint Let the tour be of x days.

According to the question,

$$\frac{18000}{x} - \frac{18000}{x+4} = 750 \quad \textbf{Ans. } 8 \text{ days}$$

48. Hint Let the original number of persons be x.

Then, share of each person $= \frac{6500}{x}$

When the number of persons is increased by 15.

Then, new share $= \frac{6500}{x+15}$

Now, according to the question, we get

$$\frac{6500}{x} - \frac{6500}{x+15} = 30$$

Ans. 50

49. Hint Let Shefali's marks in Mathematics be x.

Then, Shefali's marks in English $= (30 - x)$.

According to the question,

(Marks in Mathematics + 2) $\times$ (Marks in English $-$ 3) $= 210$

Ans. When Shefali's marks in Mathematics are 12 and 13 respectively, then her marks in English are 18 and 17, respectively.

50. Hint Let the number of rows in the original arrangement be x.

Number of seats in the auditorium $= x^2$

In new arrangement, number of rows $= 2x$

Number of seats in each row $= x - 10$

$\therefore$ Number of seats after new arrangement $= 2x(x - 10)$

Ans. (i) 30 (ii) 1200

51. Hint Let the sides of square be x and y.

Then, according to the question, $x^2 + y^2 = 468$

and $4(x - y) = 24 \Rightarrow x - y = 6$

Ans. 18 m, 12 m

52. Hint Use Pythagoras theorem and use the formula,

Area of triangle $= \frac{1}{2} \times$ Base $\times$ Height

Ans. 17 cm

53. Hint Let the length and breadth of the rectangle be x and y, respectively.

According to the question,

$2(x + y) = 82$

$\Rightarrow \quad x + y = 41$

and $\quad xy = 400$

Ans. 25, 16

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1 Solve $x^2+7x=7$ and give your answer correct to two decimal places.

2 Find the value of k for which the following equation has equal roots.

$$x^2+4kx+(k^2-k+2)=0$$

3 ₹7500 were divided equally among a certain number of children. Had there been 20 less children, each would have received ₹100 more. Find the original number of children.

2017

4 Solve the equation $4x^2-5x-3=0$ and give your answer correct to two decimal places.

5 The sum of the ages of Vivek and his younger brother Amit is 47 yr. The product of their ages in years is 550. Find their ages.

2016

6 Solve the quadratic equation $x^2-3(x+3)=0$. Give your answer correct to two significant figures.

7 A bus covers a distance of 240 km at a uniform speed. Due to heavy rain, its speed gets reduced by 10 km/h and as such it takes two hours longer to cover the total distance. Assuming the uniform speed to be x km/h, form an equation and solve it to evaluate x.

2015

8 Find the value of k for which $x=3$ is a solution of the quadratic equation $(k+2)x^2-kx+6=0$. Also, find the other root of the equation.

9 The sum of two natural numbers is 8 and the difference of their reciprocals is 2/15. Find the numbers.

2014

10 Solve $(x-1)^2-3x+4=0$ for x using the quadratic formula. Write your answer correct to two significant figures.

11 A two-digit positive number is such that the product of its digits is 6. If 9 is added to the number, then the digits interchange their places. Find the number.

2013

12 Solve the following and calculate the answer correct to two decimal places.

$$x^2-5x-10=0$$

13 A shopkeeper purchases a certain number of books for ₹ 960. If the cost per book was ₹ 8 less the number of books that could be purchased for ₹ 960, would be 4 more. Write an equation, taking the original cost of each book to be ₹ x and solve it to find the original cost of the books.

14 Without solving the following quadratic equation, find the value of p for which the given equation has real and equal roots.

$$x^2+(p-3)x+p=0$$

2012

15 Solve the following equation and give your answer correct to three significant figures.

$$5x^2-3x-4=0$$

16 Without solving the following quadratic equation, find the value of m for which the given equation has real and equal roots.

$$x^2+2(m-1)x+(m+5)=0$$

17 A car covers a distance of 400 km at a certain speed. Had the speed been 12 km/h more, the time taken for the journey would have been 1 h 40 min less. Find the original speed of the car.

2011

18 ₹ 480 is divided equally among x children. If the number of children was 20 more than each would have got ₹ 12 less. Find the value of x.

19 Solve the equation $x-\frac{18}{x}=6$. Give your answer correct to two significant figures.

*All these questions are completely covered in chapter either as solved examples or in topic/chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. The linear factors of the equation $x^2 + kx + 1 = 0$ exists, if

(a) $k \geq 2$ (b) $k \leq -2$ (c) Both (a) and (b) (d) $k = 2$

2. The root of the equation $2\sin\theta + \dfrac{3}{\sin\theta} + 7 = 0$ is

(a) $\dfrac{1}{2}$ (b) $-\dfrac{1}{2}$ (c) $\dfrac{1}{3}$ (d) $\dfrac{1}{7}$

3. The equation $x^2(a^2 + b^2) + 2x(ac + bd) + (c^2 + d^2) = 0$ has no real roots, if

(a) $ad \neq bc$ (b) $ac \neq bd$ (c) $ac \neq 2bd$ (d) None of these

4. A factory kept increasing its output by the same percentage every year. Then, the percentage, if it is known that the output is doubled in the last two years, will be

(a) 44.1% (b) 14.4% (c) 44.4% (d) 41.4%

5. A trader bought a number of articles for ₹ 1200. Ten were damaged and he sold each of the rest at ₹ 2 more than what he paid for it, thus cleaning a profit of ₹ 60 on the whole transaction. If x denotes the number of articles he bought, then the value of x is

(a) 100 (b) 60 (c) 80 (d) 110

6. If the roots of the equation $x^2 + 2cx + ab = 0$ are real and unequal, then the equation $x^2 - 2(a + b)x + a^2 + b^2 + 2c^2 = 0$ has

(a) real root (b) no real roots (c) equal root (d) real and equal

7. The values of x satisfying $p^2x^2 + (p^2 - q^2)x - q^2 = 0$ are

(a) $\dfrac{q^2}{p^2}, -1$ (b) $\dfrac{p^2}{q^2}, 1$ (c) $\dfrac{q^2}{p^2}, +1$ (d) $1, \dfrac{-q}{p}$

8. From a group of Saras birds, one-fourth of the number are moving about in lotus plants, one-ninth coupled with one-fourth as well as 7 times the square root of the total number are moving on a hill, while 56 birds are sitting in the Bakula trees. Then, what is the total number of birds?

(a) 629 (b) 567 (c) 675 (d) 576

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 396.

CHAPTER

6

Ratio and Proportion

In general, ratio is a relationship between two number (or similar quantities) indicating how many times the first number (or quantity) is of the second. The equality of two ratios is called proportion. In this chapter, we will study properties, combinations and applications of ratio and proportion.

Chapter Objectives

- Ratio and Its Properties
- Proportion and Its Properties

Topic 1
Ratio and Its Properties

Ratio

A ratio is a comparison of two or more quantities of the same kind taken in the same unit.

If x and y are two quantities of same kind taken in the same unit, then fraction x/y is known as ratio of x and y. It is written as $x : y$ and read as 'x is to y'.

In the expression $x : y$, the quantities x and y are called **terms of ratio**, where x is called the **first term** (or antecedent) and y is called the **second term** (or consequent).

Ratio is always expressed in numbers, that are coprimes. It can only be calculated, if the quantities involved are in the same unit.

Properties of Ratio

1. The value of a ratio (say $x : y$) remains unchanged, if both the antecedent (x) and consequent (y) are multiplied by the same non–zero number (say a), i.e. $x : y$ is the same as $ax : ay$. $\left[\because \frac{x}{y} = \frac{ax}{ay}\right]$
2. The value of a ratio (say $x : y$) remains unchanged, if both the antecedent (x) and consequent (y) are divided by the same non–zero number (say a), i.e. $x : y$ is the same as $\frac{x}{a} : \frac{y}{a}$. $\left[\because \frac{x}{y} = \frac{x \div a}{y \div a}\right]$

Important Facts about Ratio

(i) The ratio between two or more unlike (or different) quantities does not exist.

(ii) Ratio is taken only between positive quantities.

(iii) The order of the terms in a ratio is important.

(iv) The ratio is always expressed in the lowest or simplest form.

(v) A ratio is a number, so it has no units.

(vi) Ratio $x:y$ is not equal to ratio $y:x$, i.e. $x:y \neq y:x$.

(vii) If the terms of the given ratio are in fractions, then convert the terms of the ratio in the whole numbers by multiplying each term by the LCM of their denominators.

e.g. $\frac{2}{5}:\frac{3}{4}=\frac{2}{5}\times 20:\frac{3}{4}\times 20 = 2\times 4:3\times 5 = 8:15$

Example 1. Find the ratio between 7 months and 2 years 4 months.

Sol. Clearly, 2 years 4 months $= 2\times 12$ months + 4 months [$\because$ 1 year = 12 months]

= 24 months + 4 months = 28 months

Ratio between 7 months and 2 years 4 months

= Ratio between 7 months and 28 months

$=\frac{7}{28}$

Dividing both antecedent and consequent by 7, we get

Required ratio $=\frac{1}{4}$ or 1 : 4

Example 2. Find the ratio of 50 min to $2\frac{1}{2}$ h.

Sol. Clearly, $2\frac{1}{2}\text{h}=\frac{5}{2}\text{h}=\frac{5}{2}\times 60\text{min}$ [$\because$ 1 h = 60 min]

= 150 min

$\therefore$ Ratio of 50 min to $2\frac{1}{2}$ h = Ratio of 50 min to 150 min

$=\frac{50}{150}=\frac{1}{3}$

[dividing numerator and denominator by 50]

$\therefore$ Required ratio = 1 : 3

Increase (or Decrease) a Quantity in Given Ratio

If a quantity increases (or decreases) in the ratio $x:y$.

Then, new quantity $=\frac{y}{x}\times$ Original quantity.

$\therefore$ Increase in quantity = Increased value − Original value

or Decrease in quantity = Original value − Decreased value

Example 3. When the fare of a certain journey by a train increase from ₹ 15 to ₹ 21 and the fare of original journey was ₹ 1015, then find the increase in the fare.

Sol. Given, original fare = ₹1015

and fare increases in the ratio 15 : 21 = 5 : 7

[dividing antecedent and consequent by 3]

$\therefore$ New fare of train $=\frac{7}{5}\times$ Original fare

[$\because$ new quantity $=\frac{y}{x}\times$ original quantity]

$=\frac{7}{5}\times 1015$ [$\because$ original fare = ₹ 1015]

$= 7\times 203 =$ ₹ 1421

Increase in value = Increased value − Original value

$\therefore$ Increase in the fare = New fare − Original fare

= 1421 − 1015 = ₹ 406

Comparison of Ratios

1. When Two Ratios are Given

Let $x:y$ and $z:w$ be two ratios. Then,

(i) $(x:y)>(z:w)$, i.e. $\frac{x}{y}>\frac{z}{w}$ if $xw>yz$

(ii) $(x:y)=(z:w)$, i.e. $\frac{x}{y}=\frac{z}{w}$ if $xw=yz$

(iii) $(x:y)<(z:w)$, i.e. $\frac{x}{y}<\frac{z}{w}$ if $xw<yz$

Example 4. Compare 4/5 and 7/6. Also, write the relation between them.

Sol. We have, $\frac{4}{5}$ and $\frac{7}{6}$

Consider 4×6 and 7×5 $\Rightarrow$ 24 and 35

Since, 24 < 35

$\therefore \frac{4}{5}<\frac{7}{6}$

2. When Three or More than Three Ratios are Given

In order to compare three or more than three ratios, we convert them into equivalent fractions with same denominator and then compare them.

Example 5. Compare the following ratios and write them in ascending order $\frac{4}{5},\frac{3}{2},\frac{1}{10}$.

Sol. We have, $\frac{4}{5},\frac{3}{2}$ and $\frac{1}{10}$, which can be written as

$\frac{4\times 2}{5\times 2},\frac{3\times 5}{2\times 5}$ and $\frac{1\times 1}{10\times 1}$ [$\because$ LCM of 5, 2 and 10 = 10]

$\Rightarrow \frac{8}{10},\frac{15}{10}$ and $\frac{1}{10}$

Since, $1<8<15$

$\therefore \frac{1}{10}<\frac{8}{10}<\frac{15}{10}$ [divide each term by 10]

$\Rightarrow \frac{1}{10}<\frac{4}{5}<\frac{3}{2}$

Hence, the given ratios in ascending order of their magnitude are $\frac{1}{10},\frac{4}{5}$ and $\frac{3}{2}$.

Example 6. Write the following ratios in descending order.

$$\frac{1}{6}, \frac{2}{3} \text{ and } \frac{7}{12}$$

Sol. Given ratios are $\frac{1}{6}, \frac{2}{3}$ and $\frac{7}{12}$, which can be written as

$$\frac{1\times 2}{6\times 2}, \frac{2\times 4}{3\times 4} \text{ and } \frac{7\times 1}{12\times 1} \quad [\because \text{LCM of 6, 3 and } 12 = 12]$$

$$\Rightarrow \quad \frac{2}{12}, \frac{8}{12} \text{ and } \frac{7}{12}$$

Since, $\quad 8 > 7 > 2$

$$\therefore \quad \frac{8}{12} > \frac{7}{12} > \frac{2}{12} \quad [\text{dividing each term by } 12]$$

$$\Rightarrow \quad \frac{2}{3} > \frac{7}{12} > \frac{1}{6}$$

Hence, the given ratios in descending order of their magnitude are $\frac{2}{3}, \frac{7}{12}$ and $\frac{1}{6}$.

Composition of Ratios

1. **Compounded ratio** When two or more ratios are multiplied with each other, then they are called compounded ratio.
   (i) Compounded ratio of $x : y$ and $z : w$ is $xz : yw$.
   (ii) Compounded ratio of $x : y$, $z : w$ and $s : t$ is $xzs : ywt$.
2. **Duplicate ratio** When ratio is multiplied with itself, then resulting ratio is called duplicate ratio of the given ratio.
   i.e. duplicate ratio of $x : y$ is $x^2 : y^2$.
3. **Triplicate ratio** When ratio is multiplied twice with itself, then the resulting ratio is called triplicate ratio of the given ratio.
   i.e. triplicate ratio of $x : y$ is $x^3 : y^3$.
4. **Sub-duplicate ratio** The square root of the ratio is called sub-duplicate ratio.
   i.e. sub-duplicate ratio of $x : y$ is $\sqrt{x} : \sqrt{y}$.
5. **Sub-triplicate ratio** The cube root of the ratio is called sub-triplicate ratio.
   i.e. sub-triplicate ratio of $x : y$ is $\sqrt[3]{x} : \sqrt[3]{y}$.
6. **Reciprocal ratio** The reciprocal ratio of $x : y$ is $y : x$.

Example 7. Find the compounded ratio of

$$3x : 2y,\ 4a : 5b \text{ and } z : t.$$

Sol. Required compounded ratio

$$= (3x \times 4a \times z) : (2y \times 5b \times t)$$
$$= 12xaz : 10ybt$$
$$= 6xaz : 5ybt$$

[dividing antecedent and consequent by 2]

Example 8. Find the following.

(i) The duplicate ratio of 2 : 5.
(ii) The triplicate ratio of 3 : 7.
(iii) The sub-duplicate ratio of 121 : 169.
(iv) The sub-triplicate ratio of 64 : 27.
(v) The reciprocal ratio of 13 : 17.

Sol.
(i) Duplicate ratio of 2 : 5 is $(2)^2 : (5)^2 = 4 : 25$
(ii) Triplicate ratio of 3 : 7 is $(3)^3 : (7)^3 = 27 : 343$
(iii) Sub-duplicate ratio of 121 : 169 is $\sqrt{121} : \sqrt{169} = 11 : 13$
(iv) Sub-triplicate ratio of 64 : 27 is $\sqrt[3]{64} : \sqrt[3]{27} = 4 : 3$
(v) Reciprocal ratio of 13 : 17 is 17 : 13.

Topic Exercise 1

1. Find the ratio between 7.8 cm and 8 km.
2. Find the ratio of 65 min to $1\frac{1}{5}$ h.
3. An alloy consists of $27\frac{1}{2}$ kg of copper and $2\frac{3}{4}$ kg of tin. Find the ratio by weight of tin to the alloy.
4. An alloy consists of $40\frac{1}{2}$ g of gold and $5\frac{1}{4}$ g of silver. Find the ratio by weight of the gold to the alloy.
5. The fare of a certain journey by an airline was increased in the ratio 5 : 7. Find the increase in the fare, if the original fare is ₹ 1050.
6. The current price of a box is ₹ 300. Find the original price of an article, if the price decreases from 15 to 9.
7. Compare the ratios $\frac{1}{7}, \frac{3}{4}$ and $\frac{9}{14}$ and write them in ascending order.
8. Arrange the ratios 3 : 4, 7 : 8, 10 : 12, 15 : 8 in ascending order of magnitude.
9. Compare the ratios $\frac{3}{8}, \frac{5}{24}$ and $\frac{7}{3}$ and write them in descending order.
10. Find the following.
    (i) The compounded ratio of $2a : 3b$, $mn : x^2$ and $x : n$.
    (ii) The duplicate ratio of $3\sqrt{3} : 2\sqrt{5}$.
    (iii) The triplicate ratio of $\frac{m}{2} : \frac{n}{3}$.

11. Find the following.

(i) The sub-duplicate ratio of $9p^2q^2 : 16r^2s^4$.

(ii) The sub-triplicate ratio of $125a^3 : 1000b^6$.

(iii) The reciprocal ratio of $13xyz : 14abc$.

12. Find the following.

(i) The sub-duplicate ratio of $9 : 16$.

(ii) The sub-triplicate ratio of $216 : 512$.

(iii) The reciprocal ratio of $3x : \frac{1}{y}$.

13. If $(x-9) : (3x+6)$ is the duplicate ratio of $4 : 9$, then find the value of x. *[2014]*

14. Find the following.

(i) The compounded ratio of $(a-b):(a+b)$, $(a+b)^2:(a^2+b^2)$ and $(a^4-b^4):(a^2-b^2)^2$.

(ii) The duplicate ratio of $\sqrt{5}:7$.

(iii) The triplicate ratio of $\frac{1}{2}:\frac{1}{3}$.

15. Divide ₹ 1870 into three parts in such a way that half of the first part, one-third of the second part and one-sixth of the third part are all equal.

16. A labourer earns ₹ 9000 per month and spends ₹ 6500 per month on his family and the rest he saves for the future needs. Find the ratio of his

(i) income to his expenditure.

(ii) income to his savings.

(iii) savings to his expenditure.

Hints and Answers

1. Hint $8 \text{ km} = 8 \times 1000 \text{ m}$, $8000 \text{ m} = 8000 \times 100 \text{ cm}$

$\therefore$ Required ratio $= \frac{7.8}{800000}$

Ans. 39 : 4000000

2. Do same as Example 2. **Ans.** 65 : 72

3. Hint Weight of alloy $= 27\frac{1}{2} + 2\frac{3}{4} = \frac{55}{2} + \frac{11}{4} = \frac{121}{4}$ kg

$\therefore$ Ratio by weight of the tin to the alloy

$$= 2\frac{3}{4} : \frac{121}{4} = \frac{11}{4} : \frac{121}{4}$$

Ans. 1 : 11

4. Do same as Q. 3.
Ans. 54 : 61

5. Do same as Example 3.
Ans. Increase in the fare is ₹ 420.

6. Hint Current price $= \frac{3}{5} \times$ original price

Ans. ₹ 500

7. Do same as Example 5. **Ans.** $\frac{1}{7}, \frac{9}{14}, \frac{3}{4}$

8. Do same as Example 5.
Ans. $\frac{3}{4}, \frac{10}{12}, \frac{7}{8}, \frac{15}{8}$

9. Do same as Example 6. **Ans.** $\frac{7}{3}, \frac{3}{8}, \frac{5}{24}$

10. (i) Do same as Example 7. **Ans.** $2am : 3bx$

(ii) Do same as Example 8 (i). **Ans.** 27 : 20

(iii) Do same as Example 8 (ii). **Ans.** $27m^3 : 8n^3$

11. (i) Do same as Example 8 (iii). **Ans.** $3pq : 4rs^2$

(ii) Do same as Example 8 (iv). **Ans.** $5a : 10b^2$

(iii) Do same as Example 8 (v). **Ans.** $14abc : 13xyz$

12. (i) Do same as Example 8 (iii). **Ans.** 3 : 4

(ii) Do same as Example 8 (iv). **Ans.** 6 : 8

(iii) Do same as Example 8 (v). **Ans.** $1 : 3xy$

13. Hint We know that, the duplicate ratio of $a : b$ is $a^2 : b^2$.

$$\therefore \quad \frac{x-9}{3x+6} = \left(\frac{4}{9}\right)^2$$

Ans. $x = 25$

14. (i) Do same as Example 7. **Ans.** 1 : 1

(ii) Do same as Example 8 (i). **Ans.** 5 : 49

(iii) Do same as Example 8 (ii). **Ans.** 27 : 8

15. Hint $\frac{1}{2}$(1st part) $= \frac{1}{3}$(2nd part) $= \frac{1}{6}$(3rd part) $= y$ (let)

$\Rightarrow$ 1st part $= 2y$, 2nd part $= 3y$ and 3rd part $= 6y$

According to the question, $2y + 3y + 6y = 1870$

Ans. ₹340, ₹510 and ₹1020

16. Hint Labourer's monthly income = ₹ 9000
and his monthly expenditure = ₹ 6500

$\therefore$ His monthly savings $= 9000 - 6500 =$ ₹ 2500

Ans. (i) 18 : 13 (ii) 18 : 5 (iii) 5 : 13

Topic 2

Proportion and Its Properties

Proportion

An equality of two ratios is called the **proportion**. The four non-zero quantities x, y, z and t of the same kind taken in same unit are said to be in proportion if and only if the product of **extremes** (i.e. first and fourth terms) is equal to the product of **means** (i.e. second and third terms).

Here, x and t are the extreme terms and y and z are the mean terms.

$\therefore$ x, y, z and t are in proportion, if $xt = yz$.

We can write $x : y = z : t$ as $x : y :: z : t$ and read as 'x is to y as z is to t'.

Example 1. Find the value of z, if $7 : z :: 5 : 20$.

Sol. We have, $7 : z :: 5 : 20$

$\Rightarrow \quad 5z = 7 \times 20$ [$\because$ product of means = product of extremes]

$\Rightarrow \quad z = \dfrac{7 \times 20}{5} = 28$

Hence, the required value of z is 28.

Example 2. What number must be added to each of the numbers 7, 16, 43 and 79 to make them in proportion?

Sol. Let the required number be x.

According to the question,

$(7 + x) : (16 + x) :: (43 + x) : (79 + x)$

$\Rightarrow \quad \dfrac{7 + x}{16 + x} = \dfrac{43 + x}{79 + x}$

$\Rightarrow \quad (7 + x)(79 + x) = (16 + x)(43 + x)$ [by cross-multiplication]

$\Rightarrow \quad 553 + 7x + 79x + x^2 = 688 + 16x + 43x + x^2$

$553 + 86x = 688 + 59x$

$\Rightarrow \quad 86x - 59x = 688 - 553$

$\Rightarrow \quad 27x = 135$

$\therefore \quad x = \dfrac{135}{27} = 5$

Hence, the required number is 5.

Example 3. If $(x^2 + z^2)(y^2 + t^2) = (xy + zt)^2$, then prove that x, y, z and t are in proportion.

Sol. We have, $(x^2 + z^2)(y^2 + t^2) = (xy + zt)^2$

$\Rightarrow x^2y^2 + x^2t^2 + z^2y^2 + z^2t^2 = x^2y^2 + z^2t^2 + 2xyzt$ [$\because (a + b)^2 = a^2 + b^2 + 2ab$]

$\Rightarrow \quad x^2t^2 + z^2y^2 = 2xyzt$

$\Rightarrow x^2t^2 + z^2y^2 - 2xyzt = 0$

$\Rightarrow \quad (xt - zy)^2 = 0$ [$\because a^2 + b^2 - 2ab = (a - b)^2$]

$\Rightarrow \quad xt - zy = 0$ [$\because a^2 = 0 \Leftrightarrow a = 0$]

$\Rightarrow \quad xt = zy$

$\Rightarrow \quad \dfrac{xt}{yt} = \dfrac{zy}{yt}$ [dividing both sides by yt]

$\Rightarrow \quad \dfrac{x}{y} = \dfrac{z}{t}$

which shows that x, y, z and t are in proportion.

Example 4. If a, b, c and d are in proportion, then prove that $(ma + nb) : b = (mc + nd) : d$.

Sol. Given, a, b, c and d are in proportion.

$\therefore \quad \dfrac{a}{b} = \dfrac{c}{d}$

On multiplying both sides by m in numerator and by n in denominator, we get

$$\frac{ma}{nb} = \frac{mc}{nd}$$

On adding 1 both sides, we get

$$\frac{ma}{nb} + 1 = \frac{mc}{nd} + 1$$

$\Rightarrow \quad \dfrac{ma + nb}{nb} = \dfrac{mc + nd}{nd}$

$\Rightarrow \quad \dfrac{ma + nb}{b} = \dfrac{mc + nd}{d}$ [multiplying both sides by n]

$\Rightarrow \quad (ma + nb) : b = (mc + nd) : d$ **Hence proved.**

Continued Proportion

Three non-zero quantities x, y and z of same kind taken in same unit are said to be in continued proportion, if and only if the ratio of first term (i.e. x) and second term (i.e. y) is equal to the ratio of second term (i.e. y) and third term (i.e. z).

Thus, if x, y and z are in continued proportion, then

$$x : y :: y : z, \text{ i.e. } \frac{x}{y} = \frac{y}{z} \Rightarrow y^2 = xz$$

In general, the non-zero quantities x, y, z, t, s, ..., etc., of same kind taken in same unit are said to be in continued proportion if and only if $\dfrac{x}{y} = \dfrac{y}{z} = \dfrac{z}{t} = \dfrac{t}{s} = \ldots$.

First Proportional

If x, y and z are in continued proportion, then x is called first proportional.

Mean Proportional

If x, y and z are in continued proportion, then y is called the mean proportional.

i.e. $x : y :: y : z \Rightarrow \dfrac{x}{y} = \dfrac{y}{z} \Rightarrow y^2 = xz \Rightarrow y = \sqrt{xz}$

Thus, the mean proportion between two numbers is the positive square root of their product.

Third Proportional

If x, y and z are in continued proportion, then z is called third proportional.

Fourth Proportional

If x, y, z and t are in proportion, then t is called the fourth proportional.

Example 5. If a, 4, 12 and b are in continued proportion, then find the values of a and b.

Sol. Given, a, 4, 12 and b are in continued proportion.

$$\therefore \quad \frac{a}{4} = \frac{4}{12} = \frac{12}{b} \Rightarrow \frac{a}{4} = \frac{4}{12} \text{ and } \frac{4}{12} = \frac{12}{b}$$

$$\Rightarrow \quad a = \frac{16}{12} \text{ and } 4b = 12 \times 12$$

$$\therefore \quad a = \frac{4}{3} \text{ and } b = 36$$

Hence, the required values of a and b are 4/3 and 36, respectively.

Example 6. What is the mean proportional between $10 + 2\sqrt{3}$ and $15 - 3\sqrt{3}$?

Sol. Let the mean proportional between $(10 + 2\sqrt{3})$ and $(15 - 3\sqrt{3})$ be x.

Then, $(10 + 2\sqrt{3}) : x :: x : (15 - 3\sqrt{3})$

$$\Rightarrow \quad \frac{10 + 2\sqrt{3}}{x} = \frac{x}{(15 - 3\sqrt{3})}$$

$$\Rightarrow \quad x^2 = (10 + 2\sqrt{3})(15 - 3\sqrt{3})$$

$$\Rightarrow \quad x^2 = 2(5 + \sqrt{3}) \times 3(5 - \sqrt{3}) = 6(5 + \sqrt{3})(5 - \sqrt{3})$$

$$= 6[(5)^2 - (\sqrt{3})^2] \quad [\because (a - b)(a + b) = a^2 - b^2]$$

$$= 6(25 - 3) = 132$$

$$\Rightarrow \quad x = \sqrt{132} \quad \text{[taking positive square root]}$$

$$\therefore \quad x = 2\sqrt{33}$$

Hence, the required mean proportional is $2\sqrt{33}$.

Example 7. Find the third proportional of 9 and 21.

Sol. Let the third proportional of 9 and 21 be x.

Then, $\quad 9 : 21 :: 21 : x \Rightarrow \frac{9}{21} = \frac{21}{x}$

$$\Rightarrow \quad 9x = 21 \times 21 \Rightarrow x = \frac{21 \times 21}{9} = 49$$

Hence, the required third proportional is 49.

Example 8. Find the fourth proportional to 9, 16 and 63.

Sol. Let the fourth proportional be x.

Then, $9 : 16 :: 63 : x$

$$\Rightarrow \quad \frac{9}{16} = \frac{63}{x}$$

$$\Rightarrow \quad 9x = 63 \times 16$$

$$\therefore \quad x = \frac{63 \times 16}{9} = 112$$

Hence, the required fourth proportional is 112.

Example 9. Find the mean proportional between $(a^3 - a^2b)$ and $(a - b)$.

Sol. Let x be the mean proportional between $(a^3 - a^2b)$ and $(a - b)$.

Then, $(a^3 - a^2b) : x :: x : (a - b) \Rightarrow \frac{(a^3 - a^2b)}{x} = \frac{x}{(a - b)}$

$$\Rightarrow \quad x^2 = (a^3 - a^2b)(a - b) = a^2(a - b)(a - b)$$

$$\Rightarrow \quad x^2 = a^2(a - b)^2$$

Taking positive square root on both sides, we get

$$x = \sqrt{a^2(a - b)^2}$$

$$\therefore \quad x = a(a - b)$$

Hence, the required mean proportional is $a(a - b)$.

Example 10. If x, y and z are in continued proportion, then prove that $\frac{(x + y)^2}{(y + z)^2} = \frac{x}{z}$. **[2010]**

Sol. Since x, y and z are in continued proportion, therefore

$$\frac{x}{y} = \frac{y}{z} \Rightarrow y^2 = xz$$

To prove, $\frac{(x + y)^2}{(y + z)^2} = \frac{x}{z}$

$$\text{LHS} = \frac{(x + y)^2}{(y + z)^2} = \frac{x^2 + y^2 + 2xy}{y^2 + z^2 + 2yz} = \frac{x^2 + xz + 2xy}{xz + z^2 + 2yz} \quad [\because y^2 = xz]$$

$$= \frac{x(x + z + 2y)}{z(x + z + 2y)} = \frac{x}{z} = \text{RHS}$$

Hence proved.

Example 11. If a, b and c are in continued proportion, prove that $(a + b + c)(a - b + c) = a^2 + b^2 + c^2$. **[2015]**

Sol. If a, b and c are in continued proportion, then $\frac{a}{b} = \frac{b}{c}$

or $\quad ac = b^2$

Now, $\text{LHS} = (a + b + c)(a - b + c)$

$$= a^2 + ab + ac - ab - b^2 - bc + ac + bc + c^2$$

$$= a^2 - b^2 + c^2 + 2ac = a^2 - b^2 + c^2 + 2b^2 \quad [\because ac = b^2]$$

$$= a^2 + b^2 + c^2 = \text{RHS}$$

Hence proved.

Example 12. If b is the mean proportion between a and c, then show that $\frac{a^4 + a^2b^2 + b^4}{b^4 + b^2c^2 + c^4} = \frac{a^2}{c^2}$. **[2017]**

Sol. Given, b is the mean proportional between a and c.

$$\therefore \quad \frac{a}{b} = \frac{b}{c} \Rightarrow b^2 = ac$$

Now, $\text{LHS} = \frac{a^4 + a^2b^2 + b^4}{b^4 + b^2c^2 + c^4} = \frac{a^4 + a^2(ac) + (ac)^2}{(ac)^2 + (ac)c^2 + c^4} \quad [\because b^2 = ac]$

$$= \frac{a^4 + a^3c + a^2c^2}{a^2c^2 + ac^3 + c^4}$$

$$= \frac{a^2(a^2 + ac + c^2)}{c^2(a^2 + ac + c^2)} = \frac{a^2}{c^2} = \text{RHS}$$

Hence proved.

Example 13. If $x \neq y$ and $x : y$ is the duplicate ratio of $(x + z)$ and $(y + z)$, then prove that z is the mean proportional between x and y.

Sol. We have, $x : y =$ Duplicate ratio of $(x + z)$ and $(y + z)$

$$\therefore \quad \frac{x}{y} = \frac{(x+z)^2}{(y+z)^2} \Rightarrow x(y+z)^2 = y(x+z)^2$$

$$\Rightarrow \quad x(y^2 + z^2 + 2yz) = y(x^2 + z^2 + 2xz) \quad [\because (a+b)^2 = a^2 + b^2 + 2ab]$$

$$\Rightarrow \quad xy^2 + xz^2 + 2xyz = x^2y + z^2y + 2xyz$$

$$\Rightarrow \quad xy^2 + xz^2 = x^2y + z^2y \Rightarrow xy^2 - x^2y + xz^2 - z^2y = 0$$

$$\Rightarrow \quad xy(y - x) - z^2(y - x) = 0 \Rightarrow (y - x)(xy - z^2) = 0$$

$$\Rightarrow \quad xy - z^2 = 0 \text{ or } y - x = 0 \Rightarrow z^2 = xy \quad [\because y \neq x]$$

$$\therefore \quad z = \sqrt{xy} \quad \text{[taking positive square root]}$$

Hence, z is the mean proportional between x and y.

Example 14. What is the third proportional to $\frac{x}{y} + \frac{y}{x}$ and $\sqrt{(x+y)^2 - 2xy}$?

Sol. Let the third proportional be A.

Then, $\frac{x}{y} + \frac{y}{x} : \sqrt{(x+y)^2 - 2xy} :: \sqrt{(x+y)^2 - 2xy} : A$

$$\Rightarrow \quad \frac{\left(\frac{x}{y} + \frac{y}{x}\right)}{\sqrt{(x+y)^2 - 2xy}} = \frac{\sqrt{(x+y)^2 - 2xy}}{A}$$

$$\Rightarrow \quad \left(\frac{x}{y} + \frac{y}{x}\right)A = [\sqrt{(x+y)^2 - 2xy}]^2$$

[by cross-multiplication]

$$\Rightarrow \quad \left(\frac{x^2 + y^2}{xy}\right)A = (x+y)^2 - 2xy$$

$$\Rightarrow \quad \left(\frac{x^2 + y^2}{xy}\right)A = x^2 + y^2 + 2xy - 2xy$$

$[\because (a + b)^2 = a^2 + b^2 + 2ab]$

$$\Rightarrow \quad \left(\frac{x^2 + y^2}{xy}\right)A = x^2 + y^2 \Rightarrow A = \frac{(x^2 + y^2)xy}{(x^2 + y^2)} = xy$$

Hence, the required third proportional is xy.

Example 15. Find the fourth proportional to $(a^3 + 8), (a^4 - 2a^3 + 4a^2)$ and $(a^2 - 4)$.

Sol. Let fourth proportional be x.

Then, $(a^3 + 8) : (a^4 - 2a^3 + 4a^2) :: (a^2 - 4) : x$

$$\Rightarrow \quad \frac{a^3 + 8}{a^4 - 2a^3 + 4a^2} = \frac{a^2 - 4}{x}$$

$$\Rightarrow \quad x(a^3 + 8) = (a^2 - 4)(a^4 - 2a^3 + 4a^2)$$

[by cross-multiplication]

$$\therefore \quad x = \frac{(a^2 - 2^2) \times a^2(a^2 - 2a + 4)}{(a^3 + 2^3)}$$

$$= \frac{a^2(a-2)(a+2)(a^2 - 2a + 4)}{(a+2)(a^2 - 2a + 4)} = a^2(a - 2)$$

Hence, the required value of fourth proportional is $a^2(a - 2)$.

Properties and Combinations of Proportion

Some useful properties and combinations of proportion are given below

1. **Invertendo** If two ratios are equal, then their reciprocals are also equal,

   i.e. if $x : y :: z : t$ or $\frac{x}{y} = \frac{z}{t}$, then $\frac{y}{x} = \frac{t}{z}$,

   i.e. $y : x :: t : z$.

2. **Alternendo** If two ratios are equal, then the two ratios obtained by interchanging the consequent of first ratio and the antecedent of second ratio are also equal,

   i.e. **if $x : y :: z : t$ or $\frac{x}{y} = \frac{z}{t}$, then $\frac{x}{z} = \frac{y}{t}$, i.e. $x : z :: y : t$.**

3. **Componendo** If two ratios are equal, then the two ratios obtained on replacing the antecedent of both ratios by the sum of antecedent and consequent of given ratios are also equal,

   i.e. if $x : y :: z : t$ or $\frac{x}{y} = \frac{z}{t}$, then $\frac{(x+y)}{y} = \frac{(z+t)}{t}$

   i.e. $(x + y) : y :: (z + t) : t$.

4. **Dividendo** If two ratios are equal, then the two ratios obtained on replacing the antecedent of both ratios by the difference of antecedent and consequent of given ratios are also equal,

   i.e. if $x : y :: z : t$ or $\frac{x}{y} = \frac{z}{t}$, then $\frac{x-y}{y} = \frac{z-t}{t}$,

   i.e. $(x - y) : y :: (z - t) : t$.

5. **Componendo and Dividendo** If two ratios are equal, then the two ratios obtained on replacing the antecedent of both ratios by the sum of antecedent and consequent and replacing the consequent of both ratios by the difference of antecedent and consequent of given ratios are also equal,

   i.e. if $x : y :: z : t$ or $\frac{x}{y} = \frac{z}{t}$, then $\frac{(x+y)}{(x-y)} = \frac{(z+t)}{(z-t)}$,

   i.e. $(x + y) : (x - y) :: (z + t) : (z - t)$.

6. **Convertendo** If two ratios are equal, then the two ratios obtained on replacing the consequent of both ratios by difference of antecedent and consequent of given ratios are also equal,

   i.e. if $x : y :: z : t$ or $\frac{x}{y} = \frac{z}{t}$, then $\frac{x}{(x-y)} = \frac{z}{(z-t)}$

   i.e. $x : (x - y) :: z : (z - t)$.

7. **Addendo** If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then by addendo,

   $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{a+c+e}{b+d+f} = \frac{\text{Sum of antecedents}}{\text{Sum of consequents}}.$$

8. **Subtrahendo** If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then by subtrahendo,

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{a-c-e}{b-d-f}=\frac{\text{Subtract of antecedents}}{\text{Subtract of consequents}}.$$

Example 16. If $\frac{x}{y}=\frac{z}{t}$, then prove that $\frac{y-x}{x}=\frac{t-z}{z}$.

Sol. We have, $\frac{x}{y}=\frac{z}{t}\Rightarrow\frac{y}{x}=\frac{t}{z}$ [by invertendo property]

$\Rightarrow \frac{y-x}{x}=\frac{t-z}{z}$ [by dividendo property]

Hence proved.

Example 17. If $\frac{7m+2n}{7m-2n}=\frac{5}{3}$, use the properties of proportion to find

(i) $m:n$. (ii) $\frac{m^2+n^2}{m^2-n^2}$.

[2017]

Sol. Given, $\frac{7m+2n}{7m-2n}=\frac{5}{3}$ ...(i)

(i) On applying componendo and dividendo property in Eq. (i), we get

$$\frac{7m+2n+7m-2n}{7m+2n-7m+2n}=\frac{5+3}{5-3}$$

$$\Rightarrow \frac{14m}{4n}=\frac{8}{2}\Rightarrow\frac{7m}{2n}=4\Rightarrow\frac{7m}{n}=8\Rightarrow\frac{m}{n}=\frac{8}{7}$$

(ii) From part (i), we get

$$\frac{m}{n}=\frac{8}{7}\Rightarrow\frac{m^2}{n^2}=\frac{64}{49}$$

$$\Rightarrow \frac{m^2+n^2}{m^2-n^2}=\frac{64+49}{64-49}$$

[by componendo and dividendo property]

$$\Rightarrow \frac{m^2+n^2}{m^2-n^2}=\frac{113}{15}$$

Example 18. If $\frac{a^2+b^2}{a^2-b^2}=\frac{17}{8}$, then find the value of

(i) $a:b$ (ii) $(a^3+b^3):(a^3-b^3)$. ***[2014]***

Sol.

(i) We have, $\frac{a^2+b^2}{a^2-b^2}=\frac{17}{8}\Rightarrow\frac{(a^2+b^2)+(a^2-b^2)}{(a^2+b^2)-(a^2-b^2)}=\frac{17+8}{17-8}$

[by componendo and dividendo property]

$$\Rightarrow \frac{a^2+b^2+a^2-b^2}{a^2+b^2-a^2+b^2}=\frac{25}{9}\Rightarrow\frac{2a^2}{2b^2}=\frac{25}{9}\Rightarrow\left(\frac{a}{b}\right)^2=\left(\frac{5}{3}\right)^2$$

$\therefore \frac{a}{b}=\frac{5}{3}$ [taking positive square root]

Hence, $a:b=5:3$.

(ii) From part (i), we get

$\frac{a}{b}=\frac{5}{3}\Rightarrow\frac{a^3}{b^3}=\frac{125}{27}$ [cubing both sides]

$$\Rightarrow \frac{a^3+b^3}{a^3-b^3}=\frac{125+27}{125-27}$$

[by componendo and dividendo property]

$$\Rightarrow \frac{a^3+b^3}{a^3-b^3}=\frac{152}{98}=\frac{76}{49}$$

Hence, $(a^3+b^3):(a^3-b^3)=76:49$.

Example 19. Given $\frac{x^3+12x}{6x^2+8}=\frac{y^3+27y}{9y^2+27}$, using componendo and dividendo, find $x:y$. ***[2015]***

Sol. We have, $\frac{x^3+12x}{6x^2+8}=\frac{y^3+27y}{9y^2+27}$

Using componendo and dividendo property,

$\frac{a}{b}=\frac{c}{d}\Rightarrow\frac{a+b}{a-b}=\frac{c+d}{c-d}$, we get

$$\frac{x^3+12x+6x^2+8}{x^3+12x-6x^2-8}=\frac{y^3+27y+9y^2+27}{y^3+27y-9y^2-27}$$

$$\Rightarrow \frac{(x+2)^3}{(x-2)^3}=\frac{(y+3)^3}{(y-3)^3}$$

Taking cube root on both sides, we get

$$\frac{x+2}{x-2}=\frac{y+3}{y-3}$$

Again, applying componendo and dividendo property, we get

$$\frac{(x+2)+(x-2)}{(x+2)-(x-2)}=\frac{(y+3)+(y-3)}{(y+3)-(y-3)}$$

$$\Rightarrow \frac{2x}{4}=\frac{2y}{6}\Rightarrow\frac{x}{y}=\frac{2}{3}$$

Hence, $x:y=2:3$.

Example 20. If $\frac{x}{y+z}=\frac{y}{z+x}=\frac{z}{x+y}$, then prove that each ratio is equal to $\frac{1}{2}$ or -1.

Sol. We have, $\frac{x}{y+z}=\frac{y}{z+x}=\frac{z}{x+y}$

Clearly, each ratio $=\frac{\text{Sum of antecedents}}{\text{Sum of consequents}}$ [by addendo]

$$=\frac{x+y+z}{y+z+z+x+x+y}=\frac{(x+y+z)}{2(x+y+z)}$$

Case I When $x+y+z\neq 0$, then each ratio $=\frac{1}{2}$.

Case II When $x+y+z=0$, then

$$\frac{x}{y+z}=\frac{x}{(x+y+z)-x}=\frac{x}{0-x}=\frac{x}{-x}=-1\Rightarrow\frac{x}{y+z}=-1$$

$$\frac{y}{z+x}=\frac{y}{(z+x+y)-y}=\frac{y}{0-y}=\frac{y}{-y}=-1\Rightarrow\frac{y}{z+x}=-1$$

and $\frac{z}{x+y}=\frac{z}{(x+y+z)-z}=\frac{z}{0-z}=\frac{z}{-z}=-1$

$$\Rightarrow \frac{z}{z+x}=-1$$

$$\therefore \frac{x}{y+z}=\frac{y}{z+x}=\frac{z}{x+y}=-1$$

Hence, each ratio is equal to $\frac{1}{2}$ or -1. **Hence proved.**

Topic Exercise 2

1. Find the value of n, if $n : 15 :: 4 : 30$.

2. What number must be added to each term of $2 : 5$, so that it may be equal to $5 : 6$?

3. Find whether the numbers 6, 10, 14 and 22 are in proportion or not. If not, what must be added to each of the numbers so that they become proportional?

4. What number must be added to each of the numbers 5, 11, 19 and 37, so that they are in proportion? **[2009]**

5. If $a + c = 2b$ and $\frac{1}{b} + \frac{1}{d} = \frac{2}{c}$, then prove that a, b, c and d are in proportion.

6. If $p + r = mq$ and $\frac{1}{q} + \frac{1}{s} = \frac{m}{r}$, then prove that $p : q :: r : s$.

7. If four quantities a, b, c and d are in proportion, then show that
$$(a - c)b^2 : (b - d)cd = (a^2 - b^2 - ab) : (c^2 - d^2 - cd).$$

8. If a, b, c and d are in proportion, then prove that
$$\frac{a - b}{c - d} = \sqrt{\frac{3a^2 + 8b^2}{3c^2 + 8d^2}}.$$

9. If $(4x + 5y)(4z - 5u) = (4x - 5y)(4z + 5u)$, then prove that x, y, z and u are in proportion.

10. If $x : y :: y : z$, prove that $x : z :: x^2 : y^2$.

11. (i) If $(a^2 + c^2)$, $(ab + cd)$ and $(b^2 + d^2)$ are in continued proportion, then prove that a, b, c and d are in proportion.
(ii) If x, 16, 48 and y are in continued proportion, then find the values of x and y.

12. If x, y and z are in continued proportion, then prove that $\frac{x}{z} = \frac{2x^2 - 5xy + 7y^2}{2y^2 - 5yz + 7z^2}$.

13. If a, b, c and d are in continued proportion, then prove that
$$\sqrt{ab} + \sqrt{bc} - \sqrt{cd} = \sqrt{(a + b - c)(b + c - d)}.$$

14. Find the mean proportional between the following.
(i) 5 and 80
(ii) $360x^4$ and $250x^2y^2$

15. Find the mean proportional between $\left(\frac{a - b}{a + b}\right)$ and $\left(\frac{a^2b^2}{a^2 - b^2}\right)$.

16. If q is the mean proportional between p and r, then prove that $\frac{p^3 + q^3 + r^3}{p^2q^2r^2} = \frac{1}{p^3} + \frac{1}{q^3} + \frac{1}{r^3}$.

17. If x is the mean proportional between p and q, then prove that p, q, $(p^2 + x^2)$ and $(x^2 + q^2)$ are in proportion.

18. If 6 is the mean proportional between two numbers x and y and 48 is the third proportional to x and y, then find the numbers. **[2011]**

19. Find two numbers, such that the mean proportional between them is 18 and the third proportional to them is 144.

20. Find the third proportional to
(i) 25 and 15 (ii) $5 + 2\sqrt{3}, 37 + 20\sqrt{3}$.

21. Find the third proportional to $\frac{x}{y} + \frac{y}{x}$ and $\frac{x}{y}$.

22. Find the fourth proportional of the following.
(i) 8, 14, 16 (ii) $2xy, x^2, y^2$

23. Find the fourth proportional to $(x^2 - 5x + 4)$, $(x^2 + x - 2)$ and $(x^2 - 16)$.

24. Find the fourth proportional to
$$x^3 - y^3, x^4 + x^2y^2 + y'', x - y.$$

25. Using the properties of proportion, solve for x, given $\frac{x^4 + 1}{2x^2} = \frac{17}{8}$. **[2013]**

26. Using the properties of proportion, solve the following equation $\frac{5x + (x^2 - 1)}{5x - (x^2 - 1)} = \frac{7}{5}$.

27. If $a : b :: c : d$, then show that
$$(4a + 5b) : (4a - 5b) :: (4c + 5d) : (4c - 5d).$$

28. If $\frac{4a^2 + 3b^2}{4a^2 - 3b^2} = \frac{7}{4}$, find the value of $\frac{2a^4 - 11b^4}{2a^4 + 11b^4}$.

29. If $\frac{8a - 5b}{8c - 5d} = \frac{8a + 5b}{8c + 5d}$, then prove that $\frac{a}{b} = \frac{c}{d}$. **[2008]**

30. If $x = \frac{\sqrt{a + 1} + \sqrt{a - 1}}{\sqrt{a + 1} - \sqrt{a - 1}}$, then using the properties of proportion, show that $x^2 - 2ax + 1 = 0$. **[2012]**

31. If $x = \dfrac{\sqrt{a+3b}+\sqrt{a-3b}}{\sqrt{a+3b}-\sqrt{a-3b}}$, then show that $3bx^2 - 2ax + 3b = 0$. *[2007]*

32. Using properties of proportion, solve for x. Given that x is positive.

$$\frac{2x+\sqrt{4x^2-1}}{2x-\sqrt{4x^2-1}} = 4$$ *[2018]*

33. If $x = \dfrac{4\sqrt{6}}{\sqrt{2}+\sqrt{3}}$, then find the value of

$$\frac{x+2\sqrt{2}}{x-2\sqrt{2}} + \frac{x+2\sqrt{3}}{x-2\sqrt{3}}.$$

34. If $A = \dfrac{6xy}{x+y}$, then find the value of $\dfrac{A+3x}{A-3x} + \dfrac{A+3y}{A-3y}$.

35. If $\dfrac{b+c-a}{y+z-x} = \dfrac{c+a-b}{z+x-y} = \dfrac{a+b-c}{x+y-z}$, then prove that each ratio is equal to $\dfrac{a}{x} = \dfrac{b}{y} = \dfrac{c}{z}$.

Hints and Answers

1. Do same as Example 1. **Ans.** $n = 2$

2. Hint Let the required number be x.

Then, $\dfrac{2+x}{5+x} = \dfrac{5}{6}$

Ans. $x = 13$

3. Hint Here, $6 \times 22 \neq 10 \times 14$

Now, do same as Example 2. **Ans.** 2

4. Do same as Example 2. **Ans.** 2

5. Hint Given, $\dfrac{d+b}{bd} = \dfrac{2}{c}$

$\Rightarrow \quad c(d+b) = 2bd$ [by cross-multiplication]

$\Rightarrow \quad cd + cb = (a+c)d$ [given, $a + c = 2b$]

$\Rightarrow \quad \dfrac{ad}{bd} = \dfrac{cb}{bd}$ [dividing both sides by bd]

6. Do same as Q. 5.

7. Hint Given, $a : b :: c : d \Rightarrow \dfrac{a}{b} = \dfrac{c}{d} = k$ (say)

$\Rightarrow \quad a = bk$ and $c = dk$ [by cross-multiplication] ...(i)

Now, LHS $= \dfrac{(a-c)b^2}{(b-d)cd} = \dfrac{(bk-dk)b^2}{(b-d)(dk)d}$ [from Eq. (i)]

and RHS $= \dfrac{a^2-b^2-ab}{c^2-d^2-cd} = \dfrac{(bk)^2-b^2-(bk)b}{(dk)^2-d^2-(dk)d}$

[from Eq. (i)]

8. Hint Given, $\dfrac{a}{b} = \dfrac{c}{d} = k$ (say)

Now, LHS $= \dfrac{a-b}{c-d} = \dfrac{bk-b}{dk-d}$

RHS $= \sqrt{\dfrac{3a^2+8b^2}{3c^2+8d^2}} = \sqrt{\dfrac{3b^2k^2+8b^2}{3d^2k^2+8d^2}}$

9. Hint Simplify the expression of LHS and RHS.

10. Hint We have, $\dfrac{x}{y} = \dfrac{y}{z} \Rightarrow y^2 = xz$

Now, consider $\dfrac{x^2}{y^2} = \dfrac{x^2}{xz} = \dfrac{x}{z}$

11. (i) **Hint** Here, $(ab+cd)^2 = (a^2+c^2)(b^2+d^2)$

Now, do same as Example 3.

(ii) Do same as Example 5. **Ans.** $x = \dfrac{16}{3}$, $y = 144$

12. Hint Given, $\dfrac{x}{y} = \dfrac{y}{z} \Rightarrow y^2 = xz$

[by cross-multiplication]

$\Rightarrow \quad y = \sqrt{xz}$ ...(i)

Now, RHS $= \dfrac{2x^2-5xy+7y^2}{2y^2-5yz+7z^2}$

$= \dfrac{2x^2-5x\sqrt{xz}+7xz}{2xz-5z\sqrt{xz}+7z^2}$ [from Eq. (i)]

13. Hint Given, $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d} = k$ (let)

$\Rightarrow \quad c = dk,\ b = ck = dk^2$ and $a = bk = dk^3$ ...(i)

Now, LHS $= \sqrt{ab} + \sqrt{bc} - \sqrt{cd}$

$= \sqrt{(dk^3)(dk^2)} + \sqrt{(dk^2)(dk)} - \sqrt{(dk)d}$

[from Eq. (i)]

and RHS $= \sqrt{(a+b-c)(b+c-d)}$

$= \sqrt{(dk^3+dk^2-dk)(dk^2+dk-d)}$

[from Eq. (i)]

14. Do same as Example 6. **Ans.** (i) 20 (ii) $300x^3y$

15. Do same as Example 9. **Ans.** $\dfrac{ab}{a+b}$

16. Do same as Example 12.

17. Hint To prove, $\dfrac{p}{q} = \dfrac{p^2+x^2}{x^2+q^2}$

18. Hint Given, 6 is the mean proportional between two numbers x and y.

$\therefore \quad x : 6 :: 6 : y$

$\Rightarrow \quad x = \dfrac{36}{y}$ ...(i)

Also, given 48 is the third proportional to x and y.

$\therefore \quad x : y :: y : 48$

$\Rightarrow \quad y^2 = 48x$

$\Rightarrow \quad y^2 = 48 \times \dfrac{36}{y}$ [from Eq. (i)]

Ans. $x = 3$ and $y = 12$

19. **Hint** Let two numbers be x and y.

Then, $x : 18 :: 18 : y$ and $x : y :: y : 144$

Ans. 9, 36

20. Do same as Example 7.

Ans. (i) 9 (ii) $305 + 174\sqrt{3}$

21. Do same as Example 14.

Ans. $\dfrac{x^3}{y(x^2 + y^2)}$

22. (i) Do same as Example 8. **Ans.** 28

(ii) **Hint** Let the fourth proportional be A.

Then, $2xy : x^2 :: y^2 : A$ **Ans.** $\dfrac{xy}{2}$

23. **Hint** Let the fourth proportional be A.

Then, $(x^2 - 5x + 4) : (x^2 + x - 2) :: (x^2 - 16) : A$

$$\Rightarrow \quad \frac{x^2 - 5x + 4}{x^2 + x - 2} = \frac{x^2 - 16}{A}$$

$$\Rightarrow \quad A = \frac{(x^2 - 16)(x^2 + x - 2)}{(x^2 - 5x + 4)}$$

$$= \frac{(x - 4)(x + 4)(x - 1)(x + 2)}{(x - 1)(x - 4)}$$

Ans. $x^2 + 6x + 8$

24. **Hint** Fourth proportional, A

$$= \frac{(x - y)(x^4 + x^2y^2 + y^4)}{x^3 - y^3}$$

$$= \frac{(x - y)(x^4 + x^2y^2 + y^4)}{(x - y)(x^2 + xy + y^2)}$$

$$= \frac{(x^4 + 2x^2y^2 + y^4 - x^2y^2)}{(x^2 + y^2 + xy)}$$

$$= \frac{[(x^2 + y^2)^2 - (xy)^2]}{(x^2 + y^2 + xy)}$$

Ans. $x^2 + y^2 - xy$

25. **Hint** On applying componendo and dividendo in the given proportion, we get

$$\frac{x^4 + 1 + 2x^2}{x^4 + 1 - 2x^2} = \frac{25}{9}$$

$$\Rightarrow \quad \frac{(x^2 + 1)^2}{(x^2 - 1)^2} = \left(\frac{5}{3}\right)^2$$

On taking square root both sides, we get

$$\frac{x^2 + 1}{x^2 - 1} = \frac{5}{3}$$

$$\Rightarrow \quad \frac{x^2 + 1 + x^2 - 1}{x^2 + 1 - (x^2 - 1)} = \frac{8}{2}$$

[by componendo and dividendo]

Ans. ± 2

26. Do same as Q. 25. **Ans.** $\dfrac{3}{2}, \dfrac{-2}{3}$

27. **Hint** We have, $\dfrac{a}{b} = \dfrac{c}{d} \Rightarrow \dfrac{4a}{5b} = \dfrac{4c}{5d}$

[multiplying numerator of both sides by 4 and denominator by 5]

Now, apply componendo and dividendo.

28. **Hint** We have, $\dfrac{4a^2 + 3b^2}{4a^2 - 3b^2} = \dfrac{7}{4}$

On applying componendo and dividendo, we get

$$\frac{8a^2}{6b^2} = \frac{11}{3} \Rightarrow \frac{a^2}{b^2} = \frac{11}{4}$$

$$\Rightarrow \quad \frac{a^4}{b^4} = \frac{11^2}{4^2}$$

$$\Rightarrow \quad \frac{2a^4}{11b^4} = \frac{2 \times 11^2}{11 \times 4^2} = \frac{11}{8}$$

Now, again apply componendo and dividendo.

Ans. $\dfrac{3}{19}$

29. **Hint** Given, $\dfrac{8a - 5b}{8c - 5d} = \dfrac{8a + 5b}{8c + 5d}$

$$\Rightarrow \quad \frac{8a - 5b}{8a + 5b} = \frac{8c - 5d}{8c + 5d} \quad \text{[by alternendo property]}$$

$$\Rightarrow \quad \frac{8a + 5b}{8a - 5b} = \frac{8c + 5d}{8c - 5d} \quad \text{[by invertendo property]}$$

On applying componendo and dividendo property, we get

$$\frac{8a + 5b + 8a - 5b}{8a + 5b - 8a + 5b} = \frac{8c + 5d + 8c - 5d}{8c + 5d - 8c + 5d}$$

30. Hint On applying componendo and dividendo property, we get

$$\frac{x+1}{x-1}=\frac{\sqrt{a+1}+\sqrt{a-1}+\sqrt{a+1}-\sqrt{a-1}}{\sqrt{a+1}+\sqrt{a-1}-\sqrt{a+1}+\sqrt{a-1}}$$

$$\Rightarrow \quad \frac{x+1}{x-1}=\frac{\sqrt{a+1}}{\sqrt{a-1}}$$

On squaring both sides, we get

$$\frac{x^2+2x+1}{x^2-2x+1}=\frac{a+1}{a-1}$$

31. Do same as Q. 30.

32. Hint

Given, $\dfrac{2x+\sqrt{4x^2-1}}{2x-\sqrt{4x^2-1}}=\dfrac{4}{1}$

$$\Rightarrow \quad \frac{2x+\sqrt{4x^2-1}+2x-\sqrt{4x^2-1}}{2x+\sqrt{4x^2-1}-2x+\sqrt{4x^2-1}}=\frac{4+1}{4-1}$$

[using componendo and dividendo rule]

Ans. $x=\dfrac{5}{8}$

33. Hint $\dfrac{x}{2\sqrt{2}}=\dfrac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}$

and $\dfrac{x}{2\sqrt{3}}=\dfrac{2\sqrt{2}}{\sqrt{2}+\sqrt{3}}$

Now, apply compodendo and dividendo.

Ans. 2

34. Do same as Q. 33.

Ans. 2

35. Hint Each given ratio $=\dfrac{a+b+c}{x+y+z}$

Consider, $\dfrac{b+c-a}{y+z-x}=\dfrac{a+b+c}{x+y+z}$

$$\Rightarrow \quad \frac{b+c-a}{a+b+c}=\frac{y+z-x}{x+y+z} \qquad \text{[by alternendo]}$$

$$\Rightarrow \quad \frac{-2a}{a+b+c}=\frac{-2x}{x+y+z} \qquad \text{[by dividendo]}$$

$$\Rightarrow \quad \frac{a}{x}=\frac{a+b+c}{x+y+z}$$

Similarly, $\dfrac{a+b+c}{x+y+z}=\dfrac{b}{y}$ and $\dfrac{a+b+c}{x+y+z}=\dfrac{c}{z}$.

CHAPTER EXERCISE

a 3 Marks Questions

1. Arrange the ratios $5:6$, $7:8$ and $13:6$ in descending order of magnitude.

2. Arrange the ratios 2 : 3, 17 : 21, 11 : 14 and 5 : 7 in descending order of magnitude.

3. If $(3a+2b):(5a+3b)=18:29$, then find $a:b$. *[2016]*

4. If the ratio between 2 and 3 is the same as the ratio of $(m+n)$ to $(m+3n)$, then find the value of
$$\frac{2n^2}{3m^2+mn}.$$

5. Two numbers are in the ratio 3 : 5. If 8 is added to each number, then the ratio becomes 2 : 3. Find the numbers. *[2001]*

6. What quantity must be subtracted from each term of the ratio 9 : 17 to make it equal to 1 : 3?

7. A bag contains ₹ 140 in the form of 50 paise, ₹ 1 and ₹ 2 coins in the ratio of 6 : 3 : 7. Find the total number of coins.

8. The ages of A and B are in the ratio 7 : 8. Six years ago, their ages were in the ratio 5 : 6. Find their present ages.

9. Two positive numbers are in the ratio 3 : 5 and the difference between their squares is 400. Find the numbers.

10. The monthly pocket money of Ravi and Sanjeev are in the ratio 5 : 7. Their expenditures are in the ratio 3 : 5. If each save ₹ 80 every month, then find their monthly pocket money. *[2012]*

11. The monthly income of P and Q are in the ratio $4:3$ and their expenditures are in the ratio $3:2$. If each of them saves ₹ 600 per month, then find the monthly income of each.

12. Find the following.
    (i) The compounded ratio of $\sqrt{2}:1$, $3:\sqrt{5}$ and $\sqrt{20}:9$.
    (ii) The duplicate ratio of $5\sqrt{7}:7\sqrt{5}$.
    (iii) The triplicate ratio of $1:3xy$.

13. Find the following.
    (i) The sub-duplicate ratio of $9x^2:49y^2$.
    (ii) The sub-triplicate ratio of $64a^6:27b^3$.
    (iii) The reciprocal ratio of $9x^2a^2:25z^2b^6$.

14. Find the following.
    (i) The sub-duplicate ratio of $(x-y)^4:(x+y)^6$.
    (ii) The sub-triplicate ratio of
    $$(a^3+b^3)^3:(a^2+b^2-ab)^3\,ab.$$
    (iii) The reciprocal ratio of $\frac{x}{m}:\frac{y}{n}$.

15. If $(x+2y):(2x-y)$ is equal to the duplicate ratio of $3:2$, then find $x:y$.

16. If $x:y$ is the duplicate ratio of $(x+a):(y+a)$, then show that $a^2=xy$.

17. If $(3a+1):(5a+3)$ is the triplicate ratio of $3:4$, then find the value of a.

18. If $(a+2):(2a+10)$ is the triplicate ratio of 3 : 5, then find the value of a.

19. What least number must be added to each of the numbers 16, 7, 79 and 43, so that the resulting numbers are in proportion?

20. What number must be added to each of the numbers 6, 15, 20 and 43 to make them proportional? *[2013, 2005]*

21. What must be subtracted from each of the four numbers 23, 40, 57 and 108 to make them in proportion?

22. If x, y, z and u are in proportion, then prove that
$$\frac{5x^2+12z^2}{5y^2+12u^2}=\sqrt{\frac{3x^4-7z^4}{3y^4-7u^4}}.$$

23. If $ax=by=cz$, then prove that
$$\frac{x^2}{yz}+\frac{y^2}{zx}+\frac{z^2}{xy}=\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}.$$

24. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then prove that
$$\left(\frac{a^2b^2+c^2d^2+e^2f^2}{ab^3+cd^3+ef^3}\right)^{3/2}=\sqrt{\frac{ace}{bdf}}.$$

25. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, then show that
$$\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{z^3}{c^3}=\frac{3xyz}{abc}.$$ **[2016]**

26. If a, b, c and d are in continued proportion, then prove that

(i) $\frac{a}{d}=\frac{a^3+b^3+c^3}{b^3+c^3+d^3}$

(ii) $abc(a+b+c)^3=(ab+bc+ca)^3$.

27. Find the mean proportional between

(i) $\frac{x^2}{4ab}$ and $\frac{a}{by^2}$

(ii) $(\sqrt{27}-\sqrt{18})$ and $(\sqrt{27}+\sqrt{18})$.

28. Find the third proportional to

(i) $x^2-\frac{1}{x^2}, x+\frac{1}{x}$ (ii) ab, ab^2.

29. Find two numbers, such that the mean proportional between them is 28 and third proportional to them is 224.

30. Find the fourth proportional to
$(a^2-ab+b^2), (a^3+b^3)$ and $(a-b)$.

31. Show that if $a>b>0$, then the ratio $(a^2-b^2):(a^2+b^2)$ is greater than $(a-b):(a+b)$.

32. If $(a-b):(a+b)=1:11$, then find the ratio $(5a+4b+15):(5a-4b+3)$.

33. Using componendo and dividendo, find the value of x in the equation $\frac{\sqrt{3x+4}+\sqrt{3x-5}}{\sqrt{3x+4}-\sqrt{3x-5}}=9$. **[2011]**

b 4 Marks Questions

34. The work done by $(3x+1)$ workers in $(2x-1)$ days and the work done by $(2x-1)$ workers in $(x+8)$ days are in the ratio 5 : 2. Find the value of x.

35. Mr. Kamal reduces the number of workers of his factory in the ratio 9 : 7 and increases their wages in the ratio 13 : 20. In what ratio, the wages bill is increased or decreased?

36. An employer reduces the number of employees in the ratio 11 : 7 and increases their wages in the ratio 10 : 13. In what ratio, the wages bill is increased or decreased?

37. (i) If $x, 9, y$ and 16 are in continued proportion, then find the values of x and y.

(ii) If a, b and c are in continued proportion, then prove that
$$\frac{a+b+c}{a-b+c}=\frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

38. Find the mean proportional of
$(a^4-b^4)^2$ and $[(a^2+b^2)(a+b)]^{-2}$.

39. Find the third proportional to
(i) 16 and 36 (ii) $(x^2+y^2+xy)^2$ and (x^3-y^3).

40. If y is the mean proportional between x and z, then prove that $(xy+yz)$ is the mean proportional between (x^2+y^2) and (y^2+z^2).

41. Find the fourth proportional of the following.

(i) $3a^2b^2, a^3, b^3$

(ii) a^2-5a+6, a^2+a-6, a^2-9

42. What quantity should be added to the terms of the ratio $(p+q):(p-q)$ to make it equal to $(p+q)^2:(p-q)^2$?

43. If $\frac{a^3+3ab^2}{3a^2b+b^3}=\frac{x^3+3xy^2}{3x^2y+y^3}$, then prove that $\frac{x}{a}=\frac{y}{b}$.

44. If $\frac{ay-bx}{c}=\frac{cx-az}{b}=\frac{bz-cy}{a}$, then show that $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

45. If $7a-15b=4a+b$, then find the value of $\frac{3a^2+2b^2}{3a^2-2b^2}$.

46. If $\frac{ap+bq}{ar+bs}=\frac{ap-bq}{ar-bs}$, then show that $\frac{p}{q}=\frac{r}{s}$.

47. If x, y, z and u are in proportion, then prove that $(lx+my):(lz+mu)::(lx-my):(lz-mu)$.

48. If $p=\frac{4xy}{x+y}$, then find the value of
$$\frac{p+2x}{p-2x}+\frac{p+2y}{p-2y}.$$

49. If $y=\frac{10ab}{a+b}$, then find the value of $\frac{y+5a}{y-5a}+\frac{y+5b}{y-5b}$.

50. If $\frac{x}{r^2-pq}=\frac{y}{p^2-qr}=\frac{z}{q^2-pr}$, then prove that $px+qy+rz=0$.

Hints and Answers

1. Do same as Example 6 of Topic 1.
Ans. $\frac{13}{6}, \frac{7}{8}, \frac{5}{6}$

2. Do same as Example 6 of Topic 1. **Ans.** $\frac{17}{21}, \frac{11}{14}, \frac{5}{7}, \frac{2}{3}$

3. Hint $\frac{3a+2b}{5a+3b} = \frac{18}{29} \Rightarrow \frac{3(a/b)+2}{5(a/b)+3} = \frac{18}{29}$

$$\Rightarrow \quad 87\left(\frac{a}{b}\right)+58 = 90\left(\frac{a}{b}\right)+54$$

Ans. $a:b = 4:3$

4. Hint $\frac{m+n}{m+3n} = \frac{2}{3} \Rightarrow 3m+3n = 2m+6n \Rightarrow \frac{m}{n} = \frac{3}{1}$

Also, $\frac{2n^2}{3m^2+mn} = \frac{2}{3\left(\frac{m^2}{n}\right)+\left(\frac{m}{n}\right)}$ **Ans.** 1 : 15

5. Hint Let two numbers be $3x$ and $5x$, respectively.

Then, $\frac{3x+8}{5x+8} = \frac{2}{3}$ **Ans.** 24, 40

6. Hint Let x must be subtracted from each term of 9 : 17 to make it equal to 1 : 3.

Then, $\frac{9-x}{17-x} = \frac{1}{3}$ **Ans.** 5

7. Hint Let number of 50 paise coins $= 6x$.
number of ₹ 1 coins $= 3x$
and number of ₹ 2 coins $= 7x$
According to the question,

$$6x \times \frac{1}{2} + 3x \times 1 + 7x \times 2 = 140$$

Ans. 112

8. Hint Let the present age of $A = 7x$
and the present age of $B = 8x$

$$\therefore \quad \frac{7x-6}{8x-6} = \frac{5}{6}$$

Ans. 21 yr and 24 yr

9. Hint Let two numbers be $3x$ and $5x$.
Then, $(5x)^2 - (3x)^2 = 400$ **Ans.** 15 and 25

10. Hint Let Ravi's monthly pocket money = ₹ $5x$
and Sanjeev's monthly pocket money = ₹ $7x$
Again, let Ravi's expenditure = ₹ $3y$
and Sanjeev's expenditure = ₹ $5y$
According to the question, each save ₹ 80.
$\therefore$ Pocket money − Expenditure = 80, which gives

$$5x - 3y = 80 \qquad \ldots\text{(i)}$$

and

$$7x - 5y = 80 \qquad \ldots\text{(ii)}$$

On solving Eqs. (i) and (ii), we get the required value.
Ans. ₹ 200 and ₹ 280

11. Do same as Q. 10. **Ans.** ₹ 2400, ₹ 1800

12. (i) Do same as Example 7 of Topic 1. **Ans.** $2\sqrt{2}:3$
(ii) Do same as Example 8 (i) of Topic 1. **Ans.** 5 : 7
(iii) Do same as Example 8 (ii) of Topic 1.
Ans. $1:27x^3y^3$

13. (i) Do same as Example 8 (iii) of Topic 1. **Ans.** $3x:7y$
(ii) Do same as Example 8 (iv) of Topic 1. **Ans.** $4a^2:3b$
(iii) Do same as Example 8 (v) of Topic 1.
Ans. $25z^2b^6 : 9x^2a^2$

14. (i) Do same as Example 8 (iii) of Topic 1.
Ans. $(x-y)^2:(x+y)^3$
(ii) **Hint** Do same as Example 8 (iv) of Topic 1 and use the formula $a^3+b^3 = (a+b)(a^2+b^2-ab)$
Ans. $(a+b):\sqrt[3]{ab}$
(iii) Do same as Example 8 (v) of Topic 1.
Ans. $my:nx$

15. Hint $\frac{x+2y}{2x-y} = \frac{3^2}{2^2}$ **Ans.** 17 : 14

16. Hint $\frac{(x+a)^2}{(y+a)^2} = \frac{x}{y}$

17. Hint $\frac{3a+1}{5a+3} = \frac{(3)^3}{(4)^3}$ **Ans.** 17 : 57

18. Hint $\frac{a+2}{2a+10} = \frac{3^3}{5^3}$ **Ans.** $a = \frac{20}{71}$

19. Do same as Example 2 of Topic 2. **Ans.** 5

20. Do same as Example 2 of Topic 2. **Ans.** 3

21. Hint Let the required number be x.
According to the question,

$$(23-x):(40-x)::(57-x):(108-x)$$

Ans. 6

22. Let $\frac{x}{y} = \frac{z}{u} = k \Rightarrow x = yk$ and $z = uk$

Now, LHS $= \frac{5y^2k^2+12u^2k^2}{5y^2+12u^2} = k^2$

and RHS $= \sqrt{\frac{3y^4k^4-7u^4k^4}{3y^4-7u^4}} = k^2$

23. **Hint** LHS $= \left(\frac{x}{y}\right)\left(\frac{x}{z}\right)+\left(\frac{y}{z}\right)\left(\frac{y}{x}\right)+\left(\frac{z}{x}\right)\left(\frac{z}{y}\right)$

$$= \left(\frac{b}{a}\right)\left(\frac{c}{a}\right)+\left(\frac{c}{b}\right)\left(\frac{a}{b}\right)+\left(\frac{a}{c}\right)\left(\frac{b}{c}\right)$$

$$\left[\begin{array}{l} \because ax = by \Rightarrow \frac{x}{y}=\frac{b}{a},\ by = cz \Rightarrow \frac{y}{z}=\frac{c}{b} \\ \text{and } ax = cz \Rightarrow \frac{x}{z}=\frac{c}{a} \end{array}\right]$$

24. **Hint** $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$ (say)

$\Rightarrow \quad a = bk,\ c = dk$ and $e = fk$ $\quad$ ...(i)

Now, LHS $=\left(\frac{(bk)^2 b^2+(dk)^2 d^2+(fk)^2 f^2}{(bk)b^3+(dk)d^3+(fk)f^3}\right)^{3/2}$ [from Eq. (i)]

and RHS $=\sqrt{\frac{(bk)(dk)(fk)}{bdf}}$ [from Eq. (i)]

25. **Hint** Let $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$ (say)

$x = ak,\ \ y = bk$ and $z = ck$ $\quad$ ...(i)

Now, find LHS and RHS separately.

26. (i) **Hint** Let $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=k$ (say)

$\Rightarrow \quad c = dk,\ b = ck = dk^2$ and $a = bk = dk^3$ $\quad$ ...(i)

Now, LHS $=\frac{a}{d}=\frac{dk^3}{d}$ [from (i)]

and RHS $=\frac{(dk^3)^3+(dk^2)^3+(dk)^3}{(dk^2)^3+(dk)^3+d^3}$ [from (i)]

(ii) Similar as part (i).

27. (i) **Hint** Let z be the required mean proportional between $\frac{x^2}{4ab}$ and $\frac{a}{by^2}$. Then, $\frac{x^2}{4ab}:z::z:\frac{a}{by^2}$

Ans. $\frac{x}{2by}$

(ii) Similar as part (i). **Ans.** 3

28. (i) **Hint** Let the required third proportional be z.

Then, $\left(x^2-\frac{1}{x^2}\right):\left(x+\frac{1}{x}\right)::\left(x+\frac{1}{x}\right):z$

$$\Rightarrow \left(\frac{x^4-1}{x^2}\right):\left(\frac{x^2+1}{x}\right)=\left(\frac{x^2+1}{x}\right):z$$

Ans. $\left(\frac{x^2+1}{x^2-1}\right)$

(ii) Similar as part (i). **Ans.** ab^3

29. **Hint** Let the required two numbers be x and y.

According to the question,

28 is mean proportional between x and y.

$\therefore \quad x:28::28:y \Rightarrow x=\frac{784}{y}$ $\quad$ ...(i)

and 224 is third proportional to x and y.

$\therefore \quad x:y::y:224 \Rightarrow y^2 = 224x$

[by cross-multiplication] ...(ii)

Ans. 14 and 56

30. **Hint** Let the required fourth proportional be x.

Then, $(a^2-ab+b^2):(a^3+b^3)::(a-b):x$

Ans. a^2-b^2

31. **Hint** (i) $\frac{a}{b}>\frac{1}{1}\Rightarrow\frac{a^2}{b^2}>\frac{a}{b}$

(ii) Apply componendo and dividendo.

32. **Hint** $\frac{a+b}{a-b}=\frac{11}{1}$

$\Rightarrow \quad \frac{2a}{2b}=\frac{12}{10}$ [by componendo and dividendo]

$\Rightarrow \quad \frac{a}{b}=\frac{6}{5}\Rightarrow\frac{5a}{4b}=\frac{30}{20}=\frac{3}{2}$

$\Rightarrow \quad \frac{5a+4b}{5a-4b}=\frac{5}{1}\Rightarrow 5a+4b=5(5a-4b)$

Now, consider $\frac{5a+4b+15}{5a-4b+3}=\frac{5(5a-4b)+15}{(5a-4b+3)}$

$$=\frac{5[5a-4b+3]}{[5a-4b+3]}$$

Ans. 5 : 1

33. **Hint** On applying componendo and dividendo, we get

$$\frac{\sqrt{3x+4}}{\sqrt{3x-5}}=\frac{5}{4}$$

On squaring both sides, we get $\frac{3x+4}{3x-5}=\frac{25}{16}$

Ans. 7

34. **Hint** Let a man do A amount of work in one day.

Then, the quantity of work done by $(3x+1)$ workers in $(2x-1)$ days $= A(3x+1)(2x-1)$

and the quantity of work done by $(2x-1)$ workers in $(x+8)$ days $= A(2x-1)(x+8)$

According to the question,

$$\frac{A(3x+1)(2x-1)}{A(2x-1)(x+8)}=\frac{5}{2}\Rightarrow\frac{(3x+1)(2x-1)}{(2x-1)(x+8)}=\frac{5}{2}$$

Ans. $x = 38$

35. Hint Let the number of workers initially and at present be $9x$ and $7x$, respectively.

Again, let the wages per worker initially and at present be ₹ $13y$ and ₹ $20y$, respectively.

Then, total wages bill at initially $= 9x \times 13y = ₹\,117xy$

and total wages bill at present $= 7x \times 20y = ₹\,140xy$

Ans. Wages bill is increased in the ratio 117 : 140.

36. Hint Do same as Q. 35.

Ans. The wages bill is decreased in the ratio 110 : 91.

37. (i) Do same as Example 5 of Topic 2.

Ans. $x = \dfrac{27}{4}$ and $y = 12$

(ii) Do same as Example 11 of Topic 2.

38. Do same as Example 9 of Topic 2.

Ans. $(a-b)$

39. Do same as Example 14 of Topic 2.

Ans. (i) 81 (ii) $(x-y)^2$

40. Hint As y is the mean proportion between two numbers x and z.

$\therefore \quad x : y :: y : z \Rightarrow y^2 = xz$

To prove, $(xy + yz)^2 = (x^2 + y^2)(y^2 + z^2)$.

41. Do same as Example 15 of Topic 2.

Ans. (i) $\dfrac{ab}{3}$ (ii) $(a+3)^2$

42. Hint Let the required added number be x.

According to the question,

$$(p+q+x) : (p-q+x) :: (p+q)^2 : (p-q)^2$$

$$\Rightarrow \frac{p+q+x}{p-q+x} = \frac{(p+q)^2}{(p-q)^2}$$

Now, on applying componendo and dividendo, we get the required value.

Ans. $\dfrac{q^2 - p^2}{2p}$

43. Hint On applying componendo and dividendo, we get

$$\frac{a^3 + 3ab^2 + 3a^2b + b^3}{a^3 + 3ab^2 - 3a^2b - b^3} = \frac{x^3 + 3xy^2 + 3x^2y + y^3}{x^3 + 3xy^2 - 3x^2y - y^3}$$

$$\Rightarrow \frac{(a+b)^3}{(a-b)^3} = \frac{(x+y)^3}{(x-y)^3} \Rightarrow \frac{a+b}{a-b} = \frac{x+y}{x-y}$$

[taking cube root on both sides]

Again, applying componendo and dividendo, we get

$$\frac{a+b+a-b}{a+b-a+b} = \frac{x+y+x-y}{x+y-x+y} \Rightarrow \frac{a}{b} = \frac{x}{y}$$

44. Hint $\dfrac{c(ay-bx)}{c^2} = \dfrac{b(cx-az)}{b^2} = \dfrac{a(bz-cy)}{a^2}$

$$\Rightarrow \frac{cay - cbx}{c^2} = \frac{cbx - abz}{b^2} = \frac{abz - cay}{a^2}$$

Clearly, each ratio $= \dfrac{\text{Sum of antecedents}}{\text{Sum of consequents}}$

$$= \frac{cay - cbx + cbx - abz + abz - cay}{c^2 + b^2 + a^2} = 0$$

$$\therefore \quad \frac{ay - bx}{c} = \frac{cx - az}{b} = \frac{bz - cy}{a} = 0$$

$\Rightarrow ay - bx = 0, cx - az = 0$ and $bz - cy = 0$

45. Hint $7a - 15b = 4a + b$

$$\Rightarrow \frac{a}{b} = \frac{16}{3} \Rightarrow \frac{a^2}{b^2} = \frac{256}{9} \Rightarrow \frac{3a^2}{2b^2} = \frac{768}{18}$$

Now, apply componendo and dividendo.

Ans. 131 : 125

46. Hint $\dfrac{ap + bq}{ap - bq} = \dfrac{ar + bs}{ar - bs}$.

Now, apply componendo and dividendo.

47. Hint Given, $\dfrac{x}{y} = \dfrac{z}{u}$

$$\Rightarrow \frac{lx}{my} = \frac{lz}{mu} \Rightarrow \frac{lx + my}{lx - my} = \frac{lz + mu}{lz - mu}$$

[by componendo and dividendo]

48. Hint $p = \dfrac{(2x)(2y)}{(x+y)}$ ...(i)

Now, $\dfrac{p}{2x} = \dfrac{2y}{x+y}$ [from Eq. (i)]

On applying componendo and dividendo, we get

$$\frac{p+2x}{p-2x} = \frac{3y+x}{y-x} \quad \text{...(ii)}$$

Also, $\dfrac{p}{2y} = \dfrac{2x}{x+y}$ [from Eq. (i)]

On applying componendo and dividendo, we get

$$\frac{p+2y}{p-2y} = \frac{3x+y}{x-y} \quad \text{...(iii)}$$

On adding Eqs. (ii) and (iii), we get the required value.

Ans. 2

49. Do same as Q. 48. **Ans.** 2

50. Hint Let $\dfrac{x}{r^2 - pq} = \dfrac{y}{p^2 - qr} = \dfrac{z}{q^2 - pr} = k$. Then,

$$\text{LHS} = pk(r^2 - pq) + qk(p^2 - qr) + rk(q^2 - pr)$$

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1 Using properties of proportion, solve for x. Given that x is positive.

$$\frac{2x+\sqrt{4x^2-1}}{2x-\sqrt{4x^2-1}}=4$$

2017

2 If $\frac{7m+2n}{7m-2n}=\frac{5}{3}$, use properties of proportion to find

(i) $m:n$ (ii) $\frac{m^2+n^2}{m^2-n^2}$.

3 If b is the mean proportion between a and c, then show that $\frac{a^4+a^2b^2+b^4}{b^4+b^2c^2+c^4}=\frac{a^2}{c^2}$.

2016

4 If $(3a+2b):(5a+3b)=18:29$, then find $a:b$.

5 If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, then show that $\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{z^3}{c^3}=\frac{3xyz}{abc}$.

2015

6 Given $\frac{x^3+12x}{6x^2+8}=\frac{y^3+27y}{9y^2+27}$, using componendo and dividendo, find $x:y$.

7 If a, b and c are in continued proportion, then prove that $(a+b+c)(a-b+c)=a^2+b^2+c^2$.

2014

8 If $\frac{a^2+b^2}{a^2-b^2}=\frac{17}{8}$, then find the value of

(i) $a:b$ (ii) $(a^3+b^3):(a^3-b^3)$.

9 If $(x-9):(3x+6)$ is the duplicate ratio of 4 : 9, then find the value of x.

2013

10 What number must be added to each of the numbers 6, 15, 20 and 43 to make them proportional?

11 Using the properties of proportion, solve for x, given $\frac{x^4+1}{2x^2}=\frac{17}{8}$.

2012

12 The monthly pocket money of Ravi and Sanjeev are in the ratio 5 : 7. Their expenditures are in the ratio 3 : 5. If each save ₹ 80 every month, then find their monthly pocket money.

13 If $x=\frac{\sqrt{a+1}+\sqrt{a-1}}{\sqrt{a+1}-\sqrt{a-1}}$, then using the properties of proportion, show that $x^2-2ax+1=0$.

2011

14 If 6 is the mean proportional between two numbers x, y and 48 is the third proportional to x and y, then find the numbers.

15 Using componendo and dividendo, find the value of x in the equation $\frac{\sqrt{3x+4}+\sqrt{3x-5}}{\sqrt{3x+4}-\sqrt{3x-5}}=9$.

* All these questions are completely covered in chapter either as solved examples or in topic/chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. The ratio of the price of coffee to that of tea, when coffee costs ₹ 24 per 100 g and the tea costs ₹ 80 per kg, is

(a) 1 : 2 (b) 2 : 1 (c) 1 : 3 (d) 3 : 1

2. If $2xy$ is subtracted from the antecedent and consequent of $(x^2+y^2):(x^2+y^2+xy)$, then new ratio will be $(x^2-y^2):1$. The value of $x:y$ is

(a) $(1+y^2):(1-x^2)$ (b) $(1-y^2):(1+x^2)$ (c) $(1-y^2):(1-x^2)$ (d) $(1+y^2):(1+x^2)$

3. The number of all possible proportions from the numbers 15, 18, 35 and 42 is

(a) 1 (b) 2 (c) 3 (d) 4

4. If $\dfrac{x+y}{a^3-b^3}=\dfrac{y+z}{b^3-c^3}=\dfrac{z+x}{c^3-a^3}$, then the value of $x+y+z$ is

(a) 1 (b) 0 (c) 2 (d) 3

5. If $\dfrac{3k+4l+6m+7n}{3k+4l-6m-7n}=\dfrac{3k-4l+6m-7n}{3k-4l-6m+7n}$, then $k:2m$ is proportional to

(a) $4l/7n$ (b) $5l/8n$ (c) $6l/7n$ (d) $4l/9n$

6. If a, b, c and d are in proportion, then $\dfrac{a^3+3ab^2}{3a^2b+b^3}$ is equal to

(a) $\dfrac{c^3-3cd^2}{3c^2d+d^3}$ (b) $\dfrac{c^3+3cd^2}{3c^2d+d^3}$ (c) $\dfrac{3c^2d-d^3}{c^3+3cd^2}$ (d) $\dfrac{3c^2d+d^3}{c^3+3cd^2}$

7. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=\dfrac{121}{125}$, then the value of $\dfrac{3a+4c+9e}{3b+4d+9f}$ is

(a) 120/125 (b) 125/120 (c) 121/125 (d) 125/121

8. If $x=\dfrac{\sqrt[3]{m+1}+\sqrt[3]{m-1}}{\sqrt[3]{m+1}-\sqrt[3]{m-1}}$, then the value of x^3-3mx^2+3x is

(a) $-m$ (b) m (c) $2m$ (d) $3m$

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 397.

CHAPTER

7

Factorisation of Polynomials

In this chapter, we will study the polynomial in one variable and their classification. Here, we also study about zero of a polynomial, remainder and factor theorems and their use in factorisation of a polynomial.

Chapter Objectives

- Introduction to Polynomials and Remainder Theorem
- Factor Theorem

Topic 1

Introduction to Polynomials and Remainder Theorem

Polynomial

An algebraic expression which have only whole numbers or non-negative integers as the exponent of the variable(s), is called a polynomial. e.g. x^2+1; x^2+y^2+xy; xyz.

Polynomial in One Variable

An algebraic expression of the form $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$, where $a_0, a_1, a_2, \ldots, a_n$ are real numbers and n is a non-negative integer, is called a **polynomial** in the variable x.

Or

It is denoted by the symbols $f(x)$ or $g(x)$ or $p(x)$ or $q(x)$ etc.

Thus, $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$.

Here, $a_0x^n, a_1x^{n-1}, a_2x^{n-2}, \ldots, a_{n-1}x, a_n$ are **terms** of the polynomial $f(x)$; a_n is called the **constant term** and $a_0, a_1, a_2, \ldots, a_n$ are **coefficients** of the polynomial.

e.g. (i) $p(x)=3x^3+2x^2-7x+5$ is a polynomial in one variable x.

(ii) $q(y)=5y^2+3y$ is a polynomial in one variable y.

Important Terms Related to Polynomial

(i) **Constant polynomial** A polynomial containing only constant term, is called a **constant polynomial**, e.g. $f(x) = c$, where c is a constant.

(ii) **Zero polynomial** The constant polynomial 0 is called the **zero polynomial**.

Or

If all the coefficients of polynomial f(x) are zero, then f(x) is called the **zero polynomial**.

(iii) **Degree** The highest power of the variable in a polynomial, is known as **degree** of the polynomial.

e.g. (i) Degree of $f(x) = x^2 + 2x + 3$ is 2.

(ii) Degree of $g(x) = x^4 + x^3 + x$ is 4.

- The degree of a polynomial is zero if and only if it contains only non-zero constant term, i.e. when $f(x) = c$, where $c \neq 0$, then degree of $f(x) = 0$.
- The degree of zero polynomial is never defined.
- The degree of a polynomial cannot be negative.
- If degree of a polynomial is n and $a_0 \neq 0$, then $a_0 x^n$ is called the **leading term** and a_0 is called the **leading coefficient** of polynomial.

Polynomial Equation

If $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ is a polynomial of degree n, then $f(x) = 0$,

i.e. $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$ is called a polynomial equation of degree n.

e.g. (i) $3x^2 - 5x + \sqrt{3} = 0$ is a polynomial equation of degree 2.

(ii) $4x^3 + \sqrt{2}x^2 + 7x - 8 = 0$ is a polynomial equation of degree 3.

Equality of Two Polynomials

Let two polynomials be

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$$

and $g(x) = c_0 x^m + c_1 x^{m-1} + c_2 x^{m-2} + \ldots + c_{m-1} x + c_m$

Then, two polynomials $f(x)$ and $g(x)$ are equal, if and only if degree of $f(x)$ is equal to degree of $g(x)$ and corresponding elements are equal, i.e. $n = m$ and $a_i = c_i$ for all i.

Value of a Polynomial

The value obtained on putting a particular value of the variable in the polynomial is called the value of the polynomial. The value of a polynomial $f(x)$ at $x = a$ is denoted by $f(a)$.

e.g. Let $f(x) = 5x^3 - 2x^2 + 3x - 2$.

Then, at $x = 1$, $f(1) = 5(1)^3 - 2(1)^2 + 3(1) - 2$
$= 5 - 2 + 3 - 2 = 8 - 4 = 4$

So, 4 is the value of $f(x)$ at $x = 1$.

Example 1. Find the value of quadratic polynomial $f(x) = 8x^2 - 3x + 7$ at $x = -1$ and $x = 2$.

Sol. Given, $f(x) = 8x^2 - 3x + 7$

At $x = -1$, $f(-1) = 8(-1)^2 - 3(-1) + 7$
$= 8 \times 1 + 3 + 7 = 8 + 10 = 18$

At $x = 2$, $f(2) = 8(2)^2 - 3(2) + 7$
$= 8 \times 4 - 6 + 7 = 32 + 1 = 33$

Zero (Root) of a Polynomial

Zero of a polynomial $f(x)$ is a number α, such that $f(\alpha) = 0$. Zero of a polynomial $f(x)$ is also called a root of the polynomial equation $f(x) = 0$.

e.g. Let $f(x) = 5x + 7$.

Then, at $x = \frac{-7}{5}$, $f\left(\frac{-7}{5}\right) = 5\left(\frac{-7}{5}\right) + 7 = -7 + 7 = 0$

Hence, $x = -\frac{7}{5}$ is a zero of $f(x)$ and root of $f(x) = 0$.

Example 2. Prove that $x = 1$ is one of the root of $f(x) = 0$, where $f(x) = 3x^2 - 5x + 2$.

Sol. Given, $f(x) = 3x^2 - 5x + 2$

At $x = 1$, $f(1) = 3(1)^2 - 5(1) + 2 = 3 \times 1 - 5 + 2 = 3 - 3 = 0$

So, $x = 1$ is one of the root of quadratic equation $3x^2 - 5x + 2 = 0$.

Example 3. If one zero of the polynomial $2x^2 - 5x - (2k + 1)$ is twice the other, then find both the zeroes of the polynomial and the value of k.

Sol. Let α and 2α are the zeroes of the polynomial $2x^2 - 5x - (2k + 1)$.

Then, $2\alpha^2 - 5\alpha - (2k + 1) = 0$

and $2(2\alpha)^2 - 5(2\alpha) - (2k + 1) = 0$

$\Rightarrow$ $2\alpha^2 - 5\alpha = 2k + 1$ ...(i)

and $8\alpha^2 - 10\alpha = 2k + 1$ ...(ii)

From Eqs. (i) and (ii), we get

$$2\alpha^2 - 5\alpha = 8\alpha^2 - 10\alpha \Rightarrow 6\alpha^2 = 5\alpha \Rightarrow \alpha = \frac{5}{6} \quad [\because \alpha \neq 0]$$

$\therefore$ $2\alpha = \frac{5}{6} \times 2 = \frac{5}{3}$

Thus, the zeroes of the polynomial are $\frac{5}{6}$ and $\frac{5}{3}$.

Now, substituting $\alpha = \frac{5}{6}$ in Eq. (i), we get

$$2 \times \frac{25}{36} - \frac{25}{6} = 2k + 1$$

$\Rightarrow$ $2k + 1 = \frac{50 - 150}{36} \Rightarrow 2k + 1 = -\frac{100}{36}$

$\Rightarrow$ $2k = -\frac{100}{36} - 1 \Rightarrow 2k = -\frac{136}{36}$

$\Rightarrow$ $k = -\frac{68}{36} = \frac{-17}{9}$

Factor of a Polynomial

A non-zero polynomial $g(x)$ is said to be a factor of a polynomial $f(x)$, if and only if $f(x)$ is completely divisible by $g(x)$, i.e. remainder is 0. e.g.

(i) As $2x^2 + 7x + 6 = (2x + 3)(x + 2)$

Therefore, $(2x + 3)$ is a factor of $2x^2 + 7x + 6$.

(ii) As $x^3 - 5x^2 + 7x - 3 = (x - 3)(x^2 - 2x + 1)$

Therefore, $(x - 3)$ is a factor of $(x^3 - 5x^2 + 7x - 3)$.

Example 4. Check whether $g(x) = x + 1$ be a factor of $f(x) = 3x + 3x^2 + x^3 + 1$.

Sol. Write the dividend and divisor in the standard form as

$f(x) = x^3 + 3x^2 + 3x + 1$ and $g(x) = x + 1$

$$\begin{array}{r|l} & x^2 + 2x + 1 \\ \hline x+1 & x^3 + 3x^2 + 3x + 1 \\ & \underline{\underset{-}{x^3} + \underset{-}{x^2}} \\ & \quad 2x^2 + 3x + 1 \\ & \quad \underline{\underset{-}{2x^2} + \underset{-}{2x}} \\ & \qquad x + 1 \\ & \qquad \underline{\underset{-}{x} + \underset{-}{1}} \\ & \qquad\ 0 \end{array}$$

Here, the remainder is zero, so $g(x)$ be a factor of $f(x)$.

Remainder Theorem

Let $f(x)$ be a polynomial of degree n greater than or equal to 1 (i.e. $n \geq 1$) and a be any real number. If $f(x)$ is divided by the linear polynomial $(x - a)$, then the remainder is $f(a)$.

Note (i) If $f(x)$ is divided by $(x + a)$, then remainder is $f(-a)$.

(ii) If $f(x)$ is divided by $(ax - b)$, then remainder is $f\left(\frac{b}{a}\right)$, $a \neq 0$.

(iii) If $f(x)$ is divided by $(ax + b)$, then remainder is $f\left(-\frac{b}{a}\right)$, $a \neq 0$.

Example 5. Using remainder theorem, find the remainder when $x^3 - 5x^2 + 3x + 1$ is divided by $\left(x - \frac{1}{3}\right)$.

Sol. Let $f(x) = x^3 - 5x^2 + 3x + 1$.

Then, the remainder, when $f(x)$ is divided by $\left(x - \frac{1}{3}\right)$, is given by

$$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - 5\left(\frac{1}{3}\right)^2 + 3\left(\frac{1}{3}\right) + 1 \quad \text{[by remainder theorem]}$$

$$= \frac{1}{27} - 5\left(\frac{1}{9}\right) + \frac{3}{3} + 1 = \frac{1}{27} - \frac{5}{9} + 1 + 1$$

$$= \frac{1}{27} - \frac{5}{9} + 2 = \frac{1 - 15 + 54}{27} = \frac{55 - 15}{27} = \frac{40}{27}$$

Example 6. Using remainder theorem, find the value of x, if on dividing $7y^3 + 2y^2 - xy + 14$ by $y + 3$, leaves a remainder -49.

Sol. Let $f(y) = 7y^3 + 2y^2 - xy + 14$

Then, the remainder, when $f(y)$ is divided by $(y + 3)$, is given by

$$f(-3) = 7(-3)^3 + 2(-3)^2 - x(-3) + 14$$

[by remainder theorem]

But its remainder $= -49$ [given]

$\therefore \quad 7(-27) + 2(9) + 3x + 14 = -49$

$\Rightarrow \quad -189 + 18 + 3x + 14 = -49$

$\Rightarrow \quad -157 + 3x = -49$

$\Rightarrow \quad 3x = -49 + 157 = 108 \Rightarrow x = \frac{108}{3} = 36$

Hence, the value of x is 36.

Example 7. If the polynomials $ax^3 + 3x^2 + 5x - 4$ and $x^3 - 4x + a$ leave the same remainder, when divided by $(x - 2)$, then find the value of a.

Sol. Let $p_1(x) = ax^3 + 3x^2 + 5x - 4$ and $p_2(x) = x^3 - 4x + a$

Then, the remainders, when both $p_1(x)$ and $p_2(x)$ are divided by $(x - 2)$ are

$$p_1(2) = a(2)^3 + 3(2)^2 + 5(2) - 4$$
$$= 8a + 12 + 10 - 4 = 8a + 18$$

and $\quad p_2(2) = (2)^3 - 4(2) + a = 8 - 8 + a = a$

[by remainder theorem]

Now, according to the question,

$p_1(2) = p_2(2) \Rightarrow 8a + 18 = a$

$\Rightarrow \quad 8a - a = -18 \Rightarrow 7a = -18$

$\therefore \quad a = \frac{-18}{7}$

which is the required value of a.

Topic Exercise 1

1. Write whether the following expressions are polynomials or not.

(i) $x^3 + \frac{1}{x^3} + \frac{1}{x} + 1$ (ii) $\sqrt{2}\, y^3 + \sqrt{5}\, z$

(iii) $z^{-1/2} - 13z + 11$

2. Write the degree of the following polynomials.

(i) $5x^2 - 4x + 2$ (ii) 7

(iii) $13z^3 - 12z^2 + 11z + 5$

3. Find the value of the following polynomial at $x = -1$.

(i) $2x^2 + 5x + 7$ (ii) $\frac{2}{5}x + 1 - x^2$

4. A function f is defined by $f(x) = 144 - 16x^2$, calculate $f(2)$. Also, find the value of x, when $f(x) = 0$.

5. Find the zeroes of the following polynomials.
(i) $7y-1$ (ii) x^2-4x-5 (iii) $x^2+7x+10$

6. Find the remainder when $f(x)=2x^3-3x^2-4x-5$ is divided by $g(x)=2x+1$.

7. Find the remainder when $f(x)=12x^3-13x^2-5x+7$ is divided by $3x+2$.

8. Using remainder theorem, find the value of k, if on dividing $2x^3+3x^2-kx+5$ by $x-2$, leaves a remainder 7. ***[2016]***

9. The polynomial $p(x)=kx^3+9x^2+4x-8$, when divided by the polynomial $q(x)=x+3$, leaves the remainder (-20). Find the value of k.

10. What number should be added to $2x^3-3x^2-8x$ so that the resulting polynomial leaves the remainder 10 when divided by $2x+1$.

11. Find 'a', if the two polynomials ax^3+3x^2-9 and $2x^3+4x+a$ leave the same remainder when divided by $x+3$. ***[2015]***

12. If $2x^3+ax^2-11x+b$ leaves remainder 0 and 42 when divided by $(x-2)$ and $(x-3)$ respectively, find the values of a and b.

13. Use remainder theorem to factorize the polynomial $2x^3+3x^2-9x-10$. ***[2018]***

Hints and Answers

1. Hint In a polynomial, exponents of variable(s) are always a non-negative integer.
Ans. (i) Not a polynomial (ii) Polynomial
(iii) Not a polynomial

2. Hint The highest power of the variable occurring in the polynomial, is called the degree of the polynomial.
Ans. (i) 2 (ii) 0 (iii) 3

3. Hint (i) Let $f(x)=2x^2+5x+7$
Then, $f(-1)=2(-1)^2+5(-1)+7$
(ii) Let $f(x)=\frac{2}{5}x+1-x^2$
Then, $f(-1)=\frac{2}{5}(-1)+1-(-1)^2$
Ans. (i) 4 (ii) $\frac{-2}{5}$

4. Hint (i) Put $x=2$ in $f(x)$
(ii) $f(x)=0 \Rightarrow x^2=\frac{144}{16}=9$
Ans. $f(2)=80;\ x=\pm 3$

5. Hint Equate the given polynomial to zero and then solve it.
Ans. (i) $\frac{1}{7}$ (ii) $-1, 5$ (iii) $-2, -5$

6. Hint Required remainder $=f\left(-\frac{1}{2}\right)$ **Ans.** -4

7. Hint Required remainder $=f\left(\frac{-2}{3}\right)$ **Ans.** 1

8. Do same as Example 6. **Ans.** $k=13$

9. Do same as Example 6. **Ans.** $k=3$

10. Hint Let the number to be added be k and let the resulting polynomial be $f(x)$, i.e.
$f(x)=2x^3-3x^2-8x+k$
Now, according to given condition $f(x)$ will leave the remainder 10 when divided by $2x+1$.
Thus, $f\left(-\frac{1}{2}\right)=10.$
Ans. 7

11. Do same as Example 8.
Ans. $a=3$

12. Hint Let $f(x)=2x^3+ax^2-11x+b$.
Then, according to given conditions, we have
$$f(2)=0 \quad \text{and} \quad f(3)=42$$
$$\Rightarrow \quad 4a+b=b \quad \text{and} \quad 9a+b=21$$
On solving above equations, we get the required values of a and b.
Ans. $a=3$ and $b=-6$.

13. Hint Let $f(x)=2x^3+3x^2-9x-10$
For $x=-1$,
$$f(x)=f(-1)=2(-1)^3+3(-1)^2-9(-1)-10$$
$$=-2+3+9-10=12-12=0$$
$\therefore\ 2x^3+3x^2-9x-10=(x+1)(2x^2+x-10)$
Ans. $(x+1)(x-2)(2x+5)$

Topic 2
Factor Theorem

Let $f(x)$ be a polynomial of degree $n \geq 1$ and a be any real number.

(i) If $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.
(ii) If $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$.

Note (i) If $(x + a)$ is a factor of $f(x)$, then $f(-a) = 0$.
(ii) If $(ax - b)$ is a factor of $f(x)$, then $f\left(\frac{b}{a}\right) = 0$.
(iii) If $(ax + b)$ is a factor of $f(x)$, then $f\left(\frac{-b}{a}\right) = 0$.
(iv) If $(x - a)(x - b)$ is a factor of $f(x)$, then $f(a) = 0$ and $f(b) = 0$.

Example 1. Using factor theorem, show that
(i) $(x + 1)$ is a factor of $x^{19} + 1$.
(ii) $(x - 3)$ is a factor of $x^4 - 81$.

Sol.
(i) Let $f(x) = x^{19} + 1$
Clearly, $(x + 1)$ will be a factor of $f(x)$, if $f(-1) = 0$.
On putting $x = -1$ in $f(x)$, we get
$$f(-1) = (-1)^{19} + 1 = -1 + 1 = 0$$
Hence, by factor theorem, $(x + 1)$ is a factor of $x^{19} + 1$.
(ii) Let $f(x) = x^4 - 81$
Clearly, $(x - 3)$ will be a factor of $f(x)$, if $f(3) = 0$.
On putting $x = 3$ in $f(x)$, we get
$$f(3) = 3^4 - 81 = 81 - 81 = 0$$
Hence, by factor theorem, $(x - 3)$ is a factor of $f(x)$.

Example 2. What number should be subtracted from $2x^3 - 5x^2 + 5x$, so that the resulting polynomial has a factor $2x - 3$?

Sol. Let the number to be subtracted from the given polynomial be k and let the resulting polynomial be $f(x)$, i.e.
$$f(x) = 2x^3 - 5x^2 + 5x - k \quad \text{...(i)}$$
Then, according to given condition, $(2x - 3)$ will be factor of $f(x)$.
So, by factor theorem, $f\left(\frac{3}{2}\right) = 0$.
Now, putting the value of $x = \frac{3}{2}$ in Eq. (i), we get
$$\Rightarrow \quad 2\left(\frac{3}{2}\right)^3 - 5\left(\frac{3}{2}\right)^2 + 5\left(\frac{3}{2}\right) - k = 0$$
$$\Rightarrow \quad 2 \times \frac{27}{8} - 5 \times \frac{9}{4} + \frac{15}{2} - k = 0$$
$$\Rightarrow \quad k = \frac{27}{4} - \frac{45}{4} + \frac{15}{2} = \frac{27 - 45 + 30}{4}$$
$$= \frac{27 - 15}{4} = \frac{12}{4} = 3$$
$$\therefore \quad k = 3$$
Hence, the required value of k is 3.

Example 3. Given that $x + 2$ and $x + 3$ are factors of $2x^3 + ax^2 + 7x - b$. Determine the values of a and b. ***[2009]***

Sol. Let $f(x) = 2x^3 + ax^2 + 7x - b$ ...(i)
Since, $(x + 2)$ is a factor of $f(x)$.
$$\therefore \quad f(-2) = 0 \quad \text{[by factor theorem]}$$
$$\Rightarrow \quad 2(-2)^3 + a(-2)^2 + 7(-2) - b = 0$$
[putting $x = -2$ in Eq. (i)]
$$\Rightarrow \quad 2(-8) + 4a - 14 - b = 0$$
$$\Rightarrow \quad -16 + 4a - 14 - b = 0$$
$$\Rightarrow \quad -30 + 4a - b = 0$$
$$\Rightarrow \quad 4a - b = 30 \quad \text{...(ii)}$$
Also, $(x + 3)$ is a factor of $f(x)$.
$$\therefore \quad f(-3) = 0 \quad \text{[by factor theorem]}$$
$$\Rightarrow \quad 2(-3)^3 + a(-3)^2 + 7(-3) - b = 0$$
[putting $x = -3$ in Eq. (i)]
$$\Rightarrow \quad 2 \times (-27) + 9a - 21 - b = 0$$
$$\Rightarrow \quad -54 + 9a - 21 - b = 0$$
$$\Rightarrow \quad -75 + 9a - b = 0$$
$$\Rightarrow \quad 9a - b = 75 \quad \text{...(iii)}$$
On subtracting Eq. (ii) from Eq. (iii), we get
$$\begin{array}{r} 9a - b = 75 \\ 4a - b = 30 \\ - \quad + \quad - \\ \hline 5a = 45 \\ \hline \end{array}$$
$$\Rightarrow \quad a = \frac{45}{5} = 9$$
On putting $a = 9$ in Eq. (ii), we get
$$4 \times 9 - b = 30$$
$$\Rightarrow \quad 36 - b = 30$$
$$\Rightarrow \quad -b = 30 - 36$$
$$\Rightarrow \quad -b = -6 \Rightarrow b = 6$$
Hence, $a = 9$ and $b = 6$.

Example 4. Using the factor theorem, factorise the polynomial $2x^2 + 10x + 12$.

Sol. Let $f(x) = 2x^2 + 10x + 12$
Using hit and trial method,
putting $x = 1$, we get
$$f(1) = 2(1)^2 + 10(1) + 12 = 2 + 10 + 12 = 24 \neq 0$$
putting $x = -1$, we get
$$f(-1) = 2(-1)^2 + 10(-1) + 12 = 2 - 10 + 12 = 4 \neq 0$$
putting $x = 2$, we get
$$f(2) = 2(2)^2 + 10(2) + 12 = 8 + 20 + 12 = 40 \neq 0$$
putting $x = -2$, we get
$$f(-2) = 2(-2)^2 + 10(-2) + 12$$
$$= 8 - 20 + 12 = 20 - 20 = 0$$
So, by factor theorem, $(x + 2)$ will be a factor of $f(x)$.

Now, to find other factor, let us divide $f(x)$ by $(x+2)$ using long division method.

$$\begin{array}{r|l} & 2x+6 \\ \hline x+2 & 2x^2+10x+12 \\ & 2x^2+4x \\ & - \quad - \\ \hline & \quad 6x+12 \\ & \quad 6x+12 \\ & \quad - \quad - \\ \hline & \quad\quad 0 \end{array}$$

Clearly, $(2x+6)$ is another factor of $f(x)$.

Hence, $f(x)=(x+2)(2x+6)$.

Example 5. Using the factor theorem, factorise completely the polynomial

$$3x^3+2x^2-19x+6 \qquad \textbf{[2012]}$$

Sol. Let $f(x)=3x^3+2x^2-19x+6$

Using hit and trial method,

putting $x=1$, we get

$$f(1)=3(1)^3+2(1)^2-19(1)+6$$
$$=3+2-19+6=-8\neq 0$$

putting $x=-1$, we get

$$f(-1)=3(-1)^3+2(-1)^2-19(-1)+6$$
$$=-3+2+19+6$$
$$=-3+27=24\neq 0$$

putting $x=2$, we get

$$f(2)=3(2)^3+2(2)^2-19(2)+6$$
$$=3\times 8+2\times 4-38+6$$
$$=24+8-38+6$$
$$=38-38=0$$

So, by factor theorem, $(x-2)$ will be a factor of $f(x)$.

Now, to find other factor, let us divide $f(x)$ by $(x-2)$ using long division method.

$$\begin{array}{r|l} & 3x^2+8x-3 \\ \hline x-2 & 3x^3+2x^2-19x+6 \\ & 3x^3-6x^2 \\ & - \quad + \\ \hline & \quad 8x^2-19x+6 \\ & \quad 8x^2-16x \\ & \quad - \quad + \\ \hline & \quad\quad -3x+6 \\ & \quad\quad -3x+6 \\ & \quad\quad + \quad - \\ \hline & \quad\quad\quad 0 \end{array}$$

Clearly, $(3x^2+8x-3)$ is another factor of $f(x)$.

$\therefore \qquad f(x)=(x-2)(3x^2+8x-3) \qquad$ ...(i)

Now, factorise $3x^2+8x-3=3x^2+9x-x-3$

[by splitting the middle term]

$$=3x(x+3)-1(x+3)=(3x-1)(x+3)$$

Hence, from Eq. (i), we get

$$f(x)=(x-2)(3x-1)(x+3)$$

Example 6. The polynomial px^3+4x^2-3x+q is completely divisible by x^2-1, find the values of p and q. Also, for these values of p and q factorise the given polynomial completely.

Sol. Let $f(x)=px^3+4x^2-3x+q$

$\because$ $f(x)$ is divisible by x^2-1

$\Rightarrow f(x)$ is divisible by $(x+1)(x-1)$

$\Rightarrow$ $(x+1)$ and $(x-1)$ both are factors of $f(x)$.

Now, by factor theorem, we get $f(-1)=0$

$\Rightarrow \quad p(-1)^3+4(-1)^2-3(-1)+q=0$

$\Rightarrow \quad -p+4+3+q=0$

$\Rightarrow \quad -p+q=-7 \qquad$ ...(i)

Also, we get $f(1)=0$

$\Rightarrow \quad p(1)^3+4(1)^2-3(1)+q=0$

$\Rightarrow \quad p+4-3+q=0$

$\Rightarrow \quad p+q=-1 \qquad$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$\begin{array}{r} -p+q=-7 \\ p+q=-1 \\ \hline 2q=-8 \\ \hline \end{array}$$

$\Rightarrow \quad q=-4$

On putting $q=-4$ in Eq. (i), we get

$$-p+(-4)=-7 \Rightarrow -p=-7+4 \Rightarrow -p=-3 \Rightarrow p=3$$

Hence, $p=3$ and $q=-4$

$\therefore$ $f(x)$ becomes $3x^3+4x^2-3x-4$.

Now, let us divide $f(x)$ by (x^2-1) using long division method.

$$\begin{array}{r|l} & 3x+4 \\ \hline x^2-1 & 3x^3+4x^2-3x-4 \\ & 3x^3 \qquad\quad -3x \\ & - \qquad\quad\ + \\ \hline & \quad\quad 4x^2-4 \\ & \quad\quad 4x^2-4 \\ & \quad\quad - \quad + \\ \hline & \quad\quad\quad 0 \end{array}$$

Hence, $f(x)=(x^2-1)(3x+4)=(x+1)(x-1)(3x+4)$.

Topic Exercise 2

1. Using factor theorem, check whether the polynomial $g(x) = x - 2$ is a factor of $f(x) = x^3 - 4x^2 + x + 6$ or not.

2. If $(x + 2)$ and $(x + 3)$ are factors of $x^3 + ax + b$, find the values of 'a' and 'b'. ***[2018]***

3. Find the value of k, if $(x - 2)$ is a factor of $x^3 + 2x^2 - kx + 10$. Hence, determine whether $(x + 5)$ is also a factor. ***[2011]***

4. For what value of 'a', the polynomial $g(x)$ is a factor of $f(x)$. $f(x) = x^3 + a(x^2 + 1) - 2x + 4$, $g(x) = x + a$

5. What must be subtracted from $16x^3 - 8x^2 + 4x + 7$, so that the resulting expression has $(2x + 1)$ as a factor? ***[2017]***

6. Show that $(x - 2)$ is a factor of $7x^2 - 10x - 8$. Hence, factorise $7x^2 - 10x - 8$.

7. Show that $(2x + 7)$ is a factor of $2x^3 + 7x^2 - 4x - 14$. Hence, factorise $2x^3 + 7x^2 - 4x - 14$.

8. Use factor theorem to factorise completely $x^3 + x^2 - 4x - 4$.

9. Use factor theorem to factorise the polynomial $x^3 - 13x - 12$ completely. Hence, solve the corresponding polynomial equation.

10. If $(x - 2)$ is a factor of the expression $2x^3 + ax^2 + bx - 14$ and when the expression is divided by $(x - 3)$, it leaves a remainder 52. Find the values of a and b. ***[2013]***

Hints and Answers

1. Do same as Example 1.

Ans. Yes, $(x - 2)$ is a factor of $f(x)$.

2. Hint Let $f(x) = 6x^3 + 11x^2 - 3x - 2$

Then, show that $f(-2) = 0$ and $f\left(-\frac{1}{3}\right) = 0$.

3. Hint Let $f(x) = x^3 + ax + b$

Since, $(x + 2)$ is a factor of $f(x)$, therefore $f(-2) = 0$

Also, $(x + 3)$ is factor of $f(x)$, therefore $f(-3) = 0$

Ans. $a = -19$ and $b = -30$

4. Hint Let $g(x)$ is factor of $f(x)$.

Then, $f(-a) = 0$.

$\Rightarrow -a^3 + a(a^2 + 1) + 2a + 4 = 0$

Ans. $g(x) = -\frac{4}{3}$.

5. Do same as Example 2.

Ans. 1

6. Hint Let $f(x) = 7x^2 - 10x - 8$. Then, show that $f(2) = 0$. Further divide $f(x)$ by $(x - 2)$ using long division method.

Ans. $(x - 2)(7x + 4)$.

7. Hint Let $f(x) = 2x^3 + 7x^2 - 4x - 14$.

Then, show that $f\left(-\frac{7}{2}\right) = 0$.

Further divide $f(x)$ by $(2x + 7)$ using long division method.

Ans. $(x + \sqrt{2})(x - \sqrt{2})(2x + 7)$.

8. Do same as Example 5.

Ans. $(x - 2)(x + 1)(x + 2)$

9. Hint

(i) Do same as Example 5.

(ii) Consider, $x^3 - 13x - 12 = 0$

$\Rightarrow (x + 1)(x + 3)(x - 4) = 0$

$\Rightarrow \quad x = -1, -3, 4$

Ans. $(x + 1)(x + 3)(x - 4); -1, -3, 4$

10. Hint Here, we have $f(2) = 0$ and $f(3) = 52$

$\Rightarrow \quad 2a + b + 1 = 0 \quad \ldots(i)$

and $\quad 3a + b - 4 = 0 \quad \ldots(ii)$

Ans. $a = 5$ and $b = -11$

CHAPTER EXERCISE

a 3 Marks Questions

1. Find the remainder (without division) on dividing $2x^3 + x^2 - 7x + 4$ by $(x - 2)$.
2. Without actual division, find the remainder, if $p(x) = 9x^2 - 6x + 2$ is divided by $(3x - 2)$.
3. Using remainder theorem, find the remainder when $f(x) = 3x^4 + 2x^3 - \frac{x^2}{3} - \frac{x}{9} + \frac{2}{27}$ is divided by $g(x) = x + \frac{2}{3}$.
4. When $x^3 - 3x^2 - kx + 4$ is divided by $(x - 3)$, then the remainder is k. Find the value of the constant k.
5. When divided by $(y - 2)$, the polynomials $py^3 + 3y^2 - 13$ and $2y^3 - 5y + p$ leaves the same remainder. Find the value of p.
6. The polynomials $3x^3 - ax^2 + 5x - 13$ and $(a+1)x^2 - 7x + 5$ leaves the same remainder when divided by $(x - 3)$. Find the value of a.
7. What number must be added to $2x^3 - 7x^2 + 2x$, so that the resulting polynomial leaves the remainder -2 when divided by $2x - 3$?
8. Determine, whether the polynomial $g(x) = x - 7$ is a factor of $f(x) = x^3 - 6x^2 - 19x + 84$ or not.
9. Show that $(x - 5)$ is a factor of $2x^2 - 9x - 5$. Hence, factorise $2x^2 - 9x - 5$.
10. Show that $(x - 1)$ is a factor of $x^3 - 7x^2 + 14x - 8$. Hence, completely factorise the above expression. ***[2007]***
11. Use the factor theorem to factorise the following expression $2x^3 + x^2 - 13x + 6$. ***[2010]***
12. Using the remainder and factor theorem, factorise the following polynomial
$$x^3 + 10x^2 - 37x + 26$$ ***[2014]***
13. Using the remainder and factor theorem, factorise the polynomial $2y^3 + y^2 - 2y - 1$.
14. Factorise the expression
$$f(x) = 2x^3 - 7x^2 - 3x + 18$$
Hence, find all possible values of x for which $f(x) = 0$.
15. If $(y - p)$ is a common factor of the polynomials $f(y) = y^2 + ay + b$ and $g(y) = y^2 + my + n$, then show that $p = \frac{n - b}{a - m}$.
16. Find the value of the constants a and b, if $(x - 2)$ and $(x + 3)$ are both factors of the expression $x^3 + ax^2 + bx - 12$.
17. Find b and c such that $(y + 1)$ and $(y + 2)$ are the factors of the polynomial $y^3 + by^2 - cy + 10$.
18. $(x - 2)$ is a factor of expression $x^3 + ax^2 + bx + 6$. When this expression is divided by $(x - 3)$, it leaves the remainder 3. Find the values of a and b.

b 4 Marks Questions

19. Show that $(2x + 3)$ is a factor of $4x^3 + 20x^2 + 33x + 18$. Hence, factorise the given polynomial completely by using factor theorem.
20. Show that $(2x + 7)$ is factor of $2x^3 + 5x^2 - 11x - 14$. Hence, factorise the given expression completely, using factor theorem.
21. Show that $(3z + 10)$ is a factor of $9z^3 - 27z^2 - 100z + 300$. Factorise the given polynomial completely by using factor theorem. $(3z + 10)$ is a factor of it.
22. Factorise $x^3 - 23x^2 + 142x - 120$, using factor theorem.
23. Using factor theorem, factorise the polynomial $x^3 + 6x^2 + 11x + 6$. Hence, solve the given polynomial equation.
24. If $(-px^2 + qx + 6 + x^3)$ has $(x - 1)$ as a factor and leaves a remainder 4 when divided by $(x + 1)$, then find the values of p and q.
25. If $2y^3 + py^2 + qy - 6$ has a factor $(2y + 1)$ and leaves the remainder 12 when divided by $(y + 2)$.

    Calculate the values of p and q, hence factorise the expression completely.

Hints and Answers

1. **Hint** Put $x = 2$ in $2x^3 + x^2 - 7x + 4$. **Ans.** 10

2. **Hint** If a polynomial $f(x)$ is divided by $ax - b$, $a \neq 0$, then remainder $= f\left(\frac{b}{a}\right)$. **Ans.** 2

3. **Hint** Required remainder $= f\left(-\frac{2}{3}\right)$ **Ans.** 0

4. **Hint** Here, we have $f(3) = k$ **Ans.** $k = 1$

5. Do same as Example 7 of Topic 1. **Ans.** $p = 1$

6. Do same as Example 7 of Topic 1. **Ans.** $a = 5$

7. **Hint** Let $f(x) = 2x^3 - 7x^2 + 2x + k$, where k is the required number.
Then, according to the questions $f\left(\frac{3}{2}\right) = -2$. **Ans.** 4

8. **Hint** Check whether $f(7) = 0$ or not.
Ans. Yes

9. **Hint** Let $f(x) = 2x^2 - 9x - 5$. Then, show that $f(5) = 0$. Further divide $f(x)$ by $(x - 5)$ using long division method.
Ans. $(2x + 1)(x - 5)$.

10. **Hint** Let $f(x) = x^3 - 7x^2 + 14x - 8$.
Then, show that $f(1) = 0$. Further, divide $f(x)$ by $(x - 1)$ using long division method.
Ans. $(x - 1)(x - 2)(x - 4)$

11. Do same as Example 5 of Topic 2.
Ans. $(x - 2)(2x - 1)(x + 3)$.

12. Do same as Example 5 of Topic 2.
Ans. $(x - 1)(x - 2)(x + 13)$

13. Do same as Example 5 of Topic 2.
Ans. $(y - 1)(y + 1)(2y + 1)$.

14. **Hint** First, do same as Example 5 of Topic 2. Secondly, put each factor of $f(x)$ equal to zero, for obtaining the value of x.
Ans. $f(x) = (x - 3)(x - 2)(2x + 3)$; $x = 3, 2, \frac{-3}{2}$.

15. **Hint** Here, we have $f(p) = 0$ and $g(p) = 0$
$\Rightarrow \quad f(p) = g(p)$

16. Do same as Example 3 of Topic 2.
Ans. $a = 3, b = -4$

17. Do same as Example 3 of Topic 2.
Ans. $b = 8, c = -17$

18. **Hint** Let $f(x) = x^3 + ax^2 + bx + 6$
Then, we have $f(2) = 0$ and $f(3) = 3$

$$\Rightarrow \quad 2a + b + 7 = 0 \quad \ldots(i)$$

$$\text{and} \quad 3a + b + 10 = 0 \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get the required values.
Ans. $a = -3, b = -1$

19. **Hint** Let $f(x) = 4x^3 + 20x^2 + 33x + 18$
Then, show that $f\left(-\frac{3}{2}\right) = 0$.
Further, divide $f(x)$ by $(2x + 3)$ using long division method.
Ans. $(x + 2)(2x + 3)(2x + 3)$

20. Do same as Q. 19.
Ans. $(2x + 7)(x - 2)(x + 1)$

21. Do same as Q. 19.
Ans. $(3z + 10)(3z - 10)(z - 3)$

22. Do same as Example 5 of Topic 2.
Ans. $(x - 1)(x - 10)(x - 12)$

23. **Hint** First, do same as Example 5 of Topic 2 and then put each factor equal to zero.
Ans. $(x + 1)(x + 2)(x + 3); -1, -2, -3$

24. Do same as Q. 18.
Ans. $p = 4, q = -3$

25. **Hint** Let $f(y) = 2y^3 + py^2 + qy - 6$. Then, we have

$$f\left(-\frac{1}{2}\right) = 0 \text{ and } f(-2) = 12$$

$$\Rightarrow \quad p - 2q = 25$$

$$\text{and} \quad 2p - q = 17$$

On solving above equations, we get the required values of p and q.
Further, divide the given polynomial by $2y + 1$.
Ans. $p = 3, q = -11; (y - 2)(2y + 1)(y + 3)$

CHAPTER

8

Matrices

The evolution of concept of matrices is the result of an attempt to obtain compact and simple method of solving system of linear equations. It is compact notation for describing sets of data, sets of equations and electronic spreadsheet. The word 'Matrices' is plural of 'Matrix'. In this chapter, we will study the fundamentals of matrix and matrix algebra.

Chapter Objectives

- Matrix and Its Types
- Operations on Matrices

Topic 1
Matrix and Its Types

Matrix

A set of mn numbers arranged in the form of a rectangular array of m rows (horizontal lines) and
n columns (vertical lines), is called $m \times n$ matrix. It is read as 'm by n' matrix and it is denoted by capital letters. Each number in the matrix, i.e. $a_{11}, a_{12}, a_{13}, \ldots$, etc., is known as the element of the matrix. Horizontal lines are called **rows** and vertical lines are called **columns**. The elements of matrix are always enclosed in brackets [ ] or ( ).

The general form of matrix is given as $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

In the compact form, the above matrix is represented as

$$A = [a_{ij}]_{m \times n} \quad \text{or} \quad A = [a_{ij}], \text{ where } 1 \le i \le m \text{ and } 1 \le j \le n.$$

The element a_{ij} belongs to ith row and jth column and it is called (i, j)th element of the matrix $A = [a_{ij}]$. In notation, $A = [a_{ij}]_{m \times n}$, the first subscript m always denotes the number of rows and the second subscript n always denotes the number of columns.

Order of a Matrix

A matrix having m rows and n columns is called a matrix of order $m \times n$ and read as 'm by n'. By convention, number of rows are listed first and columns, second.

e.g. Order of matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$ is 2×2.

Number of Elements in a Matrix

The total number of elements in a matrix is equal to the product of its number of rows and columns. e.g. if a matrix has 2 rows and 3 columns, then the number of elements $= 2 \times 3 = 6$.

Note The number of elements in an $m \times n$ **matrix** will be equal to mn.

Example 1. If a matrix has 30 elements, then what are the possible orders it can have?

Sol. Given, number of elements = 30

$\therefore \quad mn = 30$

All factors of 30 are 1, 2, 3, 5, 6, 10, 15 and 30.

Now, the possible ordered pairs whose element's product is 30, are (1, 30), (2, 15), (3, 10), (5, 6), (30, 1), (15, 2), (10, 3) and (6, 5).

Hence, all possible orders of matrix are

1×30, 2×15, 3×10, 5×6, 30×1, 15×2, 10×3 and 6×5.

Formation of a Matrix

Sometimes, a relation between the elements and its position is given to us. Then, to form a matrix, whose elements satisfies this relation, we put different values of i and j according to the given order and find all the elements of matrix and then write the matrix of given order.

Example 2. Construct a matrix of order 3×2, whose elements are determined by $a_{ij} = \dfrac{i+j}{2}$.

Sol. Let A be a matrix of order 3×2.

Then, $$A = [a_{ij}]_{3 \times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad \text{...(i)}$$

Given, $$a_{ij} = \frac{i+j}{2} \quad \text{...(ii)}$$

On putting $i = 1, 2, 3$ and $j = 1, 2$ in Eq. (ii), we get

$$a_{11} = \frac{1+1}{2} = 1$$
$$a_{12} = \frac{1+2}{2} = \frac{3}{2}$$
$$a_{21} = \frac{2+1}{2} = \frac{3}{2}$$
$$a_{22} = \frac{2+2}{2} = 2$$
$$a_{31} = \frac{3+1}{2} = 2 \text{ and } a_{32} = \frac{3+2}{2} = \frac{5}{2}$$

On putting all the values in Eq. (i), we get

$$A = \begin{bmatrix} 1 & 3/2 \\ 3/2 & 2 \\ 2 & 5/2 \end{bmatrix}$$

which is the required 3×2 order matrix.

Example 3. Construct a 3×4 matrix, whose elements are given by $a_{ij} = \dfrac{1}{2}(-3i + j)$.

Sol. Let A be a matrix of order 3×4.

Then, $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3 \times 4} \quad \text{...(i)}$$

Given, $$a_{ij} = \frac{1}{2}(-3i + j) \quad \text{...(ii)}$$

On putting the values of $i = 1, 2, 3$ and $j = 1, 2, 3, 4$ in Eq. (ii), we get

$$a_{11} = \frac{1}{2}(-3 \times 1 + 1) = \frac{1}{2}(-3+1) = \frac{1}{2}(-2) = -1$$
$$a_{12} = \frac{1}{2}(-3 \times 1 + 2) = \frac{1}{2}(-3+2) = \frac{1}{2}(-1) = -\frac{1}{2}$$
$$a_{13} = \frac{1}{2}(-3 \times 1 + 3) = \frac{1}{2}(-3+3) = 0$$
$$a_{14} = \frac{1}{2}(-3 \times 1 + 4) = \frac{1}{2}(-3+4) = \frac{1}{2}$$
$$a_{21} = \frac{1}{2}(-3 \times 2 + 1) = \frac{1}{2}(-6+1) = -\frac{5}{2}$$
$$a_{22} = \frac{1}{2}(-3 \times 2 + 2) = \frac{1}{2}(-6+2) = \frac{-4}{2} = -2$$
$$a_{23} = \frac{1}{2}(-3 \times 2 + 3) = \frac{1}{2}(-6+3) = -\frac{3}{2}$$
$$a_{24} = \frac{1}{2}(-3 \times 2 + 4) = \frac{1}{2}(-6+4) = \frac{1}{2}(-2) = -1$$
$$a_{31} = \frac{1}{2}(-3 \times 3 + 1) = \frac{1}{2}(-9+1) = \frac{1}{2}(-8) = -4$$
$$a_{32} = \frac{1}{2}(-3 \times 3 + 2) = \frac{1}{2}(-9+2) = -\frac{7}{2}$$
$$a_{33} = \frac{1}{2}(-3 \times 3 + 3) = \frac{1}{2}(-9+3) = \frac{1}{2}(-6) = -3$$
$$a_{34} = \frac{1}{2}(-3 \times 3 + 4) = \frac{1}{2}(-9+4) = -\frac{5}{2}$$

On putting the above values in Eq. (i), we get

$$A = \begin{bmatrix} -1 & -1/2 & 0 & 1/2 \\ -5/2 & -2 & -3/2 & -1 \\ -4 & -7/2 & -3 & -5/2 \end{bmatrix}_{3 \times 4}$$

which is the required matrix of order 3×4.

Types of Matrices

There are various types of matrices, which are as follow

1. **Row matrix** A matrix having only one row and some columns, is called a row matrix.

   e.g. $[1 \;\; 2 \;\; 3]_{1 \times 3}$ [one row and three columns]

   In general, row matrix $A = [a_{ij}]_{1 \times n}$

   where, n is the number of columns.

2. **Column matrix** A matrix having only one column and some rows, is called a column matrix.

   e.g. $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3 \times 1}$ [one column and three rows]

   In general column matrix $B = [b_{ij}]_{m \times 1}$, where m is number of rows.

3. **Zero matrix or null matrix** If all the elements of a matrix are zero, then it is called a zero or null matrix and it is denoted by capital latter 'O'.

e.g. $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2 \times 3}$

4. **Square matrix** A matrix in which number of rows and number of columns are equal, is called a square matrix. e.g. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2 \times 2}$ [here, order is 2×2]

In general square matrix $A = [a_{ij}]_{m \times m}$ or $[a_{ij}]_{n \times n}$.

where, number of rows = number of columns = m or n.

Note (i) We call a matrix A of order $n \times n$ a square matrix of order n.
(ii) A matrix, which is not a square, is called **rectangular matrix**.

5. **Diagonal matrix** A square matrix $A = [a_{ij}]_{n \times n}$ is said to be a diagonal matrix, if all the elements lying outside the diagonal elements are zero.

e.g. $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}_{3 \times 3}$ Principal diagonal

Note If $A = [a_{ij}]$ is any square matrix of order n, then elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called the diagonal elements and they constitute the diagonal of the matrix A.

6. **Scalar matrix** A diagonal matrix in which all diagonal elements are equal, is called a scalar matrix. In other words, a square matrix $A = [a_{ij}]_{n \times n}$ is said to be scalar, if $a_{ij} = \begin{cases} \text{constant } (k) & \text{when } i = j \\ 0, & \text{when } i \neq j \end{cases}$

e.g. $A = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}_{3 \times 3}$

7. **Identity matrix or Unit matrix** A square matrix having 1 (one) on its principal diagonal and 0 (zero) elsewhere, is called an identity matrix. In other words, a square matrix $A = [a_{ij}]_{n \times n}$ is an identity matrix, if $a_{ij} = \begin{cases} 1, \text{ when } i = j \\ 0, \text{ when } i \neq j \end{cases}$.

Identity matrix of order n is denoted by I_n or simply I.

e.g. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$

Note Identity matrix is always a scalar matrix.

Example 4. Classify the types of matrices and also write their orders.

(i) $\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 7 \\ 3 \\ -2 \end{bmatrix}$

(iv) $[1 \ 2 \ 3 \ 4]$ (v) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(vii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Sol.
(i) The given matrix is a square matrix of order 2×2.
(ii) The given matrix is a diagonal matrix of order 3×3.
(iii) The given matrix is a column matrix of order 3×1.
(iv) The given matrix is a row matrix of order 1×4.
(v) The given matrix is an identity matrix of order 3×3.
(vi) The given matrix is a scalar matrix of order 3×3.
(vii) The given matrix is a null matrix of order 2×3.

Example 5. Write the null matrix denoted by each of the following.

(i) $O_{1 \times 3}$ (ii) $O_{2 \times 1}$ (iii) $O_{3 \times 4}$ (iv) $O_{2 \times 2}$

Sol. (i) $O_{1 \times 3} = [0 \ 0 \ 0]$ (ii) $O_{2 \times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(iii) $O_{3 \times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (iv) $O_{2 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Equality of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal, if their orders are same and their corresponding elements are also equal, i.e. $a_{ij} = b_{ij}, \ \forall \ i$ and j. Symbolically, if two matrices A and B are equal, then we write it as $A = B$.

e.g. $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ are equal matrices, but $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}$ are not equal matrices.

Example 6. If $\begin{bmatrix} x - y & 2x + z \\ 2x - y & 3z + w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$, then find the values of x, y, z and w.

Sol. Given, $\begin{bmatrix} x - y & 2x + z \\ 2x - y & 3z + w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

Here, both matrices are equal, so to find x, y, z and w, we equate the corresponding elements.

On equating the corresponding elements, we get

$$x - y = -1 \quad \text{...(i)}$$
$$2x + z = 5 \quad \text{...(ii)}$$
$$2x - y = 0 \quad \text{...(iii)}$$

and $\qquad 3z + w = 13 \qquad$ ...(iv)

On solving Eqs. (i) and (iii), we get

$$x = 1 \text{ and } y = 2$$

On putting the value of x in Eq. (ii), we get

$$2 \times 1 + z = 5 \Rightarrow z = 5 - 2 = 3$$

On putting the value of z in Eq. (iv), we get

$$3 \times 3 + w = 13 \Rightarrow w = 13 - 9 = 4$$

Hence, $x = 1$, $y = 2$, $z = 3$ and $w = 4$.

Topic Exercise 1

1. In the matrix $A = \begin{bmatrix} a & x^2 \\ y^2 & b \end{bmatrix}$, write

(i) the order of the matrix.

(ii) the number of elements.

(iii) the elements a_{21}, a_{12} and a_{22}.

2. Write the order of the following matrix.

$$B = \begin{bmatrix} 3 & -2 & 5 & 4 \\ 1 & 0 & 9 & -5 \\ -4 & 2 & -3 & -4 \end{bmatrix}$$

Also, find the elements b_{24} and b_{33}. Show that $b_{13} + b_{24} = b_{14} + b_{34}$.

3. If a matrix has 24 elements, then what are the possible orders it can have ?

4. If a matrix has 28 elements, then what are the possible orders it can have ? What, if it has 13 elements?

5. Construct a 2×2 matrix, whose elements are given by $a_{ij} = i \cdot j$.

6. Construct a 2×2 matrix, whose elements a_{ij} are given by $a_{ij} = \left(\dfrac{i-j}{i+j}\right)^2$.

7. Construct a 3×2 matrix, whose elements are given by $a_{ij} = e^{ix} \cdot \sin jx$.

8. Identify the types of matrices given below.

(i) $[1 \;\; 5 \;\; 7]$ (ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$

9. Find the values of x, y, z and u, if

$$\begin{bmatrix} x & -2 \\ y & 7 \end{bmatrix} = \begin{bmatrix} 2 & z \\ 3 & z+u \end{bmatrix}.$$

10. Find the value of each variable in the following matrix.

$$\begin{bmatrix} 5-x \\ y+6 \\ 3z \\ 6-2u \end{bmatrix} = O_{4 \times 1}.$$

Hints and Answers

1. (i) **Hint** Here, A has two rows and two columns.
Ans. 2×2

(ii) **Hint** Number of elements in an $m \times n$ matrix $= mn$.
Ans. 4

(iii) **Hint** a_{21} = Element of second row and first column.
a_{12} = Element of first row and second column.
a_{22} = Element of second row and second column.
Ans. $a_{21} = y^2$, $a_{12} = x^2$ and $a_{22} = b$

2. **Hint** Here, B has 3 rows and 4 columns.
Now, b_{ij} = Element of ith row and jth column.
Ans. 3×4; $b_{24} = -5$, $b_{33} = -3$

3. Do same as Example 1.
Ans.
$1 \times 24, 2 \times 12, 3 \times 8, 4 \times 6, 6 \times 4, 8 \times 3, 12 \times 2, 24 \times 1$.

4. Do same as Example 1.
Ans. 1×28, 2×14, 4×7, 7×4, 14×2 and 28×1;
1×13 and 13×1

5. Do same as Example 2. **Ans.** $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{2 \times 2}$

6. Do same as Example 2. **Ans.** $\begin{bmatrix} 0 & 1/9 \\ 1/9 & 0 \end{bmatrix}_{2 \times 2}$

7. Do same as Example 2. **Ans.** $\begin{bmatrix} e^x \sin x & e^x \sin 2x \\ e^{2x} \sin x & e^{2x} \sin 2x \\ e^{3x} \sin x & e^{3x} \sin 2x \end{bmatrix}_{3 \times 2}$

8. (i) Do same as Example 4 (iv). **Ans.** Row matrix

(ii) Do same as Example 4 (v). **Ans.** Identity matrix

(iii) Do same as Example 4 (iii). **Ans.** Column matrix

9. Do same as Example 6.
Ans. $x = 2$, $y = 3$, $z = -2$ and $u = 9$

10. **Hint** $\begin{bmatrix} 5-x \\ y+6 \\ 3z \\ 6-2u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Now, proceed as in Example 6.
Ans. $x = 5$, $y = -6$, $z = 0$ and $u = 3$

Topic 2
Operations on Matrices

Here, we will discuss certain operations on matrices, namely addition of matrices, subtraction of matrices, scalar multiplication of a matrix and multiplication of matrices.

Addition of Matrices

Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be two matrices of same order, say $m \times n$. Then, the sum of two matrices A and B is defined as a matrix $C=[c_{ij}]_{m\times n}$, where $c_{ij}=a_{ij}+b_{ij}$ for all possible values of i and j, i.e. the sum of matrices A and B is a matrix whose elements are obtained by adding the corresponding elements of A and B.

e.g. $\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}+\begin{bmatrix} 6 & 9 \\ -4 & 1 \end{bmatrix}=\begin{bmatrix} 2+6 & 3+9 \\ 0+(-4) & -1+1 \end{bmatrix}=\begin{bmatrix} 8 & 12 \\ -4 & 0 \end{bmatrix}$

Compatibility for Addition

For addition of two matrices, both matrices should be of same order, otherwise we cannot add them.

e.g. If $A=\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, then $A+B$ is not defined, because A and B are not of same order.

Properties of Addition of Matrices

Let $A=[a_{ij}]$, $B=[b_{ij}]$ and $C=[c_{ij}]$ be the three matrices of same order $m \times n$. Then, the following properties hold

(i) **Commutative law** Matrix addition is commutative, i.e. $A+B=B+A$.

(ii) **Associative law** Matrix addition is associative, i.e. $A+(B+C)=(A+B)+C$.

(iii) **Existence of additive identity** For every matrix A, there exists zero matrix O, of same order such that
$$A+O=O+A=A.$$
Here, the matrix O is called the additive identity.

(iv) **Existence of additive inverse** For every matrix A, there exists another matrix $(-A)$, obtained by multiplying the elements of A by (-1), such that
$$A+(-A)=O=(-A)+A.$$
Here, the matrix $(-A)$ is called the additive inverse of A.

Example 1. If $A=\begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 4 & -3 \\ 1 & 5 \end{bmatrix}$, then find $A+B$.

Sol. Given, $A=\begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 4 & -3 \\ 1 & 5 \end{bmatrix}$

$$\therefore \quad A+B=\begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}+\begin{bmatrix} 4 & -3 \\ 1 & 5 \end{bmatrix}$$
$$=\begin{bmatrix} (-2)+4 & 5+(-3) \\ 3+1 & 4+5 \end{bmatrix}$$
$$=\begin{bmatrix} -2+4 & 5-3 \\ 4 & 9 \end{bmatrix}=\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$$

Subtraction of Matrices

Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be two matrices of same order, say $m \times n$. Then, the difference of these matrices, $A-B$ is defined as a matrix $D=[d_{ij}]$, where $d_{ij}=a_{ij}-b_{ij}$ for all values of i and j.

e.g. Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B=\begin{bmatrix} e & f \\ g & h \end{bmatrix}$.

Then, $A-B=\begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$

Note $B-A$ is defined as a matrix $D=[d_{ij}]$, where $d_{ij}=b_{ij}-a_{ij}$ for all values of i and j.

Example 2. If $A=\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$, then find $A-B$.

Sol. Given, $A=\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$

$$\therefore \quad A-B=\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}-\begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}=\begin{bmatrix} 3-2 & -2-4 \\ 1-(-3) & 4-1 \end{bmatrix}$$
$$=\begin{bmatrix} 1 & -6 \\ 1+3 & 3 \end{bmatrix}=\begin{bmatrix} 1 & -6 \\ 4 & 3 \end{bmatrix}$$

Scalar Multiplication [Multiplication of a Matrix by a Scalar (Number)]

Let $A=[a_{ij}]_{m\times n}$ be a matrix and k be a scalar. Then, kA is another matrix, which is obtained by multiplying each element of A by the scalar k.

Thus, $\quad kA=k\,[a_{ij}]_{m\times n}=[k(a_{ij})]_{m\times n}$

i.e. (i, j)th element of kA is ka_{ij} for all possible values of i and j.

Properties of Scalar Multiplication

Let A and B be two matrices of same order. Then,

(i) $1 \cdot A = A$ (ii) $(-1)A = -A$ (iii) $0 \cdot A = O$

(iv) $l(kA) = k(lA)$, where k and l are scalars.

(v) $-kA = k(-A) = -(kA)$, where k is a scalar.

(vi) $k(A+B) = (A+B)k = kA + kB$, where k is a scalar.

(vii) $(k_1 + k_2)A = k_1A + k_2A$, where k_1 and k_2 are scalars.

Example 3. If $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$, then find the following.

(i) $3A$ (ii) $(-2)A$

Sol. Given, $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$

(i) $3A = 3\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2\times 3 & -1\times 3 \\ 0\times 3 & 3\times 3 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 0 & 9 \end{bmatrix}$

(ii) $(-2)A = (-2)\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2\times 2 & -2\times(-1) \\ -2\times 0 & -2\times 3 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 0 & -6 \end{bmatrix}$

Example 4. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$, then show that $4(A+B) = 4A + 4B$.

Sol. Given, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$

Now, $\text{LHS} = 4(A+B) = 4\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}\right)$

$= 4\begin{bmatrix} 1-2 & 2+3 \\ 3+0 & 4+4 \end{bmatrix} = 4\begin{bmatrix} -1 & 5 \\ 3 & 8 \end{bmatrix}$

$= \begin{bmatrix} 4\times(-1) & 4\times 5 \\ 4\times 3 & 4\times 8 \end{bmatrix} = \begin{bmatrix} -4 & 20 \\ 12 & 32 \end{bmatrix}$ ...(i)

and $\text{RHS} = 4A + 4B = 4\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 4\begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 4\times 1 & 4\times 2 \\ 4\times 3 & 4\times 4 \end{bmatrix} + \begin{bmatrix} 4\times(-2) & 4\times 3 \\ 4\times 0 & 4\times 4 \end{bmatrix}$

$= \begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix} + \begin{bmatrix} -8 & 12 \\ 0 & 16 \end{bmatrix}$

$= \begin{bmatrix} 4-8 & 8+12 \\ 12+0 & 16+16 \end{bmatrix} = \begin{bmatrix} -4 & 20 \\ 12 & 32 \end{bmatrix}$ ...(ii)

From Eqs. (i) and (ii), we get LHS = RHS **Hence proved.**

Multiplication of a Matrix by a Matrix (Multiplication of Matrices)

The product AB of two matrices A and B is defined, if the number of columns of A is equal to the number of rows of B. Let $A = [a_{ij}]$ be $m \times n$ matrix and $B = [b_{jk}]$ be $n \times p$ matrix. Then, the product of the matrices A and B is the matrix C of order $m \times p$.

Thus,

$$C = [c_{ik}]_{m\times p} = AB = [a_{ij}]_{m\times n} \times [b_{jk}]_{n\times p}$$

Same number

Order of resulting matrix

To get the (i, k)th element c_{ik} of the matrix C, consider the ith row of A and kth column of B, then multiply them elementwise and take the sum of all these products.

Compatibility for Multiplication of Matrices

We can multiply two matrices, if the number of columns of first matrix is equal to the number of rows of second matrix, otherwise we cannot multiply them.

e.g. If $A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, then AB is not defined, because number of columns of first matrix is not same as the number of rows of second matrix. But BA is defined.

Working Rule for Multiplication of Two Matrices

Suppose we have two matrices $A = [a_{ij}]$ and $B = [b_{jk}]$, then to multiply them (i.e. for finding AB), we proceed as follow

(i) First, write the given two matrices (A and B), then find the number of columns of matrix A and number of rows of matrix B. If both are same, then go to the next step, otherwise product is not possible.

(ii) Multiply first row (R_1) of A with first column (C_1) of B elementwise and take the sum of all these products. e.g. If $[a_{11} \;\; a_{12}]$ is first row of A and $\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$ is first column of B, then their multiplication is $a_{11}b_{11} + a_{12}b_{21}$. This gives the first element c_{11} of product matrix C. Now, similarly multiplying first row (R_1) of A with second column (C_2) of B to get c_{12} of product matrix C. Repeat this process to multiply each row of matrix A with each column of B.

In general, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$, then

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}.$$

(iii) Write the matrix $C = [c_{ij}]_{m\times p}$, whose elements are obtained in (ii).

Example 5. Find AB, if $A=\begin{bmatrix}6&9\\2&3\end{bmatrix}$ and $B=\begin{bmatrix}2&0\\7&8\end{bmatrix}$.

Sol. Given, $A=\begin{bmatrix}6&9\\2&3\end{bmatrix}_{2\times 2}$ and $B=\begin{bmatrix}2&0\\7&8\end{bmatrix}_{2\times 2}$

Here, number of columns in A = number of rows in $B=2$. So, the product of A and B is possible and it will be of order 2×2.

$$\therefore\quad AB=\begin{bmatrix}6&9\\2&3\end{bmatrix}\begin{bmatrix}2&0\\7&8\end{bmatrix}$$
$$=\begin{bmatrix}6\times 2+9\times 7 & 6\times 0+9\times 8\\2\times 2+3\times 7 & 2\times 0+3\times 8\end{bmatrix}$$
$$=\begin{bmatrix}12+63 & 0+72\\4+21 & 0+24\end{bmatrix}=\begin{bmatrix}75&72\\25&24\end{bmatrix}$$

Example 6. If $A=\begin{bmatrix}3&7\\2&4\end{bmatrix}$, $B=\begin{bmatrix}0&2\\5&3\end{bmatrix}$ and $C=\begin{bmatrix}1&-5\\-4&6\end{bmatrix}$, then find $AB-5C$. ***[2015]***

Sol. Given, $A=\begin{bmatrix}3&7\\2&4\end{bmatrix}$, $B=\begin{bmatrix}0&2\\5&3\end{bmatrix}$ and $C=\begin{bmatrix}1&-5\\-4&6\end{bmatrix}$

Now, $AB=\begin{bmatrix}3&7\\2&4\end{bmatrix}\begin{bmatrix}0&2\\5&3\end{bmatrix}$
$$=\begin{bmatrix}0+35&6+21\\0+20&4+12\end{bmatrix}=\begin{bmatrix}35&27\\20&16\end{bmatrix}$$

and $5C=5\begin{bmatrix}1&-5\\-4&6\end{bmatrix}=\begin{bmatrix}5&-25\\-20&30\end{bmatrix}$

$$\therefore AB-5C=\begin{bmatrix}35&27\\20&16\end{bmatrix}-\begin{bmatrix}5&-25\\-20&30\end{bmatrix}=\begin{bmatrix}30&52\\40&-14\end{bmatrix}$$

Example 7. If $A=\begin{bmatrix}2&1\\0&-2\end{bmatrix}$, $B=\begin{bmatrix}4&1\\-3&-2\end{bmatrix}$ and $C=\begin{bmatrix}-3&2\\-1&4\end{bmatrix}$, then find $A^2+AC-5B$. ***[2014]***

Sol. Given, $A=\begin{bmatrix}2&1\\0&-2\end{bmatrix}$, $B=\begin{bmatrix}4&1\\-3&-2\end{bmatrix}$ and $C=\begin{bmatrix}-3&2\\-1&4\end{bmatrix}$

Now, $A^2=\begin{bmatrix}2&1\\0&-2\end{bmatrix}\begin{bmatrix}2&1\\0&-2\end{bmatrix}$
$$=\begin{bmatrix}2\times 2+1\times 0 & 2\times 1+1\times(-2)\\0\times 2+(-2)\times 0 & 0\times 1+(-2)\times(-2)\end{bmatrix}$$
[multiplying row by column]
$$=\begin{bmatrix}4+0&2-2\\0-0&0+4\end{bmatrix}=\begin{bmatrix}4&0\\0&4\end{bmatrix}\quad\text{...(i)}$$

$$AC=\begin{bmatrix}2&1\\0&-2\end{bmatrix}\begin{bmatrix}-3&2\\-1&4\end{bmatrix}$$
$$=\begin{bmatrix}2\times(-3)+1\times(-1) & 2\times 2+1\times 4\\0\times(-3)+(-2)\times(-1) & 0\times 2+(-2)\times 4\end{bmatrix}$$
[multiplying row by column]
$$=\begin{bmatrix}-6-1&4+4\\0+2&0-8\end{bmatrix}=\begin{bmatrix}-7&8\\2&-8\end{bmatrix}\quad\text{...(ii)}$$

and $5B=5\begin{bmatrix}4&1\\-3&-2\end{bmatrix}=\begin{bmatrix}5\times 4&5\times 1\\5\times(-3)&5\times(-2)\end{bmatrix}$

[multiplying each element of matrix by 5]
$$=\begin{bmatrix}20&5\\-15&-10\end{bmatrix}\quad\text{...(iii)}$$

$$\therefore\ A^2+AC-5B=\begin{bmatrix}4&0\\0&4\end{bmatrix}+\begin{bmatrix}-7&8\\2&-8\end{bmatrix}-\begin{bmatrix}20&5\\-15&-10\end{bmatrix}$$
$$=\begin{bmatrix}4-7-20&0+8-5\\0+2+15&4-8+10\end{bmatrix}=\begin{bmatrix}-23&3\\17&6\end{bmatrix}$$

Example 8. If $A=\begin{bmatrix}3&x\\0&1\end{bmatrix}$ and $B=\begin{bmatrix}9&16\\0&-y\end{bmatrix}$, then find x and y, when $A^2=B$. ***[2015]***

Sol. Given, $A=\begin{bmatrix}3&x\\0&1\end{bmatrix}$ and $B=\begin{bmatrix}9&16\\0&-y\end{bmatrix}$

It is given that, $A^2=B$ ...(i)

Here, $A^2=A\cdot A=\begin{bmatrix}3&x\\0&1\end{bmatrix}\begin{bmatrix}3&x\\0&1\end{bmatrix}$
$$=\begin{bmatrix}9+0&3x+x\\0+0&0+1\end{bmatrix}=\begin{bmatrix}9&4x\\0&1\end{bmatrix}$$

Now, from Eq. (i), we get $A^2=B$
$$\Rightarrow\quad\begin{bmatrix}9&4x\\0&1\end{bmatrix}=\begin{bmatrix}9&16\\0&-y\end{bmatrix}$$

On comparing the corresponding elements, we get
$$4x=16 \text{ and } 1=-y$$
$$\Rightarrow\quad x=\frac{16}{4} \text{ and } y=-1$$
$$\Rightarrow\quad x=4 \text{ and } y=-1$$

Example 9. If $A=\begin{bmatrix}2&0\\-1&7\end{bmatrix}$, $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $A^2=9A+mI$, then find m. ***[2016]***

Sol. Given, $A=\begin{bmatrix}2&0\\-1&7\end{bmatrix}$ and $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

Also, it is given that $A^2=9A+mI$ ...(i)

Here, $A^2=A\cdot A=\begin{bmatrix}2&0\\-1&7\end{bmatrix}\begin{bmatrix}2&0\\-1&7\end{bmatrix}$
$$=\begin{bmatrix}2\times 2+0\times(-1) & 2\times 0+0\times 7\\-1\times 2+7\times(-1) & -1\times 0+7\times 7\end{bmatrix}=\begin{bmatrix}4&0\\-9&49\end{bmatrix}$$

Now, from Eq. (i), we get
$$\begin{bmatrix}4&0\\-9&49\end{bmatrix}=9\begin{bmatrix}2&0\\-1&7\end{bmatrix}+m\begin{bmatrix}1&0\\0&1\end{bmatrix}$$
$$\Rightarrow\quad\begin{bmatrix}4&0\\-9&49\end{bmatrix}=\begin{bmatrix}18&0\\-9&63\end{bmatrix}+\begin{bmatrix}m&0\\0&m\end{bmatrix}$$
$$\Rightarrow\quad\begin{bmatrix}4&0\\-9&49\end{bmatrix}=\begin{bmatrix}18+m&0\\-9&63+m\end{bmatrix}$$

On comparing corresponding elements both sides, we get
$$18+m=4\ \Rightarrow\ m=4-18=-14$$
and $$63+m=49\ \Rightarrow\ m=49-63=-14$$
$$\therefore\quad m=-14$$

Properties of Multiplication of Matrices

Multiplication of matrices have following properties

(i) **Associative law** Matrix multiplication is associative, i.e. if A, B and C are three matrices, then
$A(BC) = (AB)C$, whenever both sides are defined.

(ii) **Distributive law** Distributive law holds in matrix multiplication, i.e. if A, B and C are three matrices, then $\quad A(B+C) = AB + AC$
or $\quad (A+B)C = AC + BC$,
whenever both sides are defined.

(iii) **Existence of multiplicative identity** For every square matrix A, there exists identity matrix I of same order, such that $AI = A = IA$.
Here, the identity matrix I is called the multiplicative identity.

(iv) **Non-commutativity** Generally, matrix multiplication is not commutative, i.e. if A and B are two matrices such that AB and BA both exist, then it is not necessary that $AB = BA$.

(v) **Zero matrix as the product of two non-zero matrices** If the product of two matrices is a zero matrix, then it is not necessary that one of the matrices is zero matrix.

e.g. $\quad A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} \neq O$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} \neq O.$

But $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$

Note Multiplication of diagonal matrices of same order will be commutative.

Example 10. If $A = \begin{bmatrix} 1 & -2 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, then show that $AB \neq BA$.

Sol. Given, $A = \begin{bmatrix} 1 & -2 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

Now, $AB = \begin{bmatrix} 1 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

$= \begin{bmatrix} 1\times 2 + (-2)\times 4 & 1\times 3 + (-2)\times 5 \\ (-4)\times 2 + 2\times 4 & (-4)\times 3 + 2\times 5 \end{bmatrix}$

[multiplying row by column]

$= \begin{bmatrix} 2-8 & 3-10 \\ -8+8 & -12+10 \end{bmatrix}$

$\Rightarrow \quad AB = \begin{bmatrix} -6 & -7 \\ 0 & -2 \end{bmatrix} \quad$ ...(i)

and $\quad BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -4 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2\times 1 + 3\times(-4) & 2\times(-2) + 3\times 2 \\ 4\times 1 + 5\times(-4) & 4\times(-2) + 5\times 2 \end{bmatrix}$

[multiplying row by column]

$= \begin{bmatrix} 2-12 & -4+6 \\ 4-20 & -8+10 \end{bmatrix}$

$\Rightarrow \quad BA = \begin{bmatrix} -10 & 2 \\ -16 & 2 \end{bmatrix} \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$AB \neq BA$ **Hence proved.**

Solving the Matrix Equation

In order to solve equation of the type $A + kX = B$, where k is a scalar and A, B, X (unknown) are matrices of same order, take the unknown matrix on LHS with unity coefficient and shift rest of the matrices to the RHS of the equation. Simplify the RHS, to get the unknown matrix.

Example 11. If $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}$ and $A^2 - 5B^2 = 5C$, then find matrix C, where C is a 2 by 2 matrix. ***[2017]***

Sol. Given, $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}$

Also, we have $A^2 - 5B^2 = 5C$

$\therefore \quad C = \frac{1}{5}A^2 - B^2$

Here, $A^2 = A \cdot A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1+9 & 3+12 \\ 3+12 & 9+16 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 15 & 25 \end{bmatrix}$

$\Rightarrow \quad \frac{1}{5}A^2 = \frac{1}{5}\begin{bmatrix} 10 & 15 \\ 15 & 25 \end{bmatrix} = \begin{bmatrix} 10\times\frac{1}{5} & 15\times\frac{1}{5} \\ 15\times\frac{1}{5} & 25\times\frac{1}{5} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

and $\quad B^2 = B \cdot B = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 4-3 & -2+2 \\ 6-6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore \quad C = \frac{1}{5}A^2 - B^2 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2-1 & 3-0 \\ 3-0 & 5-1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$

Example 12. Given matrix $B = \begin{bmatrix} 1 & 1 \\ 8 & 3 \end{bmatrix}$. Find the matrix X, if $X = B^2 - 4B$. Hence, solve for a and b, given that $X\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 50 \end{bmatrix}$. ***[2017]***

Sol. Given, $\quad B = \begin{bmatrix} 1 & 1 \\ 8 & 3 \end{bmatrix}$

We have, $X = B^2 - 4B$

Now, $B^2 = B \cdot B = \begin{bmatrix} 1 & 1 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 8 & 3 \end{bmatrix} = \begin{bmatrix} 1+8 & 1+3 \\ 8+24 & 8+9 \end{bmatrix}$

$= \begin{bmatrix} 9 & 4 \\ 32 & 17 \end{bmatrix}$

and $\quad 4B = 4\begin{bmatrix} 1 & 1 \\ 8 & 3 \end{bmatrix} = \begin{bmatrix} 4\times 1 & 4\times 1 \\ 4\times 8 & 4\times 3 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 32 & 12 \end{bmatrix}$

$\therefore \quad X = B^2 - 4B = \begin{bmatrix} 9 & 4 \\ 32 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 4 \\ 32 & 12 \end{bmatrix}$

$= \begin{bmatrix} 9-4 & 4-4 \\ 32-32 & 17-12 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

Given, $X\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 50 \end{bmatrix}$

$\therefore \quad \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 50 \end{bmatrix} \Rightarrow \begin{bmatrix} 5a+0 \\ 0+5b \end{bmatrix} = \begin{bmatrix} 5 \\ 50 \end{bmatrix}$

$\begin{bmatrix} 5a \\ 5b \end{bmatrix} = \begin{bmatrix} 5 \\ 50 \end{bmatrix}$

On comparing corresponding elements both sides, we get

$5a = 5 \Rightarrow a = 1$ and $5b = 50 \Rightarrow b = 10$

Example 13. Given matrices

$A = \begin{bmatrix} 4\sin 30^\circ & \cos 0^\circ \\ \cos 0^\circ & 4\sin 30^\circ \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. If $AX = B$, then

(i) write the order of matrix X.

(ii) find the matrix X. **[2016]**

Sol.

(i) Given, $A = \begin{bmatrix} 4\sin 30^\circ & \cos 0^\circ \\ \cos 0^\circ & 4\sin 30^\circ \end{bmatrix}_{2\times 2}$ and $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times 1}$

Also, we have $AX = B$

Since, product of matrices A and X are possible.

$\therefore$ Number of columns in A = Number of rows in X

$\Rightarrow$ Number of rows in X is 2.

Since, product of matrices A and X results a matrix B, which have order 2×1.

$\because$ Number of columns in matrix X should be same as in matrix B.

$\therefore$ Number of columns in matrix $X = 1$

Thus, order of matrix X is 2×1.

(ii) We have, $A = \begin{bmatrix} 4\sin 30^\circ & \cos 0^\circ \\ \cos 0^\circ & 4\sin 30^\circ \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\left[\because \sin 30^\circ = \frac{1}{2}, \cos 0^\circ = 1\right]$

Let $\quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$.

Then, $\quad AX = B \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

On comparing corresponding elements both sides, we get

$2x_1 + x_2 = 4 \quad \ldots(i)$

and $\quad x_1 + 2x_2 = 5 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$x_1 = 1$ and $x_2 = 2$

$\therefore \quad X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Topic Exercise 2

1. If $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -5 \\ -1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$, then find

(i) $A + B$ (ii) $C + A$

2. If $C = \begin{bmatrix} 4 & 7 \\ 9 & 8 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $E = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, then find

(i) $C - D$ (ii) $D - E$

3. Find the values of 'x' and 'y', if

$$2\begin{bmatrix} x & 7 \\ 9 & y-5 \end{bmatrix} + \begin{bmatrix} 6 & -7 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 7 \\ 22 & 15 \end{bmatrix}.$$ **[2018]**

4. If $C = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, then find the product CD.

5. If $A = \begin{bmatrix} 3 & 5 \\ 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Is the product AB possible? Give reason. If yes, then find their product. **[2011]**

6. Evaluate $\begin{bmatrix} 4\sin 30^\circ & 2\cos 60^\circ \\ \sin 90^\circ & 2\cos 0^\circ \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$. **[2010]**

7. Evaluate $\begin{bmatrix} 2\cos 60^\circ & -2\sin 30^\circ \\ -\tan 45^\circ & \cos 0^\circ \end{bmatrix}\begin{bmatrix} \cot 45^\circ & \operatorname{cosec} 30^\circ \\ \sec 60^\circ & \sin 90^\circ \end{bmatrix}$.

8. If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, then find the value of $(A + B)(A - B)$.

9. Given, $A = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Find $AB + 2C - 4D$. **[2010]**

10. Find the value of x, given that $A^2 = B$, $A = \begin{bmatrix} 2 & 12 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & x \\ 0 & 1 \end{bmatrix}$. **[2005]**

11. If $X = \begin{bmatrix} -1 & 1 \\ x & y \end{bmatrix}$ and $X^2 = I$, then find the values of x and y.

12. If $X = \begin{bmatrix} \sin^2\theta & -1 \\ -\sec^2\theta & 2 \end{bmatrix}$ and $Y = \begin{bmatrix} -\cos^2\theta & 0 \\ -\tan^2\theta & 2 \end{bmatrix}$, then find the value of matrix A, if $A + X = Y$.

13. Given, $\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} X = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, then write

(i) the order of the matrix X.
(ii) the matrix X. ***[2012]***

Hints and Answers

1. Do same as Example 1.

Ans. (i) $\begin{bmatrix} 8 & -2 \\ 1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$

2. Do same as Example 2.

Ans. (i) $\begin{bmatrix} 1 & 7 \\ 9 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$

3. Hint Given that,

$$2\begin{bmatrix} x & 7 \\ 9 & y-5 \end{bmatrix} + \begin{bmatrix} 6 & -7 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 7 \\ 22 & 15 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x & 14 \\ 18 & 2y-10 \end{bmatrix} + \begin{bmatrix} 6 & -7 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 7 \\ 22 & 15 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+6 & 7 \\ 22 & 2y-5 \end{bmatrix} = \begin{bmatrix} 10 & 7 \\ 22 & 15 \end{bmatrix}$$

Here, both matrices are equal, so we equate the corresponding elements.

$$2x+6=10 \Rightarrow 2x=10-6$$

and $\quad 2y-5=15 \Rightarrow 2y=15+5$

Ans. $x=2$ and $y=10$

4. Do same as Example 5.

Ans. $\begin{bmatrix} -1 & 4 \\ -5 & 15 \end{bmatrix}$

5. Hint Number of columns in A = Number of rows in B.

Ans. Yes; $AB = \begin{bmatrix} 26 \\ 0 \end{bmatrix}$

6. Hint Given matrices can be written as

$$\begin{bmatrix} 4 \times \frac{1}{2} & 2 \times \frac{1}{2} \\ 1 & 2 \times 1 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$$

Ans. $\begin{bmatrix} 13 & 14 \\ 14 & 13 \end{bmatrix}$

7. Hint Write the given product as

$$\begin{bmatrix} 2 \times \frac{1}{2} & -2 \times \frac{1}{2} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Ans. $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

8. Hint $(A+B)(A-B) = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix}$

Ans. $\begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}$

9. Do same as Example 6. **Ans.** $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

10. Do same as Example 8. **Ans.** $x = 36$

11. Hint $X^2 = X \cdot X = \begin{bmatrix} -1 & 1 \\ x & y \end{bmatrix} \begin{bmatrix} -1 & 1 \\ x & y \end{bmatrix}$

$$= \begin{bmatrix} 1+x & -1+y \\ -x+xy & x+y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ans. $x = 0, \ y = 1$

12. Hint $A = Y - X$ **Ans.** $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

13. Do same as Example 13.

Ans. (i) Order $= 2 \times 1$ (ii) $X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

CHAPTER EXERCISE

(a) 3 Marks Questions

1. If the matrix $A = \begin{bmatrix} a & 1 & x \\ 2 & \sqrt{3} & x^2 - y \\ 0 & 5 & -2/5 \end{bmatrix}$, then write

(i) the order of the matrix A.
(ii) the number of elements.
(iii) elements a_{23}, a_{31} and a_{12}.

2. If the matrix $P = \begin{bmatrix} 1 & 3 & 4 \\ \sqrt{5} & x & 0 \\ y & 3/5 & -3 \end{bmatrix}$, then write

(i) the order of the matrix P.
(ii) the number of elements.
(iii) elements p_{13}, p_{32} and p_{22}.

3. Write the order of the following matrix.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Also, find the elements a_{21} and a_{22}. Show that $a_{11} \times a_{22} = 3(a_{12} + a_{21})$.

4. Write the order of the following matrix.

$$B = \begin{bmatrix} 3 & 1 & 2 \\ 7 & 4 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$

Also, find the elements b_{12} and b_{33}. Prove that $b_{11} + b_{21} = b_{32} + b_{33}$.

5. If a matrix has 32 elements, then what are the possible orders it can have?

6. If a matrix has 40 elements, then what are the possible orders it can have? What, if it has 41 elements?

7. Construct a 1×3 order matrix, whose elements are given by $c_{ij} = 4i - 3j$.

8. Construct a 3×3 matrix, whose elements are given by $a_{ij} = \sin ix \operatorname{cosec} jx$.

9. Find the values of a, b, c and d, if

$$\begin{bmatrix} a & -2 \\ b & 7 \end{bmatrix} = \begin{bmatrix} 2 & c \\ 3 & 2c + d \end{bmatrix}.$$

10. Find the values of a, b, c and d, if

$$\begin{bmatrix} a+b & 3 \\ 4 & c \end{bmatrix} = \begin{bmatrix} 2 & d \\ b & 1 \end{bmatrix}.$$

11. If $\begin{bmatrix} x-3 & x-z-4 \\ z & x+y+z \end{bmatrix} = I$, where I is the identity matrix of same order, then find the values of x, y and z.

12. Find the values of x, y, a and b, when

$$\begin{bmatrix} x+y & a-b \\ a+b & 2x-3y \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -1 & -5 \end{bmatrix}.$$

13. If $\begin{bmatrix} 39 & 5x+7y \\ 2x-5y & 41 \end{bmatrix} = \begin{bmatrix} 7u+9v & 1 \\ 16 & 9u+7v \end{bmatrix}$, then find the value of $x + y + u + v$.

14. Find the values of a, b, c and d, if

$$\begin{bmatrix} a+b+c+d \\ a+c-d \\ b-c+d \\ a+d \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$

15. If $A = \begin{bmatrix} 7 & 2 \\ -5 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} -9 & 1 \\ 0 & -7 \end{bmatrix}$, then find

(i) $A + B$ (ii) $B + C$ (iii) $C + A$

16. If $P = \begin{bmatrix} 4 & 0 \\ -10 & 7 \end{bmatrix}$, $Q = \begin{bmatrix} 3 & 10 \\ 0 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} 8 & 5 \\ 7 & 1 \end{bmatrix}$, then find the value of each of the following.

(i) $P - Q$ (ii) $Q - R$ (iii) $R - P$

17. If $E = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$, $F = \begin{bmatrix} -1 & 2 \\ -4 & 0 \end{bmatrix}$ and $G = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then find

(i) $E + F + G$ (ii) $E - F - G$

18. If $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ -2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 7 \\ 0 & 0 \end{bmatrix}$, then find

(i) $2A - 3B$
(ii) $3A + 2B - C$

19. If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, then find the values of x and y.

20. Find x, y, z and t, if

$$2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}.$$

21. If $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$, then prove that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

22. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, then find $C(B - A)$.

23. If $P = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$, then prove that $6P - P^2 = 9I_2$.

24. If $P = \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$, then evaluate $(-P^2 + 6P)$.

25. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then find $A^2 - 5A + 7I$. **[2012]**

26. If $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}$, find $AC + B^2 - 10C$. **[2018]**

27. Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$, then find $A^2 + AB + B^2$. **[2007]**

28. If $A = \begin{bmatrix} 8 & 6 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 1 & 0 \end{bmatrix}$, then solve for 2×2 matrix X, such that $X - B = A$.

29. If $\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix} + 2M = 3\begin{bmatrix} 3 & 2 \\ 0 & -3 \end{bmatrix}$, then find the matrix M. **[2008]**

30. Given, $A = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 4 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then find a matrix X, such that $A + X = 2B + C$.

31. Given, $A = \begin{bmatrix} 2 & -6 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 4 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$. Find the matrix X, such that $A + 2X = 2B + C$. **[2013]**

32. Find a and b, if $\begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = \begin{bmatrix} b \\ -3 \end{bmatrix}$.

33. Find x and y, if $\begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2x \end{bmatrix} + 3\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} y \\ 3 \end{bmatrix}$. **[2014]**

34. Find x and y, if $\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2x \\ 1 \end{bmatrix} + 2\begin{bmatrix} -4 \\ 5 \end{bmatrix} = 4\begin{bmatrix} 2 \\ y \end{bmatrix}$. **[2003]**

35. If $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ and $AB = (A + B)$, then find the values of x, y and z.

36. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} x & 1 \\ 4 & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, then find the value of x.

b 4 Marks Questions

37. Find x and y, if $\begin{bmatrix} x & 3x \\ y & 4y \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$. **[2013, 02]**

38. Find x and y, if $\begin{bmatrix} 2x & x \\ y & 3y \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 16 \\ 9 \end{bmatrix}$. **[2009]**

39. If $A = \begin{bmatrix} \cot 45° & 0 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & \cos 0° \\ 4 & 5 \end{bmatrix}$, then find the value of the following.

(i) AB (ii) $B^2 - A^2$

40. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 3 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 7 \\ 1 & 5 \end{bmatrix}$, then show that $AB = AC$. What conclusion can you draw?

41. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, then show that $A^3 - 4A^2 + A = O$.

42. If $A = \begin{bmatrix} 6 & 2 \\ 5 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix}$, then find a matrix X, such that $2A + 3B - 5X = O$.

43. If $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 3\begin{bmatrix} x \\ 2y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$, find the values of x and y.

44. If $A = \begin{bmatrix} p & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$ and $BA = C^2$, then find the values of p and q. *[2008]*

45. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, then find the value of α, for which $A^2 = B$.

46. If $A = \begin{bmatrix} 2 & -3 \\ p & q \end{bmatrix}$, then find p and q, so that $A^2 = I$.

47. Let $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 14 & 3 \\ 2 & 4 \end{bmatrix}$, then find the matrix C, such that $AC = B$.

48. Solve the following matrix equation.

$$[x \quad 1]\begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x \\ 5 \end{bmatrix} = 0$$

49. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and I is the unit matrix of same order as A, then find the value of k, so that $A^2 = 8A + kI$.

50. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then find the real values of x and y, such that $(xI + yA)^2 = A$.

Hints and Answers

1. Do same as Q. 1 of Topic Exercise 1.
Ans. (i) 3×3 (ii) 9
(iii) $a_{23} = x^2 - y$, $a_{31} = 0$ and $a_{12} = 1$.

2. Do same as Q. 1 of Topic Exercise 1.
Ans. (i) 3×3 (ii) 9 (iii) $p_{13} = 4$, $p_{32} = \frac{3}{5}$, $p_{22} = x$

3. Do same as Q. 2 of Topic Exercise 1.
Ans. Order $= 2 \times 2$; $a_{21} = 1$ and $a_{22} = 2$

4. Do same as Q. 2 of Topic Exercise 1.
Ans. Order $= 3 \times 3$; $b_{12} = 1$ and $b_{33} = 7$

5. Do same as Example 1 of Topic 1.
Ans. 1×32, 2×16, 4×8, 8×4, 16×2, 32×1

6. Do same as Example 1 of Topic 1.
Ans. 1×40, 2×20, 4×10, 5×8, 8×5, 10×4, 20×2, 40×1 and 1×41, 41×1

7. Do same as Example 2 of Topic 1.
Ans. $[1 \quad -2 \quad -5]$

8. Do same as Example 3 of Topic 1.
Ans. $\begin{bmatrix} 1 & \sin x \operatorname{cosec} 2x & \sin x \operatorname{cosec} 3x \\ \sin 2x \operatorname{cosec} x & 1 & \sin 2x \operatorname{cosec} 3x \\ \sin 3x \operatorname{cosec} x & \sin 3x \operatorname{cosec} 2x & 1 \end{bmatrix}$

9. Do same as Example 6 of Topic 1.
Ans. $a = 2$, $b = 3$, $c = -2$, $d = 11$

10. Do same as Example 6 of Topic 1.
Ans. $a = -2$, $b = 4$, $c = 1$, $d = 3$

11. **Hint** $\begin{bmatrix} x-3 & x-z-4 \\ z & x+y+z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Now, proceed as in Example 6 of Topic 1.
Ans. $x = 4$, $y = -3$, $z = 0$

12. Do same as Example 6 of Topic 1.
Ans. $x = 2$, $y = 3$, $a = 1$, $b = -2$

13. Do same as Example 6 of Topic 1.
Ans. $x + y + u + v = 6$

14. Do same as Example 6 of Topic 1.
Ans. $a = 1$, $b = 1$, $c = 1$, $d = 1$

15. Do same as Example 1 of Topic 2.
Ans. (i) $A + B = \begin{bmatrix} 7 & 3 \\ -4 & 5 \end{bmatrix}$ (ii) $B + C = \begin{bmatrix} -9 & 2 \\ 1 & -7 \end{bmatrix}$
(iii) $C + A = \begin{bmatrix} -2 & 3 \\ -5 & -2 \end{bmatrix}$

16. Do same as Example 2 of Topic 2.
Ans. (i) $P - Q = \begin{bmatrix} 1 & -10 \\ -10 & 6 \end{bmatrix}$ (ii) $Q - R = \begin{bmatrix} -5 & 5 \\ -7 & 0 \end{bmatrix}$
(iii) $R - P = \begin{bmatrix} 4 & 5 \\ 17 & -6 \end{bmatrix}$

17. (i) **Hint** $E + F + G = \begin{bmatrix} 3-1+3 & 1+2+0 \\ 4-4+0 & 1+0+3 \end{bmatrix}$
Ans. $E + F + G = \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$
(ii) **Hint** $E - F - G = \begin{bmatrix} 3-(-1)-3 & 1-2-0 \\ 4-(-4)-0 & 1-0-3 \end{bmatrix}$
Ans. $E - F - G = \begin{bmatrix} 1 & -1 \\ 8 & -2 \end{bmatrix}$

18. (i) **Hint**

$$2A=\begin{bmatrix}2\times2 & 2\times0\\ 2\times(-1) & 2\times2\end{bmatrix}$$

and $3B=\begin{bmatrix}3\times(-3) & 3\times1\\ 3\times(-2) & 3\times2\end{bmatrix}$

Ans. $2A-3B=\begin{bmatrix}13 & -3\\ 4 & -2\end{bmatrix}$

(ii) $3A=\begin{bmatrix}3\times2 & 3\times0\\ 3\times(-1) & 3\times2\end{bmatrix}$ and $2B=\begin{bmatrix}2\times(-3) & 2\times1\\ 2\times(-2) & 2\times2\end{bmatrix}$

Ans. $3A+2B-C=\begin{bmatrix}-4 & -5\\ -7 & 10\end{bmatrix}$

19. **Hint** $\begin{bmatrix}7 & 8+y\\ 10 & 2x+1\end{bmatrix}=\begin{bmatrix}7 & 0\\ 10 & 5\end{bmatrix}$

Ans. $x=2,\ y=-8$

20. **Hint** $\begin{bmatrix}2x+3 & 2z-3\\ 2y & 2t+6\end{bmatrix}=\begin{bmatrix}9 & 15\\ 12 & 18\end{bmatrix}$

Ans. $x=3,\ y=6,\ z=9,\ t=6$

21. **Hint** $A^2=\begin{bmatrix}ab & b^2\\ -a^2 & -ab\end{bmatrix}\begin{bmatrix}ab & b^2\\ -a^2 & -ab\end{bmatrix}$

22. **Hint** $C(B-A)=\begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}\left(\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}-\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}\right)$

Ans. $\begin{bmatrix}4 & -4\\ 4 & -4\end{bmatrix}$

23. **Hint** $6P-P^2=6P-P\cdot P=\begin{bmatrix}24 & 6\\ -6 & 12\end{bmatrix}-\begin{bmatrix}15 & 6\\ -6 & 3\end{bmatrix}$

24. **Hint** $-\left(\begin{bmatrix}2 & -2\\ -3 & 4\end{bmatrix}\begin{bmatrix}2 & -2\\ -3 & 4\end{bmatrix}\right)+6\begin{bmatrix}2 & -2\\ -3 & 4\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$

Ans. $2I_2$

25. **Hint** $A^2=A\times A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}$

$$A^2-5A+7I=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}-5\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}+7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Ans. $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$

26. **Hint**

$AC+B^2-10C$

$$=\begin{bmatrix}-1 & 12\\ -2 & 28\end{bmatrix}+\begin{bmatrix}-4 & 28\\ -7 & 45\end{bmatrix}-10\begin{bmatrix}1 & 0\\ -1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 12\\ -2 & 28\end{bmatrix}+\begin{bmatrix}-4 & 28\\ -7 & 45\end{bmatrix}-\begin{bmatrix}10 & 0\\ -10 & 40\end{bmatrix}$$

Ans. $\begin{bmatrix}-15 & 40\\ 1 & 33\end{bmatrix}$

27. **Hint** $A^2+AB+B^2=A\cdot A+A\cdot B+B\cdot B$

$$=\begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix}+\begin{bmatrix}2 & 3\\ 3 & 6\end{bmatrix}+\begin{bmatrix}1 & 6\\ -2 & -3\end{bmatrix}$$

Ans. $\begin{bmatrix}4 & 9\\ 5 & 4\end{bmatrix}$

28. **Hint** $X=A+B$ **Ans.** $X=\begin{bmatrix}5 & 11\\ -1 & 4\end{bmatrix}$

29. **Hint** $2M=3\begin{bmatrix}3 & 2\\ 0 & -3\end{bmatrix}-\begin{bmatrix}1 & 4\\ -2 & 3\end{bmatrix}$

Ans. $M=\begin{bmatrix}4 & 1\\ 1 & -6\end{bmatrix}$

30. **Hint** $X=2B+C-A$

Ans. $X=\begin{bmatrix}-7 & 5\\ 6 & 2\end{bmatrix}$

31. **Hint** $2X=2B+C-A$

Ans. $X=\begin{bmatrix}-2 & 5\\ 3 & 1\end{bmatrix}$

32. **Hint** $\begin{bmatrix}1\\ -2+4a\end{bmatrix}=\begin{bmatrix}b\\ -3\end{bmatrix}$

Ans. $a=-\frac{1}{4},\ b=1$

33. **Hint** $\begin{bmatrix}-2\times(-1)+0\times2x\\ 3\times(-1)+1\times2x\end{bmatrix}+\begin{bmatrix}-2\times3\\ 1\times3\end{bmatrix}=\begin{bmatrix}2\times y\\ 2\times3\end{bmatrix}$

$\Rightarrow \begin{bmatrix}-4\\ 2x\end{bmatrix}=\begin{bmatrix}2y\\ 6\end{bmatrix}$

On comparing the corresponiding elements both sides, we get the required result.

Ans. $x=3$ and $y=-2$

34. **Hint** We have,

$$\begin{bmatrix}3\times2x+(-2)\times1\\ -1\times2x+4\times1\end{bmatrix}+\begin{bmatrix}-4\times2\\ 5\times2\end{bmatrix}=\begin{bmatrix}4\times2\\ 4\times y\end{bmatrix}$$

$$\begin{bmatrix}6x-10\\ -2x+14\end{bmatrix}=\begin{bmatrix}8\\ 4y\end{bmatrix}$$

On comparing the corresponding elements both sides, we get the required result.

Ans. $x=3$ and $y=2$

35. Hint $\begin{bmatrix} 3x & 3y \\ 0 & 2z \end{bmatrix} = \begin{bmatrix} 3+x & y \\ 0 & 2+z \end{bmatrix}$

Now, proceed as in Example 6 of Topic 1.

Ans. $x = \frac{3}{2},\ y = 0,\ z = 2$

36. Hint $\left(\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} x & 1 \\ 4 & -1 \end{bmatrix}\right)^2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} x & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x & 1 \\ 4 & -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} (1+x)^2 & 0 \\ 6x-6 & 4 \end{bmatrix} = \begin{bmatrix} x^2+3 & x-1 \\ 4x-4 & 4 \end{bmatrix}$

Now, proceed as in Example 6 of Topic 1.

Ans. $x = 1$

37. Hint $\begin{bmatrix} x \times 2 + 3x \times 1 \\ y \times 2 + 4y \times 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

Ans. $x = 1,\ y = 2$

38. Do same as Q. 37.

Ans. $x = 2$ and $y = 1$

39. (i) **Hint** $AB = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$

(ii) **Hint** $B^2 - A^2 = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

Ans. $\begin{bmatrix} 4 & 6 \\ 26 & 28 \end{bmatrix}$

40. Hint $A \cdot B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = AC$

Ans. The conclusion is cancellation law may not hold for multiplication of matrices, i.e. $AB = AC$ may not imply $B = C$, $A \neq O$.

41. Hint $A^3 - 4A^2 + A = A \cdot A \cdot A - 4A \cdot A + A$

$= A^2 \cdot A - 4A \cdot A + A$

$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} - 4\begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

42. Hint $5X = 2A + 3B$

Ans. $X = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$

43. Hint $\begin{bmatrix} x^2 - 3x \\ y^2 - 6y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$

Ans. $x = 1, 2$ and $y = 3 \pm 3\sqrt{2}$

44. Hint $\begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & -2q \\ p & 0 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}$

Ans. $p = 8,\ q = 4$

45. Hint $\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

Ans. There is no value of α for which $A^2 = B$ is true.

46. Hint $\begin{bmatrix} 2 & -3 \\ p & q \end{bmatrix}\begin{bmatrix} 2 & -3 \\ p & q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ans. $p = 1,\ q = -2$

47. Hint Let the matrix $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ 2 & 4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 3a+2c & 3b+2d \\ -a+c & -b+d \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ 2 & 4 \end{bmatrix}$

Now, proceed as in Example 6 of Topic 1.

Ans. $C = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$

48. Hint $[x \quad 1]\begin{bmatrix} x \\ -2x-15 \end{bmatrix} = 0$

$\Rightarrow \quad x^2 - 2x - 15 = 0 \Rightarrow (x-5)(x+3) = 0$

Ans. $x = 5, -3$

49. Hint $\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$

Ans. $k = -7$

50. Hint $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}\begin{bmatrix} x & y \\ -y & x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Ans. $x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}}$ or $x = \frac{-1}{\sqrt{2}},\ y = \frac{-1}{\sqrt{2}}$

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1 Find the values of 'x' and 'y', if

$$2\begin{bmatrix} x & 7 \\ 9 & y-5 \end{bmatrix} + \begin{bmatrix} 6 & -7 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 7 \\ 22 & 15 \end{bmatrix}.$$

2 If $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}$, find $AC + B^2 - 10C$

2017

3 If $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}$ and $A^2 - 5B^2 = 5C$. Find the matrix C, where C is a 2 by 2 matrix.

4 Given matrix $B = \begin{bmatrix} 1 & 1 \\ 8 & 3 \end{bmatrix}$. Find the matrix X, if $X = B^2 - 4B$.

Hence solve for a and b, given that $X\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 50 \end{bmatrix}$.

2016

5 If $A = \begin{bmatrix} 2 & 0 \\ -1 & 7 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^2 = 9A + mI$, find m.

6 Given matrices $A = \begin{bmatrix} 4\sin 30° & \cos 0° \\ \cos 0° & 4\sin 30° \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

If $AX = B$, then
(i) write the order of matrix X.
(ii) find the matrix X.

2015

7 If $A = \begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 5 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -5 \\ -4 & 6 \end{bmatrix}$, then find $AB - 5C$.

8 If $A = \begin{bmatrix} 3 & x \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 16 \\ 0 & -y \end{bmatrix}$, then find x and y when $A^2 = B$.

2014

9 Find x and y, if $\begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2x \end{bmatrix} + 3\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} y \\ 3 \end{bmatrix}$.

10 If $A = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 2 \\ -1 & 4 \end{bmatrix}$, then find $A^2 + AC - 5B$.

2013

11 Given, $A = \begin{bmatrix} 2 & -6 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 4 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$.

Find the matrix X such that $A + 2X = 2B + C$.

12 Find x and y, if $\begin{bmatrix} x & 3x \\ y & 4y \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$.

2012

13 If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then find $A^2 - 5A + 7I$.

14 Given, $\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} X = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, write

(i) the order of the matrix X. (ii) the matrix X.

2011

15 If $A = \begin{bmatrix} 3 & 5 \\ 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Is the product AB possible? Give reason. If yes, then find their product.

* All these questions are completely covered in chapter either as solved examples or in topic/chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. If α and β are the roots of the equation $x^2 + x - 6 = 0$ such that $\beta > \alpha$, then the product of the matrices $\begin{bmatrix} 0 & \alpha \\ \alpha & \beta \end{bmatrix}$ and $\begin{bmatrix} \beta+1 & 0 \\ -\beta & \alpha \end{bmatrix}$ is

(a) $\begin{bmatrix} 6 & 9 \\ -13 & -6 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 4 \\ 9 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 6 & 13 \\ 9 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} -5 & 4 \\ -9 & -2 \end{bmatrix}$

2. The matrices A and B, such that $AB = O$, but $A \neq O$ and $B \neq O$, are

(a) $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then the value of matrix A^5 is

(a) $\begin{bmatrix} 62 & 149 \\ -149 & -87 \end{bmatrix}$ (b) $\begin{bmatrix} 87 & 149 \\ 149 & 62 \end{bmatrix}$

(c) $\begin{bmatrix} -62 & -149 \\ 149 & 87 \end{bmatrix}$ (d) $\begin{bmatrix} 87 & 149 \\ 149 & -62 \end{bmatrix}$

4. If both $A + B$ and AB are defined, then which one of the following is true?

(a) A and B are rectangular matrices of same order
(b) A and B are square matrices of same order
(c) A and B are rectangular matrices of different order
(d) A and B are square matrices of different order

5. If $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then A^n (where, n is a natural number) is equal to

(a) $\begin{bmatrix} 3n & 0 \\ 0 & 3n \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $3^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $I_{2 \times 2}$

6. If $\begin{bmatrix} a^x \\ a^{-x} \end{bmatrix} [1 \quad 2] = \begin{bmatrix} p & a^{-2} \\ q & \log_2 2 \end{bmatrix}$, $a > 0$, then a^{p-q} is equal to

(a) $2^{3/2}$ (b) $2^{-3/2}$
(c) 1 (d) $4^{3/2}$

7. If $A = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 5 & 5 \end{bmatrix}$ and $A^n = \begin{bmatrix} 5^{200} & 5^{200} \\ 0 & 0 \end{bmatrix}$, then the value of n is

(a) 100 (b) 50
(c) 25 (d) None of these

8. If $\begin{bmatrix} 7 & -6 \\ 8 & -7 \end{bmatrix}^{2016} = \begin{bmatrix} 7 & -6 \\ 8 & -7 \end{bmatrix}^{2018}$, then $\begin{bmatrix} 7 & -6 \\ 8 & -7 \end{bmatrix}^{2017}$ equals

(a) $\begin{bmatrix} 7 & -6 \\ 8 & -7 \end{bmatrix}^{2016} + \begin{bmatrix} 7 & -6 \\ 8 & -7 \end{bmatrix}^{2018}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & -6 \\ 8 & -7 \end{bmatrix}$

(d) None of the above

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 399.

CHAPTER

9

Arithmetic and Geometric Progression

Chapter Objectives

- Arithmetic Progression (AP)
- Sum of First n Terms of an AP
- Geometric Progression (GP)
- Sum of First n Terms of a GP

In this chapter, we will study the sequence (list of numbers) in which succeeding terms are obtained by adding or multiplying a fixed number to the preceding term. We will also study the formulae to find the value of its nth term and the sum of terms of these types of sequence.

Sequence

A list of numbers arranged in a definite order according to some definite rule, is called a **sequence**.

e.g. (i) 1, 10, 12, 16, 100, ... In this sequence, numbers are arranged in ascending order.

(ii) 1, 2, 3, 4, 5, 6, ... In this sequence, each number is 1 more than the number preceding it, except first number.

(iii) 2, 4, 8, 16, ... In this sequence, each number is obtained on multiplying the preceding number by 2, except first number.

The various numbers occurring in a sequence are called its **terms**. Consider a sequence $a_1, a_2, a_3, a_4, \dots$. Here, a_1 is called **first term**, a_2 is called **second term** and so on. Its nth term can be written as a_n, which is known as the **general term** of a sequence. A sequence is said to be **finite** or **infinite** accordingly it has **finite** or **infinite** number of terms.

Series

When the numbers or terms in a sequence are connected to each other by positive or negative signs, then the sequence becomes series. A series is finite or infinite, according as the number of terms in the series is finite or infinite.

e.g. (i) $2+4+6+8+\dots+102$ is finite series. (ii) $-3-9-27-\dots$ is infinite series.

Progression

Those sequences whose terms always follow certain pattern are called **progressions**.

e.g. (i) The sequence 3, 5, 7, 9, ... is a progression, as each term can be found by adding 2 to the term preceding it.

(ii) The sequence 3, 10, 18, 21, 35, ... is not a progression as its terms are not following any certain pattern.

Note All progressions are sequences but all sequences need not be progressions.

In this chapter, we will study about two types of progressions, **arithmetic** and **geometric progression.**

Topic 1

Arithmetic Progression (AP)

An **Arithmetic Progression** (AP) is a list of numbers in which each term is obtained by adding a fixed number to the preceding term, except the first term. This fixed number is called the **common difference** (d) of the AP. It can be positive, negative or zero.

Thus, a list of numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called an arithmetic progression (AP), if there exists a constant number d (called common difference) such that

$$\begin{aligned} a_2 &= a_1 + d \\ a_3 &= a_2 + d \\ a_4 &= a_3 + d \\ \vdots \quad & \quad \vdots \quad \vdots \\ a_n &= a_{n-1} + d \text{ and so on.} \end{aligned}$$

In other words, a list of numbers is called an arithmetic progression (AP) if and only if the difference of any term from its preceding term is constant. Thus, $a_1, a_2, a_3, a_4, \ldots$ is an AP if and only if $a_{n+1} - a_n = d$, a constant (independent of n).

In general, $a, a+d, a+2d, a+3d, \ldots$ represent an arithmetic progression, where a is the first term and d is the common difference. This is called **general form of AP.**

If number of terms in an AP is finite, then it is called a **finite AP**, otherwise it is called an **infinite AP**.

Example 1. Examine that the sequence 13, 10, 7, 4,... is an AP.

Sol. Here, $a_2 - a_1 = 10 - 13 = -3, a_3 - a_2 = 7 - 10 = -3,$

$a_4 - a_3 = 4 - 7 = -3$ and so on.

Since, difference of any term from its preceding term is constant. So, the given sequence is an AP.

Example 2. Examine that the list of numbers obtained from following situation, will be in the form of an AP.

"Amount left with Sandeep (in ₹) out of the total amount of ₹ 12000 which he had in the beginning, when he spends ₹ 500 in the beginning of every month."

Sol. Given, total amount Sandeep had = ₹ 12000

In the beginning of every month, he spend = ₹ 500

So, in the beginning of 1st month, he had amount,

t_1 = ₹ 12000

In the beginning of 2nd month, he had amount,

$t_2 = 12000 - 500$ = ₹ 11500

In the beginning of 3rd month, he had amount,

$t_3 = 11500 - 500$ = ₹ 11000

In the beginning of 4th month, he had amount,

$t_4 = 11000 - 500$ = ₹ 10500 and so on.

Now, the list of amounts is 12000, 11500, 11000, 10500, ...

Here, $t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = -500$

i.e. the difference of any term from its preceding term is constant.

So, the above list of numbers forms an AP.

Method to Find the Common Difference of an AP

Let an AP $a_1, a_2, a_3, a_4, \ldots$ be given. Then, common difference of this AP will be the difference of any term from its preceding term, i.e. common difference

$$(d) = a_2 - a_1 \text{ or } a_3 - a_2 \text{ or } a_4 - a_3 \text{ and so on.}$$

If the common difference of an AP is zero, i.e. $d = 0$, then each term of the AP will be same as the first term of the AP.

Example 3. Find the common difference of the following AP.

(i) $3, -2, -7, -12, \ldots$ (ii) 11, 11, 11, 11, ...

(iii) $5\frac{1}{2}, 9\frac{1}{2}, 13\frac{1}{2}, 17\frac{1}{2}, \ldots$

Sol.

(i) Given AP is $3, -2, -7, -12, \ldots$

Here, $a_1 = 3,\ a_2 = -2,\ a_3 = -7,\ a_4 = -12$ and so on.

$\therefore$ Common difference $(d) = a_2 - a_1 = -2 - 3 = -5$

(ii) Given AP is 11, 11, 11, 11, ...

Here, $a_1 = 11, a_2 = 11, a_3 = 11, a_4 = 11$ and so on.

$\therefore$ Common difference $(d) = a_2 - a_1 = 11 - 11 = 0$

(iii) Given AP is $5\frac{1}{2}, 9\frac{1}{2}, 13\frac{1}{2}, 17\frac{1}{2}, \ldots$

Here, $a_1 = 5\frac{1}{2},\ a_2 = 9\frac{1}{2},\ a_3 = 13\frac{1}{2},\ a_4 = 17\frac{1}{2}$ and so on.

$\therefore$ Common difference (d)

$$= a_2 - a_1 = 9\frac{1}{2} - 5\frac{1}{2} = \frac{19}{2} - \frac{11}{2} = \frac{8}{2} = 4$$

Method to Write an AP, When First Term and Common Difference are Given

To write an AP, the minimum information required is to know the first term a and the common difference d of the arithmetic progression. Then, put the values of a and d in $a, a+d, a+2d, a+3d, \ldots$ to get required AP.

Example 4. Write an AP having 4 as the first term and –3 as the common difference.

Sol. Given, first term $(a) = 4$ and common difference $(d) = -3$

On putting the values of a and d in general form of AP, i.e. $a, a+d, a+2d, a+3d, \ldots$, we get

$$4,\ 4-3,\ 4+2(-3),\ 4+3(-3), \ldots$$

or $\quad 4,\ 1,\ 4-6,\ 4-9, \ldots$

or $\quad 4,\ 1, -2, -5, \ldots$

which is the required AP.

Example 5. Which of the following are AP's? If they form an AP, then find the common difference d and write three more terms.

(i) $2, 4, 8, 16, \ldots$ (ii) $2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$

(iii) $-1.2, -3.2, -5.2, -7.2, \ldots$

Sol.

(i) Here, $a_2 - a_1 = 4 - 2 = 2$

and $a_3 - a_2 = 8 - 4 = 4$

Since, $a_2 - a_1 \neq a_3 - a_2$. Therefore, the given list of numbers does not form an AP.

(ii) Here, $a_2 - a_1 = \frac{5}{2} - 2 = \frac{5-4}{2} = \frac{1}{2}$,

$a_3 - a_2 = 3 - \frac{5}{2} = \frac{6-5}{2} = \frac{1}{2}$,

$a_4 - a_3 = \frac{7}{2} - 3 = \frac{7-6}{2} = \frac{1}{2}$ and so on.

Since, the difference of any term from its preceding term constant. Therefore, the given list of numbers forms an AP and its common difference (d) is $\frac{1}{2}$.

Now, next three terms of this AP are

$a_5 = a_4 + d = \frac{7}{2} + \frac{1}{2}$

$= \frac{7+1}{2} = \frac{8}{2} = 4$

$a_6 = a_5 + d = 4 + \frac{1}{2} = \frac{8+1}{2} = \frac{9}{2}$

and $a_7 = a_6 + d = \frac{9}{2} + \frac{1}{2} = \frac{9+1}{2} = \frac{10}{2} = 5$

(iii) Here, $a_2 - a_1 = -3.2 - (-1.2) = -3.2 + 1.2 = -2$,

$a_3 - a_2 = -5.2 - (-3.2) = -5.2 + 3.2 = -2$,

$a_4 - a_3 = -7.2 - (-5.2) = -7.2 + 5.2 = -2$ and so on.

Since, the difference of any term from its preceding term is constant. Therefore, the given list of numbers forms an AP and its common difference, d is -2.

Now, the next three terms of this AP are

$a_5 = a_4 + d = -7.2 + (-2) = -9.2$

$a_6 = a_5 + d = -9.2 + (-2) = -11.2$

and $a_7 = a_6 + d = -11.2 + (-2) = -13.2$

Properties of AP

(i) If a fixed number is added or subtracted from each term of an AP, then the resulting sequence is also an AP.

(ii) If each term of an AP is multiplied or divided by a fixed non-zero number, then the resulting sequence is also an AP.

General Term or *n*th Term of an AP

If the first term of an AP is 'a' and its common difference is 'd', then its nth term is given by the formula

$$\boxed{a_n = a + (n-1)d}$$

The nth term of an AP is also called its **general term**.

If there are n terms in an AP, then nth term is known as last term of an AP and it is denoted by l, which is given by the formula

$$\boxed{l = a + (n-1)d}$$

where, a is first term and d is common difference.

Example 6. Find the 20th term of the sequence $7, 3, -1, -5, \ldots$.

Sol. Given sequence is $7, 3, -1, -5, \ldots$.

Here, $3 - 7 = -4$, $-1 - 3 = -4$, $-5 + 1 = -4$

So, given sequence is an AP, in which $a = 7$ and $d = -4$.

$\because$ nth term, $a_n = a + (n-1)d$

On putting $n = 20$, we get

$a_{20} = a + (20-1)d = 7 + 19(-4)$ $\quad [\because a = 7, d = -4]$

$= 7 - 19 \times 4 = 7 - 76 = -69$

Hence, 20th term of the given sequence is -69.

Problems Based on *n*th Term of an AP

Some types of these problems are given below

Type I Problems Based on Finding *n*, When *n*th Term (or Last Term) and AP are Given

In this type of problems, an AP with last term is given or AP with nth term is given and we have to find the number of terms or the value of n.

It can be easily understood with the help of following examples

Example 7. How many terms are there in the sequence 3, 6, 9, 12, ..., 111?

Sol. Given sequence is 3, 6, 9, 12, ..., 111.

Here, $6 - 3 = 9 - 6 = 12 - 9 = 3$

So, it is an AP with first term, $a = 3$ and common difference, $d = 3$.

Let there be n terms in the given sequence.

Then, nth term $(a_n) = 111$

$\Rightarrow \quad a + (n-1)d = 111 \quad [\because a_n = a + (n-1)d]$

$\Rightarrow \quad 3 + (n-1) \times 3 = 111 \quad [\because a = d = 3]$

$\Rightarrow \quad 3(1 + n - 1) = 111$

$\therefore \quad n = \frac{111}{3} = 37$

Hence, the given sequence contains 37 terms.

Example 8. Which term of an AP 21, 18, 15, ... is -81?

Sol. Given AP is $21, 18, 15, \ldots$.

Here, $a = 21$ and $d = 18 - 21 = -3$

Let nth term of the given AP be -81.

Then, $a_n = -81 \Rightarrow a + (n-1)d = -81$ $[\because a_n = a + (n-1)d]$

On putting the values of a and d, we get

$$21 + (n-1)(-3) = -81 \Rightarrow 21 - 3n + 3 = -81$$

$$\Rightarrow \quad 24 - 3n = -81 \Rightarrow -3n = -81 - 24 = -105$$

$$\Rightarrow \quad n = \frac{-105}{-3} = 35$$

Hence, 35th term of the given AP is – 81.

Type II Problems Based on Finding the AP or nth Term or Both, When its Two Terms are Given

In this type of problems, two terms (or a relation between two terms of an AP) are (is) given and we have to find the AP and the indicated term. It can be easily understood with the help of following example.

Example 9. Determine the general term of an AP whose 7th term is – 1 and 16th term is 17.

Sol. Let a be the first term and d be the common difference of the AP, whose 7th term is –1 and 16th term is 17.

Thus, $a_7 = -1$ and $a_{16} = 17$

Now, $a + (7-1)d = -1 \Rightarrow a + 6d = -1$ ...(i)

and $a + (16-1)d = 17 \Rightarrow a + 15d = 17$ ...(ii)

$[\because a_n = a + (n-1)d]$

On subtracting Eq. (i) from Eq. (ii), we get

$$a + 15d - a - 6d = 17 + 1$$

$$\Rightarrow \quad 9d = 18 \Rightarrow d = 2$$

On substituting $d = 2$ in Eq. (i), we get

$$a + 6 \times 2 = -1 \Rightarrow a + 12 = -1$$

$$\Rightarrow \quad a = -13$$

Hence, general term, $a_n = a + (n-1)d$

$= -13 + (n-1)2$ $[\because a = -13 \text{ and } d = 2]$

$= -13 + 2n - 2 = 2n - 15$

Type III Number/Word Problems

In this type of problems, first we write the AP or list of numbers (sequence) with the help of given information and then find the required value.

It can be easily understood with the help of following examples

Example 10. How many numbers of two digits are divisible by 7?

Sol. Two-digits numbers are 10, 11, 12, 13, 14, 15, ..., 97, 98, 99 in which only 14, 21, 28, ..., 98 are divisible by 7.

Here, $21 - 14 = 28 - 21 = 7$

So, the list of numbers 14, 21, 28, ..., 98 forms an AP, whose first term $(a) = 14$ and common difference $(d) = 7$.

Let there are n terms in the above sequence, then

$$a_n = 98$$

$$\Rightarrow \quad a + (n-1)d = 98 \quad [\because a_n = a + (n-1)d]$$

$$\Rightarrow \quad 14 + (n-1)7 = 98 \quad [\because a = 14 \text{ and } d = 7]$$

$$\Rightarrow \quad 14 + 7n - 7 = 98 \Rightarrow 7n = 91$$

$$\Rightarrow \quad n = \frac{91}{7} = 13$$

Hence, 13 numbers of two digits are divisible by 7.

Example 11. A sum of ₹ 2000 is invested at 7% simple interest per year. Calculate the interest at the end of each year. Do these interest form an AP? If so, then find the interest at the end of 20 yr making use of this fact.

Sol. Given, initial money $P =$ ₹ 2000

Rate of interest, $R = 7\%$ per year; Time, $T = 1, 2, 3, 4, \dots$

We know that simple interest is given by the formula

$$\text{SI} = \frac{PRT}{100}$$

$\therefore$ SI at the end of 1st year $= \dfrac{2000 \times 7 \times 1}{100} =$ ₹ 140

SI at the end of 2nd year $= \dfrac{2000 \times 7 \times 2}{100} =$ ₹ 280

SI at the end of 3rd year $= \dfrac{2000 \times 7 \times 3}{100} =$ ₹ 420

Thus, required list of numbers is 140, 280, 420, ...

Here, $280 - 140 = 420 - 280 = 140$

So, above list of numbers forms an AP, whose first term $(a) = 140$ and common difference $(d) = 140$.

Now, SI at the end of 20 yr will be equal to 20th term of the above AP.

$\because \quad a_{20} = a + (20-1)d = 140 + 19 \times 140$ $[\because a_n = a + (n-1)d]$

$= 140 + 2660 = 2800$

Hence, the interest at the end of 20 yr will be ₹ 2800.

nth Term from the End of an AP

Let 'a' be the first term, 'd' be the common difference and 'l' be the last term of an AP, then nth term from the end can be found by the formula

$$\boxed{n\text{th term from the end} = l - (n-1)d}$$

Example 12. Determine the 10th term from the end of an AP 4, 9, 14, ..., 254.

Sol. Given, AP is 4, 9, 14,..., 254.

Here, $l =$ last term $= 254$

and $d =$ common difference $= 9 - 4 = 5$

$\because$ nth term from the end $= l - (n-1)d$

$\therefore$ 10th term from the end $= l - (10-1)d = l - 9d$

$= 254 - 9 \times 5 = 254 - 45 = 209$

Alternate Method

On reversing the given AP, new AP is 254, ..., 14, 9, 4.

Here, first term $(a) = 254$

and common difference $(d) = -5$

Now, 10th term of new AP $= a_{10} = 254 + (10-1)(-5)$

$[\because a_n = a + (n-1)d]$

$= 254 - 9 \times 5 = 209$

Hence, 10th term from the end of the given AP is 209.

Selection of Terms in an AP

Sometimes, we have to select certain number of terms in an AP. The convenient way of selecting terms is given below

Number of terms	Terms	Common difference
3	$a-d, a, a+d$	d
4	$a-3d, a-d, a+d, a+3d$	$2d$
5	$a-2d, a-d, a, a+d, a+2d$	d

Note If the terms of an AP are not given, then the terms can be chosen as a, $a+d$, $a+2d$, … .

Example 13. The sum of three numbers in AP is 15 and their product is 105. Find the numbers.

Sol. Let three numbers in AP be $a-d$, a and $a+d$.

According to the question,

Sum of numbers = 15

$\therefore \quad (a-d)+a+(a+d)=15 \Rightarrow 3a=15 \Rightarrow a=5$

and the product of numbers = 105

$\therefore \quad (a-d)(a)(a+d)=105$

$\Rightarrow \quad (5-d)(5)(5+d)=105$ [put $a=5$]

$\Rightarrow \quad (25-d^2)5=105$ [$\because (A-B)(A+B)=A^2-B^2$]

$\Rightarrow \quad 25-d^2=21$ [dividing both sides by 5]

$\Rightarrow \quad d^2=4 \Rightarrow d=\pm 2$

When $d=2$, then numbers in AP are $5-2$, 5 and $5+2$ i.e. 3, 5, 7.

When $d=-2$, then terms in AP are $5+2$, 5, $5-2$, i.e. 7, 5, 3.

Topic Exercise 1

1. Write first three terms of the following sequences.

(i) $a_n=5n+3$ (ii) $a_n=3n^2-2$ (iii) $a_n=\dfrac{n-3}{2}$

2. Write the first four terms in each of the list of numbers defined by

(i) $a_n=2n+3$ (ii) $a_n=(-1)^{n-1}n^4$ (iii) $a_n=\dfrac{(n-1)^2}{3}$

3. Which of the following form an AP? Justify your answer.

(i) 1, 1, 2, 2, 3, 3, … (ii) $\sqrt{3}, \sqrt{12}, \sqrt{27}, \sqrt{48}, \ldots$

(iii) $-1, \dfrac{1}{4}, \dfrac{3}{2}, \dfrac{11}{4}, \ldots$ (iv) 0.5, 0.8, 1.1, …

(v) $-3, -5, -6, -8, \ldots$

4. Check that the list of numbers defined by the following nth term is an AP or not. Also, give the reason.

(i) $t_n=3n+5$ (ii) $t_n=9-11n^2$

(iii) $t_n=8n^2+n$ (iv) $t_n=4n^3+3$

5. Show that the sequence defined by $a_n=3n^2-5$ is not an AP.

6. Write the common difference of the following AP.

(i) $\sqrt{3}, 2\sqrt{3}, 3\sqrt{3}, \ldots$ (ii) $\dfrac{1}{6}, \dfrac{1}{3}, \dfrac{1}{2}, \ldots$

(iii) $20, 19\dfrac{1}{4}, 18\dfrac{1}{2}, 17\dfrac{3}{4}, \ldots$

7. Examine that the sequence 7, 13, 19, 25, … is an AP. Also, find the common difference.

8. Show that the sequence, defined by its nth term $\dfrac{3+n}{4}$, forms an AP. Also, find the common difference of it.

9. In the following, find the first four terms of an AP, whose

(i) first term is -2 and the common difference is -2.

(ii) first term is $-\dfrac{1}{2}$ and the common difference is -1.

(iii) first term is 0.5 and the common difference is 1.2.

10. Find the common difference and the next two terms of the following AP.

(i) 6, 12, 18, … (ii) $\dfrac{1}{3}, \dfrac{4}{3}, \dfrac{7}{3}, \ldots$ (iii) $-15, -13, -11, \ldots$

11. If $(k-3), (2k+1), (4k+3)$ are three consecutive terms of an AP, find the value of k. ***[2018]***

12. Find the following terms of an AP.

(i) 10th term of the sequence $-5, -5/2, 0, 5/2, \ldots$

(ii) 11th term of the sequence $-3, -1/2, 2, \ldots$

(iii) 23rd term of the sequence 7, 5, 3, 1, …

13. If $d=-4$, $n=7$ and $a_n=4$, then find the value of a.

14. The first term of an AP is 5, common difference is 3 and the last term is 80. Find the number of terms.

15. Check 0 is a term of an AP : 31, 28, 25, … .

16. Find p, if the given value of x is the pth term of an AP 25, 50, 75, 100, … ; $x=1000$.

17. The 11th term of an AP is 80 and the 16th term is 110. Find the 31st term.

18. Which term of AP : 3, 8, 13, 18, … will be 130 more than its 31st term?

19. How many numbers lie between 10 and 300, which divided by 4 leave a remainder 3?

20. In a flower bed, there are 43 rose plants in the first row, 41 in the second, 39 in the third and so on. There are 11 rose plants in the last row. How many rows are there in the flower bed?

21. A man starts repaying a loan as first instalment of ₹ 100. If he increases the instalment by ₹ 5 every month, then what amount he will pay in the 30th instalment?

22. A sum of ₹ 5000 is invested at 8% simple interest per annum. Calculate the interest at the end of each year. Do these interests form an AP? Find the interest at the end of 30 yr.

23. Find the 6th term from the end of an AP 17, 14, 11, ..., – 40.

24. In an AP, if $a = 10$, $d = 5$ and $n = 100$, then find the value of a_{100} and also find the 50th term from the end.

25. Split 207 into three parts such that these are in AP and the product of two smaller parts is 4623.

26. The angles of a triangle are in AP. The greatest angle is twice the least. Find all the angles of the triangles.

Hints and Answers

1. Hint To find the first three terms of the sequence, put $n = 1$, 2 and 3.

Ans. (i) 8, 13 and 18 (ii) 1, 10, 25 (iii) $-1, -\frac{1}{2}, 0$

2. Hint To find the first four terms of the sequence, put $n = 1$, 2, 3 and 4.

Ans. (i) 5, 7, 9, 11 (ii) 1, -16, 81, -256 (iii) $0, \frac{1}{3}, \frac{4}{3}, 3$

3. (i) **Hint** Here, $a_2 - a_1 \neq a_3 - a_2$ **Ans.** No

(ii) **Hint** Here, $a_2 - a_1 = a_3 - a_2 = a_4 - a_3$ **Ans.** Yes

(iii) Do same as part (ii). **Ans.** Yes

(iv) Do same as part (ii). **Ans.** Yes

(v) Do same as part (i). **Ans.** No

4. (i) **Hint** Here, $a_{n+1} - a_n = [3(n+1)+5]-[3n+5]$

$$= 3n + 3 + 5 - 3n - 5 = 3$$

Ans. Yes, because $a_{n-1} - a_n$ is independent of n.

(ii) **Hint** Here,

$$a_{n+1} - a_n = [a - 11(n+1)^2] - [a - 11n^2]$$
$$= 11n^2 - 11(n+1)^2 = -11(2n+1)$$

Ans. No, because $a_{n+1} - a_n$ is not independent of n.

(iii) Do same as part (ii). **Ans.** No, because $a_{n+1} - a_n$ is not independent of n.

(iv) Do same as part (ii). **Ans.** No, because $a_{n+1} - a_n$ is not independent of n.

5. Hint Here, $a_{n+1} - a_n$ is not independent of n.

6. (i) **Hint** Here, $d = a_2 - a_1 = 2\sqrt{3} - \sqrt{3}$. **Ans.** $\sqrt{3}$

(ii) **Hint** Here, $d = a_2 - a_1 = \frac{1}{3} - \frac{1}{6}$. **Ans.** $\frac{1}{6}$

(iii) **Hint** Here, $d = a_2 - a_1 = 19\frac{1}{4} - 20 = \frac{77}{4} - 20$.

Ans. – 3/4

7. Hint $a_2 - a_1 = a_3 - a_2 = \ldots = 6$. **Ans.** Yes, 6

8. Hint Here, $a_{n+1} - a_n$ is independent of n. **Ans.** 1/4

9. Hint $a_2 = a_1 + d$; $a_3 = a_2 + d$ and $a_4 = a_3 + d$

Ans. (i) $-2, -4, -6, -8$ (ii) $-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}$

(iii) 0.5, 1.7, 2.9, 4.1

10. Do same as Example 5.

Ans. (i) $d = 6$; 24, 30 (ii) 1; $\frac{10}{3}, \frac{13}{3}$ (iii) 2; $-9, -7$

11. Hint

$$\therefore \quad 2k + 1 = \frac{k - 3 + 4k + 3}{2}$$

Ans. $k = 2$

12. Do same as Example 6.

Ans. (i) 17.5 (ii) 22 (iii) – 37

13. Hint Use the formula, $a_n = a + (n-1)d$. **Ans.** 28

14. Hint Let the number of terms be n.

Then, $l = a + (n-1)d$. **Ans.** 26

15. Hint Let $a_n = 0$, then by using the formula, $a_n = a + (n-1)d$, we get fractional value of n, which is not possible. **Ans.** No

16. Hint Here, $a_p = 1000$. **Ans.** 40

17. Do same as Example 9. **Ans.** 200

18. Hint Let $a_n = a_{31} + 130$

$\Rightarrow \quad a + (n-1)d = a + (31-1)d + 130$

$\Rightarrow \quad 3 + (n-1)(5) = 3 + 30 \times 5 + 130$. **Ans.** 57th

19. Hint The required numbers are 11, 15, ..., 299.

Now, let there be n numbers.

Then, $a_n = 299 \Rightarrow 299 = 11 + (n-1)\,4$ **Ans.** 73

20. Hint Here, $a = 43$, $d = 41 - 43 = -2$ and $a_n = 11$.

Now, use the formula, $a_n = a + (n-1)d$. **Ans.** 17

21. Hint Here, $a = 100$, $d = 105 - 100 = 5$ and $n = 30$.

Now, find, $a_{30} = a + (30-1)d$. **Ans.** ₹ 245

22. Do same as Example 11. **Ans.** Yes, ₹ 12000

23. Do same as Example 12. **Ans.** –25

24. Hint Use the formulae, nth term from the beginning, $a_n = a + (n-1)d$ and nth term from the end $= l - (n-1)d$. **Ans.** 505, 260

25. Hint Let three parts be $a - d$, a, $a + d$.

Then, $(a - d) + a + (a + d) = 207$

and $\quad a(a - d) = 4623$ **Ans.** 67, 69, 71

26. Hint Let three angles be $(a-d)^0, a^0, (a+d)^0$.

Then, we have $(a - d) + a + (a + d) = 180$

and $\quad (a + d) = 2(a - d)$ **Ans.** 40°, 60°, 80°

Topic 2

Sum of First n Terms of an AP

If first term of an AP is 'a' and its common difference is 'd', then the sum of its first n terms S_n, is given by the formula

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$$

This formula can also be written as

$$\boxed{S_n = \frac{n}{2}(a + a_n)}$$

where, a_n = nth term of the AP

If l is the last term of an AP of n terms, then sum of all the terms is given by the formula

$$\boxed{S_n = \frac{n}{2}(a + l)}$$

Example 1. Find the sum of the first 22 terms of the AP 8, 3, – 2, … .

Sol. Given AP is 8, 3, – 2, … .

Here, first term $(a) = 8$,

common difference $(d) = 3 - 8 = -5$ and $n = 22$

$\because$ Sum of first n terms $(S_n) = \frac{n}{2}[2a + (n-1)d]$

$\therefore$ Sum of first 22 terms $(S_{22}) = \frac{22}{2}[2 \times 8 + (22-1) \times (-5)]$

$= 11[16 + 21 \times (-5)]$

$= 11(16 - 105)$

$= 11(-89) = -979$

Hence, the sum of first 22 terms of an AP is – 979.

Example 2. Find the sum of first 20 terms of an AP in which $a = 1$ and 20th term = 58.

Sol. Given, $a = 1$, $l = 58$ and $n = 20$

Now, sum of 20 terms, $S_{20} = \frac{20}{2}(1 + 58)$ $\left[\because S_n = \frac{n}{2}(a + l)\right]$

$= 10(59) = 590$

Example 3. Find the sum of first 24 terms of an AP, whose nth term is given by $a_n = 3 + 2n$.

Sol. Given, nth term of an AP, $a_n = 3 + 2n$

Clearly, sum of first 24 terms

$S_{24} = \frac{24}{2}(a_1 + a_{24})$ $\left[\because S_n = \frac{n}{2}(a + a_n) \text{ and } a = a_1\right]$

$= \frac{24}{2}[(3 + 2 \times 1) + (3 + 2 \times 24)]$

$= \frac{24}{2}(3 + 2 + 3 + 48)$

$= 12 \times 56 = 672$

Hence, the required sum of first 24th terms of an AP is 672.

Problems Based on Sum of First n Terms of an AP

Some types of these problems are given below

Type I Problems Based on Finding the Sum of First m Terms, When Sum of First p Terms and q Terms are Given

In this type of problems, we form two equations in a and d with the help of given information and solve them to get a and d. Then, find the sum of required number of terms. It can be easily understood with the help of following example.

Example 4. The 4th term of an AP, is 22 and 15th term is 66. Find the first term and the common difference. Hence find the sum of the series to 8 terms. ***[2018]***

Sol. Given, $T_4 = 22$

$\Rightarrow \quad a + 3d = 22 \quad$ …(i)

and $\quad T_{15} = 66$

$\Rightarrow \quad a + 14d = 66 \quad$ …(ii)

On subtracting Eq. (i) from Eq. (ii), we get

$a + 14d - (a + 3d) = 66 - 22$

$\Rightarrow \quad 11d = 44$

$\Rightarrow \quad d = \frac{44}{11} = 4$

On putting the value of d in Eq. (i), we get

$a + 3 \times 4 = 22$

$\Rightarrow \quad a + 12 = 22$

$\Rightarrow \quad a = 22 - 12 = 10$

$\because \quad S_n = \frac{n}{2}[2a + (n-1)d]$

$\therefore \quad S_8 = \frac{8}{2}[2 \times 10 + (8-1) \times 4]$

$= 4[20 + 28]$

$= 4 \times 48 = 192$

Type II Problems Based on Finding the Number of Terms, When Sum of Terms and AP are Given

In this type of problems, we first find a and d and then assume that number of terms be n. After that, we use the formula of sum of first n terms to calculate n. If a is positive and d is negative, then sometime we get two values of n because in this case, sum of its some terms becomes zero.

It can be easily understood with the help of following example.

Example 5. How many terms of the AP $20, 19\frac{1}{3}, 18\frac{2}{3}$, ... must be taken, so that their sum is 300?

Sol. Given AP is $20, 19\frac{1}{3}, 18\frac{2}{3}, \ldots$.

Here, $a = 20$

and $d = 19\frac{1}{3} - 20 = \frac{58}{3} - 20 = \frac{58-60}{3} = \frac{-2}{3}$

Let n terms of the given AP be required to get the sum 300.

We know that, $S_n = \frac{n}{2}[2a + (n-1)d]$

$$\Rightarrow \quad 300 = \frac{n}{2}\left[2(20) + (n-1)\left(\frac{-2}{3}\right)\right] \quad [\because a = 20 \text{ and } d = -2/3]$$

$$\Rightarrow \quad 600 = n\left(40 - \frac{2}{3}n + \frac{2}{3}\right)$$

$$\Rightarrow \quad 600 = \frac{1}{3}(120n - 2n^2 + 2n)$$

$$\Rightarrow \quad 600 \times 3 = 122n - 2n^2$$

$$\Rightarrow \quad 1800 + 2n^2 - 122n = 0$$

$$\Rightarrow \quad 2(n^2 - 61n + 900) = 0$$

$$\Rightarrow \quad n^2 - 61n + 900 = 0 \quad [\because 2 \neq 0]$$

$$\Rightarrow \quad n^2 - 36n - 25n + 900 = 0$$

$$\Rightarrow \quad n(n-36) - 25(n-36) = 0$$

$$\Rightarrow \quad (n-36)(n-25) = 0$$

$$\Rightarrow \quad n = 36 \text{ or } 25$$

Since, a is positive and d is negative, so both values of n are possible.

Hence, sum of first 25 terms of given AP

= Sum of first 36 terms of given AP = 300.

Type III Problems Based on Finding the nth Term, When the Sum of First n Terms is Given

If S_n is the sum of first n terms of an AP, then its nth term is given by

$$\boxed{a_n = S_n - S_{n-1}}$$

It can be easily understood with the help of following example.

Example 6. If the sum of first n terms of an AP is given by $S_n = 6n + 7n^2$, then find the nth term of the AP. Also, find 10th term of the AP.

Sol. We have, $S_n = 6n + 7n^2$ ...(i)

On replacing n by $(n-1)$ in Eq. (i), we get

$$S_{n-1} = 6(n-1) + 7(n-1)^2 = 6n - 6 + 7(n^2 + 1 - 2n) \quad [\because (a-b)^2 = a^2 + b^2 - 2ab]$$

$$= 6n - 6 + 7n^2 + 7 - 14n = 7n^2 - 8n + 1 \quad \ldots\text{(ii)}$$

Clearly, nth term, $a_n = S_n - S_{n-1}$

$$= 6n + 7n^2 - 7n^2 + 8n - 1 = 14n - 1$$

Now, 10th term of the AP, [from Eqs. (i) and (ii)]

$$a_{10} = 14 \times 10 - 1 = 140 - 1 = 139$$

Applications of Arithmetic Progression (AP)

To solve word problems, first we form the list of numbers with the help of given information and check that this list is an AP or not. If it is an AP, then use the formula of sum of first n terms and calculate the missing value.

Example 7. A man repays a loan of ₹ 3250 by ₹ 20 in the first month and then increases the payment by ₹ 15 every month. How long will it take him to clear the loan?

Sol. Given, total amount of loan = ₹ 3250

Amount paid in first month = ₹ 20

and amount increases every month = ₹ 15

Clearly, the amounts of repayment form an AP with first term $(a) = 20$ and common difference $(d) = 15$.

Let the loan be cleared in n months.

Then, $S_n = 3250$

$$\Rightarrow \quad \frac{n}{2}[2a + (n-1)d] = 3250 \quad \left[\because S_n = \frac{n}{2}\{2a + (n-1)d\}\right]$$

$$\Rightarrow \quad \frac{n}{2}[2(20) + (n-1)15] = 3250$$

$$\Rightarrow \quad n(40 + 15n - 15) = 3250 \times 2$$

$$\Rightarrow \quad 25n + 15n^2 = 6500$$

$$\Rightarrow \quad 3n^2 + 5n - 1300 = 0 \quad \text{[dividing both sides by 5]}$$

$$\Rightarrow \quad 3n^2 + 65n - 60n - 1300 = 0$$

$$\Rightarrow \quad n(3n + 65) - 20(3n + 65) = 0$$

$$\Rightarrow \quad (n - 20)(3n + 65) = 0$$

$$\Rightarrow \quad n = 20 \text{ or } n = \frac{-65}{3}$$

Since, n should be a positive integer, so neglect $n = \frac{-65}{3}$.

Thus, $n = 20$

Hence, the loan will clear in 20 months.

Example 8. Kanika was given her pocket money on Jan 1st, 2018. She puts ₹ 1 on day 1, ₹ 2 on day 2, ₹ 3 on day 3 and continued doing so till the end of the month, from this money into her piggy bank. She also spent ₹ 204 of her pocket money and found that at the end of the month she still had ₹ 100 with her. How much was her pocket money for the month?

Sol. Let her pocket money be ₹x.

Given, she puts ₹ 1 on day 1, ₹ 2 on day 2, ₹ 3 on day 3 and so on till the end of the month, into her piggy bank.

Clearly, the amounts that she puts everyday of the month, form an AP, in which number of terms is 31, first term $(a) = 1$ and common difference $(d) = 2 - 1 = 1$.

Now, sum of first 31 terms,

$$S_{31} = \frac{31}{2}[2 \times 1 + (31-1) \times 1] \quad \left[\because S_n = \frac{n}{2}\{2a + (n-1)d\}\right]$$

$$= \frac{31}{2}(2+30) = \frac{31 \times 32}{2} = 31 \times 16 = 496$$

So, Kanika puts ₹496 into her piggy bank.

Also, it is given that, she spent ₹204 of her pocket money and found that at the end of the month, she still had ₹100 with her.

According to the condition,

$$(x - 496) - 204 = 100$$

$$\Rightarrow \quad x - 700 = 100$$

$$\Rightarrow \quad x = ₹\,800$$

Hence, ₹ 800 was her pocket money for the month.

Topic Exercise 2

1. Find the sum of the following AP.
   (i) 50, 46, 42, ... upto 10 terms.
   (ii) 2, 4, 6, ... upto 100 terms.
   (iii) $-37, -33, -29, \ldots$ upto 12 terms.
   (iv) 0.6, 1.7, 2.8, … upto 100 terms.
   (v) $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}, \ldots$ upto 11 terms.
2. Find the sum of first 25 terms of an AP, whose nth term is $1-4n$.
3. Find the sum of first 21 terms of an AP, whose 2nd term is 8 and 4th term is 4.
4. Find the sum of first 17 terms of an AP, where 4th and 9th terms are -15 and -30, respectively.
5. If sum of first 6 terms of an AP is 36 and that of the first 16 terms is 256, then find the sum of first 10 terms.
6. If the sum of first p terms of an AP is q and the sum of first q terms is p, then find the sum of first $(p+q)$ terms.
7. The sum of the first five terms and the sum of the first seven terms of an AP is 167. If the sum of the first ten terms of this AP is 235, then find the sum of its first twenty terms.
8. Find the number of terms of the following AP.
   (i) 64, 60, 56, ... having sum 544.
   (ii) 9, 17, 25, … having sum 636.
   (iii) $-7, -4, -1, \ldots$ having sum 210.
   (iv) 0.6, 0.9, 1.2, … having sum 69.
9. The sum of the first n terms of an AP is given by $S_n = 2n^2 + 5n$. Find the nth term of the AP.
10. If the sum of first n terms of an AP is given by $S_n = n(4n+1)$, then find the nth term of the AP. Also, find the AP.
11. The sum of first six terms of an arithmetic progression is 42. The ratio of its 10th term to its 30th term is 1 : 3. Calculate the first and the 13th terms of the AP.
12. Prove that the sum of later half of $2n$ terms of an AP is equal to one-third of the sum of the first $3n$ terms.
13. Yasmeen saves ₹ 32 during the first month, ₹ 36 in the second month and ₹ 40 in the third month. If she continues to save in this manner, in how many months will she save ₹ 2000?
14. 228 logs are to be stacked in a store in the following manner: 30 logs in the bottom, 28 in the next row, then 26 and so on, in how many rows can these 228 logs be stacked? How many logs are there in the last row?
15. The ages of the students in a class are in AP, whose common difference is 4 months. If the youngest student is 8 yr old and the sum of the ages of all the students is 168 yr, then find the number of students in the class.
16. A man arranges to pay off a debt of ₹ 3600 by 40 annual instalments which are in AP. When 30 of the instalments are paid, he dies leaving one-third of the debt unpaid. Find the value of the 8th instalment.

Hints and Answers

1. Do same as Example 1.
   Ans. (i) 320 (ii) 10100 (iii) – 180 (iv) 5505 (v) 33/20
2. Do same as Example 3. **Ans.** -1275
3. **Hint** Here, we have $a_2 = 8$ and $a_4 = 4$
   $\Rightarrow \quad a + d = 8$ and $a + 3d = 4$
   On solving these equations, we get the values of a and d.
   Now, S_{21} can be obtained by using
   $$S_n = \frac{n}{2}[2a + (n-1)d] \quad \textbf{Ans. } -210$$
4. Do same as Q. 3. **Ans.** -510

5. Do same as Example 4. **Ans.** 100

6. **Hint** We have, $S_p = q$ and $S_q = p$.

$\Rightarrow \quad S_p - S_q = q - p$

$2a + (p+q-1)d = -2$

Now, $S_{p+q} = \dfrac{p+q}{2}[2a + (p+q-1)d]$

$= \left(\dfrac{p+q}{2}\right)[-2] = -(p+q)$

Ans. $-(p+q)$

7. **Hint** Given, $S_5 + S_7 = 167$ and $S_{10} = 235$

$\therefore \; 12a + 31d = 167$ and $2a + 9d = 47$

Now, find a and d, and then S_{20}.

Ans. $S_{20} = 970$

8. Do same as Example 5.

Ans. (i) 16, 17 (ii) 12 (iii) 15 (iv) 20

9. Do same as Example 6.

Ans. $4n + 3$

10. Do same as Example 6.

Ans. $8n - 3$; 5, 13, 21, ...

11. **Hint** Given, $S_6 = 42$ and $\dfrac{a_{10}}{a_{30}} = \dfrac{1}{3}$.

This implies $2a + 5d = 14$ and $a = d$, respectively.

Ans. $a_1 = 2$ and $a_{13} = 26$

12. **Hint To prove,** $S_{2n} - S_n = \dfrac{1}{3}S_{3n}$

13. **Hint** Here, $a = 32$, $d = 36 - 32 = 4$

and $\quad S_n = 2000$

Use the formula, $S_n = \dfrac{n}{2}[2a + (n-1)d]$

Ans. 25 months

14. **Hint** Here, $a = 30$, $d = 28 - 30 = -2$ and $S_n = 228$

To find, n and l

Use the formula, $S_n = \dfrac{n}{2}[2a + (n-1)d]$ and

$l = a + (n-1)d$.

Ans. 12 and 8

15. **Hint** Here, $a = 8$, $d = \dfrac{1}{3}$ yr and $S_n = 168$

$\left[\because 4 \text{ months } = \dfrac{4}{12} \text{ yr} = \dfrac{1}{3} \text{ yr}\right]$

Use the formula, $S_n = \dfrac{n}{2}[2a + (n-1)d]$

Ans. Number of students in the class $= 16$.

16. **Hint** Let the amount of instalments are a, $a + d$, $a + 2d$, ...

Then, according to given condition,

$$S_{40} = 3600 \text{ and } S_{30} = 3600 - \frac{1}{3} \times 3600 = 2400$$

$\Rightarrow 2a + 39d = 180$ and $2a + 29d = 160$

Now, find a and d and then find a_8.

Ans. $a_8 = ₹65$

Topic 3
Geometric Progression (GP)

A sequence of non-zero numbers is said to be a geometric progression, if the ratio of each term, except the first one, by its preceding term is always constant.

In other words, we can say that a sequence $a_1, a_2, \ldots, a_n$ is called **geometric progression** (geometric sequence), if it follows the relation $\dfrac{a_{k+1}}{a_k} = r$ (constant) for all $k \in N$.

The constant ratio is called **common ratio** of the GP and it is denoted by r. In a GP, we usually denote the first term by a and the nth term by t_n or a_n.

The general form of GP can be written as $a, ar, ar^2, ar^3, \ldots$ and so on.

e.g. The sequence 4, 12, 36, 108, ... is a GP.

Here, $a = 4$

and $\quad r = \dfrac{12}{4} = \dfrac{36}{12} = \dfrac{108}{36} = \ldots = 3$

Example 1. Examine that the sequence $2, 2\sqrt{2}, 4, 4\sqrt{2}, \ldots$ is a GP.

Sol. Here, $\quad \dfrac{a_2}{a_1} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}$

$$\frac{a_3}{a_2} = \frac{4}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2 \times 2} = \sqrt{2},$$

$$\frac{a_4}{a_3} = \frac{4\sqrt{2}}{4} = \sqrt{2} \text{ and so on.}$$

Since, ratio of any term by its preceding term is constant, so the given sequence is a GP.

Example 2. Examine that the list of numbers obtained from the following situation, will be in the form of a GP. An insect starts from a point and travels in a straight path 1 mm in first second and half of the distance covered in previous second in the succeeding second.

Sol. Given distance moved by the insect in 1st second = 1mm

$\therefore$ Distance moved by the insect in 2nd second = 1/2 mm

and so on distance moved by the insect in 3rd second = 1/4 mm

Here, $\frac{a_2}{a_1} = \frac{1/2}{1} = \frac{1}{2}$

$\frac{a_3}{a_2} = \frac{1/4}{1/2} = \frac{1}{2}$ and so on

i.e. $\frac{a_{k+1}}{a_k}$ is same for all $k \in N$.

So, the above list of numbers forms a GP.

Example 3. Find the common ratio of the following GP.

(i) $1, \sqrt{3}, 3, 3\sqrt{3}, \ldots$ (ii) $12, 4, 1\frac{1}{3}, \ldots$

(iii) $17, 17, 17, 17, \ldots$

Sol.

(i) Given GP is $1, \sqrt{3}, 3, 3\sqrt{3}, \ldots$

Here, $a_1 = 1, a_2 = \sqrt{3}, a_3 = 3, a_4 = 3\sqrt{3}$ and so on.

$\therefore$ Common ratio $(r) = \frac{a_2}{a_1} = \frac{\sqrt{3}}{1} = \sqrt{3}$

(ii) Given GP is $12, 4, 1\frac{1}{3}, \ldots$

Here, $a_1 = 12, a_2 = 4, a_3 = 1\frac{1}{3}$ and so on.

$\therefore$ Common ratio $(r) = \frac{a_2}{a_1} = \frac{4}{12} = \frac{1}{3}$.

(iii) Given GP is 17, 17, 17, 17, ...

Here, $a_1 = 17, a_2 = 17, a_3 = 17, a_4 = 17$ and so on.

$\therefore$ Common ratio $(r) = \frac{a_2}{a_1} = \frac{17}{17} = 1$.

Method to Write a GP, When First Term and Common Ratio are Given

To write a GP, the minimum information required is to know the first term a and the common ratio r of the geometric progression. Then, put the values of a and r in $a, ar, ar^2, ar^3, \ldots$ to get required GP.

Example 4. Write a GP having 2 as the first term and $\frac{1}{2}$ as the common ratio.

Sol. Given, first term $(a) = 2$ and common ratio $(r) = \frac{1}{2}$

On putting the values of a and r in the general form of GP, i.e. $a, ar, ar^2, ar^3, \ldots$, we get

$$2, 2 \times \frac{1}{2}, 2 \times \left(\frac{1}{2}\right)^2, 2 \times \left(\frac{1}{2}\right)^3, \ldots \quad \text{or} \quad 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots,$$

which is the required GP.

Properties of GP

(i) If a, b and c are in GP, then $\frac{b}{a} = \frac{c}{b}$, i.e. $b^2 = ac$

$\therefore$ a, b and c are in GP $\Rightarrow b^2 = ac$

Conversely, $b^2 = ac \Rightarrow a, b$ and c are in GP.

(ii) If each term of a GP is multiplied or divided by a fixed non-zero number, the resulting sequence is also a GP.

(iii) The sequence obtained by taking the reciprocals of the terms of a GP is also a GP.

e.g. 6, 24, 96, 384, ... is a GP.

$\Rightarrow \frac{1}{6}, \frac{1}{24}, \frac{1}{96}, \frac{1}{384}, \ldots$ is also a GP.

Example 5. For what values of k, the numbers $-\frac{2}{7}, k, -\frac{7}{2}$ are in GP ?

Sol. If $-\frac{2}{7}, k, -\frac{7}{2}$ are in GP, then

$$k^2 = \left(-\frac{2}{7}\right)\left(-\frac{7}{2}\right)$$

[$\because$ if a, b and c are in GP, then $\Rightarrow b^2 = ac$]

$\Rightarrow k^2 = 1 \Rightarrow k = \pm 1$

Example 6. If a, b and c are in GP, then find the value of $\frac{a-b}{b-c}$.

Sol. Given that, a, b and c are in GP.

Then, $\frac{b}{a} = \frac{c}{b} = r$ (constant) ...(i)

$\Rightarrow b = ar$ and $c = br$

$\Rightarrow b = ar$ and $c = (ar)\,r = ar^2$

Now, $\frac{a-b}{b-c} = \frac{a - ar}{ar - ar^2} = \frac{a(1-r)}{ar(1-r)} = \frac{1}{r}$

$\Rightarrow \frac{a-b}{b-c} = \frac{1}{r} = \frac{a}{b}$ or $\frac{b}{c}$ [using Eq. (i)]

General Term of a GP

If a is the first term of a GP and r its common ratio, then general term or nth term,

$$\boxed{a_n \text{ or } T_n = ar^{n-1}}$$

Problems Based on General Term of a GP

Some types of these problems are given below

Type I Problems Based on Finding the Indicated Term

In this type of problems, a GP is given and we have to find the value of indicated term.

It can be easily understood with the help of following example.

Example 7. Find the nth term and the 12th term of the sequence $-6, 18, -54, \ldots$.

Sol. Given sequence is $-6, 18, -54, \ldots$.

Clearly, the ratio of each term by its preceding term is same.

So, the given sequence forms a GP with

first term, $a = -6$ and common ratio, $r = \frac{18}{-6} = -3$.

Now, n th term,

$$a_n = ar^{n-1} = -6(-3)^{n-1} = (-1)^n 6 \cdot 3^{n-1}$$

and $$a_{12} = (-1)^{12} 6 \cdot 3^{12-1} = (1) \cdot 6 \cdot 3^{11} = 2 \cdot 3 \cdot 3^{11} = 2 \cdot 3^{12}$$

Hence, the nth term of given GP is $(-1)^n 6 \cdot 3^{n-1}$ and 12th term is $2 \cdot 3^{12}$.

Type II Problems Based on Finding the Position of a Given Term

In this type of problems, a GP is given and a term with unknown position is given. Then, to find the position of given term, let it be nth term and find the value of n. This value of n gives its position in given GP.

It can be easily understood with the help of following example.

Example 8. Which term of the GP 5, 10, 20, 40, ... is 5120?

Sol. Given GP is 5, 10, 20, 40, ...

Here, $a = 5$ and $r = \frac{10}{5} = 2$

Let nth term of given GP = 5120, i.e. $a_n = 5120$

Now, $a_n = ar^{n-1} = 5120$

$\Rightarrow \quad 5(2)^{n-1} = 5120 \quad [\because a = 5, r = 2]$

$\Rightarrow \quad 2^{n-1} = \frac{5120}{5} = 1024$

$\Rightarrow \quad 2^{n-1} = 1024$

$\Rightarrow \quad 2^{n-1} = 2^{10}$

On equating the powers, we get

$n - 1 = 10$

$\Rightarrow \quad n = 10 + 1 = 11$

Hence, 11th term of given GP is 5120.

Note We can equate the powers from both sides only when the bases are same, i.e. if $b^n = b^m$, then we can equate the powers (here, b is base).

Type III Problems Based on Finding the GP, When its Two Terms are Given

In this type of problem, two or more than two terms are given and we have to find the GP and the indicated term. It can be easily understood with the help of following examples.

Example 9. If the 4th and 9th terms of a GP are 54 and 13122 respectively, then find the GP.

Sol. Let a be the first term and r be the common ratio of GP.

Given, 4th term, $a_4 = 54$

$\Rightarrow \quad ar^{4-1} = 54$

$\Rightarrow \quad ar^3 = 54 \quad \ldots$(i)

and 9th term, $a_9 = 13122$

$\Rightarrow \quad ar^{9-1} = 13122$

$\Rightarrow \quad ar^8 = 13122 \quad \ldots$(ii)

On dividing Eq. (ii) by Eq. (i), we get

$$\frac{ar^8}{ar^3} = \frac{13122}{54}$$

$\Rightarrow \quad r^5 = 243$

$\Rightarrow \quad r^5 = (3)^5$

$\therefore \quad r = 3$

On putting the value of r in Eq. (i), we get

$a(3)^3 = 54$

$\Rightarrow \quad 27a = 54$

$\Rightarrow \quad a = \frac{54}{27} = 2$

$\therefore$ Required GP is $a, ar, ar^2, ar^3, \ldots$, i.e. 2, 6, 18, 54, ...

Example 10. If the pth and qth terms of a GP are q and p respectively, then show that its $(p + q)$th term is $\left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$.

Sol. Let the first term and common ratio of GP be a and r, respectively.

According to the question, pth term = q

$\Rightarrow \quad ar^{p-1} = q \quad \ldots$(i)

and qth term = p

$\Rightarrow \quad ar^{q-1} = p \quad \ldots$(ii)

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{ar^{p-1}}{ar^{q-1}} = \frac{q}{p} \quad \Rightarrow \quad r^{p-1-q+1} = \frac{q}{p}$$

$$\Rightarrow \quad r^{p-q} = \frac{q}{p} \quad \Rightarrow \quad r = \left(\frac{q}{p}\right)^{\frac{1}{p-q}}$$

On substituting the value of r in Eq. (i), we get

$$a\left(\frac{q}{p}\right)^{\frac{p-1}{p-q}} = q \Rightarrow a = \frac{q}{\left(\frac{q}{p}\right)^{\frac{p-1}{p-q}}} = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}}$$

Now, $(p+q)$th term, $a_{p+q} = a \cdot r^{p+q-1}$

$$= q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \cdot (r)^{p+q-1}$$

$$= q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \times \left(\frac{q}{p}\right)^{\frac{p+q-1}{p-q}}$$

$$= \frac{q^{1-\frac{p-1}{p-q}+\frac{p+q-1}{p-q}}}{p^{\frac{p+q-1}{p-q}-\frac{p-1}{p-q}}} = \frac{q^{\frac{p-q-p+1+p+q-1}{p-q}}}{p^{\frac{p+q-1-p+1}{p-q}}}$$

$$= \frac{q^{\frac{p}{p-q}}}{p^{\frac{q}{p-q}}} = \left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$$

Hence proved.

Example 11. The $(m+n)$th and $(m-n)$th terms of a GP are p and q, respectively. Show that the mth and nth terms are $\sqrt{pq}$ and $p\left(\frac{q}{p}\right)^{m/2n}$, respectively.

Sol. Let a be the first term and r be the common ratio.

Then, $\quad a_{m+n} = p$ and $a_{m-n} = q$

$\Rightarrow \quad ar^{m+n-1} = p$ and $ar^{m-n-1} = q \quad [\because n\text{th} = ar^{n-1}]$

$$\Rightarrow \quad \frac{ar^{m+n-1}}{ar^{m-n-1}} = \frac{p}{q}$$

$$\Rightarrow \quad r^{2n} = \frac{p}{q}$$

$$\Rightarrow \quad r = \left(\frac{p}{q}\right)^{1/2n} \Rightarrow \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}$$

Now, $\quad a_m = ar^{m-1}$

$$\Rightarrow \quad a_m = ar^{(m+n-1)}\left(\frac{1}{r}\right)^n$$

$$\Rightarrow \quad a_m = a_{m+n}\left(\frac{1}{r}\right)^n \quad [\because a_{m+n} = ar^{m+n-1}]$$

$$\Rightarrow \quad a_m = p\left(\frac{q}{p}\right)^{n/2n} \quad \left[\because a_{m+n} = p \text{ and } \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}\right]$$

$$\Rightarrow \quad a_m = p\left(\frac{q}{p}\right)^{1/2} = \sqrt{pq}$$

and $\quad a_n = ar^{n-1}$

$$\Rightarrow \quad a_n = ar^{m+n-1}\left(\frac{1}{r}\right)^m = a_{m+n}\left(\frac{1}{r}\right)^m \quad [\because a_{m+n} = ar^{m+n-1}]$$

$$\Rightarrow \quad a_n = p\left(\frac{q}{p}\right)^{m/2n} \quad \left[\because a_{m+n} = p \text{ and } \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}\right]$$

Hence proved.

nth Term of a GP from the End

Let a be the first term and r be the common ratio of a GP having m terms. Then, nth term from the end is $(m-n+1)$th term from the beginning.

$\therefore$ $\boxed{n\textbf{th term from the end} = ar^{m-n+1-1} = ar^{m-n}}$

$\therefore$ **where,** $m > n$.

Also, nth term from the end $= l\left(\frac{1}{r}\right)^{-1}$, where l is the last term of the GP.

Example 12. Find the 8th term from the end of the sequence 3, 6, 12, ... 25th term.

Sol. Here, $a = 3$, $r = \frac{6}{3} = 2$, $m = 25$ and $n = 8$.

Now, 8th term from the end of sequence is equal to the $(25-8+1)$, i.e. 18th term from the beginning,

and $\quad a_{18} = ar^{18-1} = 3(2)^{18-1} \quad [\because a = 3, r = 2]$

$= 3(2)^{17} = 3 \times 131072 = 393216$

Hence, the 8th term from the end is 393216.

Selection of Terms in GP

Sometimes we have to select certain number of terms in GP. The convenient way of selecting terms is given below

Number of terms	Terms	Common ratio
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r

Note If the terms of the GP are not given, then the terms can be chosen as a , ar, ar^2, ar^3,... .

Example 13. Find four numbers forming a GP in which the third term is greater than the first term by 9 and the second term is greater than 4th by 18.

Sol. Let the GP be $a, ar, ar^2, ar^3, \ldots$

Given, third term = first term + 9

$$\Rightarrow \quad ar^2 = a + 9$$

$$\Rightarrow \quad ar^2 - a = 9 \quad \ldots\text{(i)}$$

Also, it is given that, second term = fourth term + 18

$$\Rightarrow \quad ar = ar^3 + 18$$

$$\Rightarrow \quad ar - ar^3 = 18 \quad \ldots\text{(ii)}$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{ar^2 - a}{ar - ar^3} = \frac{9}{18}$$

$$\Rightarrow \quad \frac{a(r^2-1)}{ar(1-r^2)} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{-1}{r}\frac{(1-r^2)}{(1-r^2)} = \frac{1}{2}$$

$$\Rightarrow \quad -\frac{1}{r} = \frac{1}{2} \Rightarrow r = -2$$

On putting $r = -2$ in Eq. (i), we get

$$a(-2)^2 - a = 9$$

$$\Rightarrow \quad -4a - a = 9$$

$$\Rightarrow \quad 3a = 9$$

$$\Rightarrow \quad a = 3$$

$\therefore$ GP is $3, 3(-2), 3(-2)^2, 3(-2)^3, \ldots$

Hence, the required four numbers are $3, -6, 12$ and -24.

Note Here, 'greater than' word stands for excess in term.

Example 14. Let S be the sum, P be the product and R be the sum of the reciprocals of 3 terms of a GP. Then, find $P^2R^3 : S^3$.

Sol. Let us take a GP with three terms $\frac{a}{r}, a, ar$.

Then, $S = \frac{a}{r} + a + ar = \frac{a(r^2+r+1)}{r}$

$$P = \left(\frac{a}{r}\right)(a)(ar) = a^3$$

and $R = \frac{r}{a} + \frac{1}{a} + \frac{1}{ar} = \frac{1}{a}\left(\frac{r^2+r+1}{r}\right)$

Now, $$\frac{P^2R^3}{S^3} = \frac{a^6 \cdot \frac{1}{a^3}\left(\frac{r^2+r+1}{r}\right)^3}{a^3\left(\frac{r^2+r+1}{r}\right)^3} = 1$$

Therefore, the required ratio is 1 : 1.

Topic Exercise 3

1. Show that the following progressions is a GP. Also, find the common ratio in each case.
   (i) $4, -2, 1, -1/2, \ldots$ (ii) $a, \frac{3a^2}{4}, \frac{9a^3}{16}, \ldots$
2. For what value of x are the numbers $(x+9)$, $(x-6)$ and 4 are in GP?
3. Find the term of the following GP.
   (i) 11th term of the GP : $3, 6, 12, 24, \ldots$
   (ii) 10th term of the GP : $12, 4, \frac{4}{3}, \frac{4}{9}, \ldots$
   (iii) 17th term of the GP : $2, 2\sqrt{2}, 4, 8\sqrt{2}, \ldots$
   (iv) 8th term of the GP : $0.3, 0.06, 0.012, \ldots$
   (v) 12th term of the GP $\frac{1}{a^3x^3}, ax, a^5x^5, \ldots$
4. Find the 20th and nth terms of the following GP.
$$\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$$
5. Find the 10th term of the geometric series $5 + 25 + 125 + \ldots$. Also, find its nth term.
6. Find 12th term of a GP, whose 8th term is 192 and common ratio is 2.
7. Which term of the following sequences
   (i) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729?
   (ii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ is $\frac{1}{19683}$?
   (iii) $-3, -27, -243, \ldots$ is -1594323?
   (iv) $0.2, 0.4, 0.8, \ldots$ is 204.8?
8. The 4th term of a GP is 16 and 7th term is 128. Find the first term and common ratio of the series. ***[2018]***
9. Find the geometric series whose 5th and 8th terms are 80 and 640, respectively.
10. The first term of a GP is 1. Sum of the third term and fifth term is 90. Find the common ratio of GP.
11. Find a GP for which sum of the first two terms is -4 and fifth term is 4 times the third term.
12. If the 4th, 10th and 16th terms of a GP are x, y, z respectively. Prove that x, y, z are in GP.
13. If the pth, qth and rth terms of a GP are a, b and c respectively. Prove that $a^{q-r}\, b^{r-p}\, c^{p-q} = 1$.
14. If the first and nth terms of a GP are a and b respectively and P is the product of n terms, then prove that $P^2 = (ab)^n$.
15. Find the term of the following GP.
   (i) 6th term from the end of the GP $8, 4, 2, \ldots, \frac{1}{1024}$
   (ii) 4th term from the end of the GP.
16. Prove that in a finite GP the product of the terms equidistant from the beginning and the end is always same and equal to the product of first and last term.
17. If the sum of three number in GP is 13/12 and their product is -1, then find the numbers.

18. The sum of first three terms of a GP is 39/10 and their product is 1. Find the first three terms of the GP if terms of GP are real numbers.

19. The product of first three terms of a GP is 1000. If 6 is added to its second term and 7 added to its third term, the terms become in AP. Find the GP.

20. The lengths of the sides of a triangle form a GP. If the perimeter of the triangle is 37 cm and the shortest side is of length 9 cm, then find the lengths of the other two sides.

Hints and Answers

1. Do same as Examples 1 and 3. **Ans.** (i) $-\frac{1}{2}$ (ii) $\frac{3a}{4}$

2. Hint For GP, $\frac{x-6}{x+9}=\frac{4}{x-6}$. **Ans.** 0 or 16

3. (i) **Hint** Here, $a=3$ and $r=2$. Now, use the formula $a_n=ar^{n-1}$, to find the required term. **Ans.** 3072

(ii) Do same as part (i). **Ans.** $\frac{4}{6561}$

(iii) Do same as part (i). **Ans.** 512

(iv) Do same as part (i). **Ans.** $(0.3)\,(0.2)^7$

(v) Do same as part (i). **Ans.** $(ax)^{41}$

4. Hint Here, $a=$ first term $=\frac{5}{2}$

and common ratio $(r)=\frac{5/4}{5/2}=\frac{2}{4}=\frac{1}{2}$

Use the formula, $a_n=ar^{n-1}$ to find the required terms.

Ans. $a_{20}=\frac{5}{2^{20}}$ and $a_n=\frac{5}{2^n}$

5. Hint Here, $a=5$ and $r=5$. Now, find a_{10} and a_n

Ans. $5^{10}, 5^n$

6. Hint Here, $a_8=192$ and $r=2$

$\because \quad a_8=ar^{8-1}=ar^7$

$\therefore \quad a=\frac{3}{2}$

Ans. $a_{12}=3072$

7. Do same as Example 8.

Ans. (i) 12 (ii) 9 (iii) 7 (iv) 11

8. Do same as Example 9.

Ans. First term $=2$ and common ratio $=2$

9. Hint

(i) Find GP as in Example 9.

(ii) Find the corresponding geometric series by replacing by + sign.

Ans. $5+10+20+40+\ldots$

10. Hint Given, $a=1$ and $a_3+a_5=90$

$\therefore \quad ar^2+ar^4=90$

$\Rightarrow \quad r^2+r^4=90$

$\Rightarrow \quad r^4+r^2-90=0$ [as $a=1$]

$\Rightarrow \quad (r^2+10)(r^2-9)=0 \Rightarrow r=\pm 3$

[$\because r^2=-10$ is not real number]

Ans. ± 3

11. Hint Given, $a+ar=-4$ ...(i)

and $a_5=4a_3 \Rightarrow ar^{5-1}=4ar^{3-1} \Rightarrow r=\pm 2$

If $r=2$, then from Eq. (i),

$a+a(2)=-4 \Rightarrow 3a=-4 \Rightarrow a=-\frac{4}{3}$

If $r=-2$, then from Eq. (i),

$a+a(-2)=-4 \Rightarrow -a=-4$

$\therefore \quad a=4$

Ans. GP is $\ldots -\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}, \ldots$ or $4, -8, 16, \ldots$

12. Hint Here, we have $ar^3=x$ ...(i)

$ar^9=y$ ...(ii)

and $ar^{15}=z$ ...(iii)

To prove $y^2=xz$

Now, multiply Eqs. (i) and (iii) to get the result.

13. Hint Let A be the first term and R be the common ratio.

Then, we have $AR^{P-1}=a$ ...(i)

$AR^{q-1}=b$ ...(ii)

$AR^{r-1}=c$ ...(iii)

Now, substitute the value of a, b and c in LHS, $a^{q-r}b^{r-p}c^{p-q}$ and simplify it to get the result.

14. Hint nth term, $ar^{n-1}=b$ ...(i)

Now, $P=$ Product of n terms

$\Rightarrow P=a\times ar^1\times ar^2\times ar^3\times\ldots$ to n terms

$\Rightarrow P=a^{1+1+1+\ldots+n \text{ terms}}\, r^{1+2+3+\ldots+(n-1)\text{ terms}}$

$\Rightarrow P=a^n r^{\frac{n(n-1)}{2}}$

$\Rightarrow P^2=a^n a^n r^{n(n-1)}=(a\cdot ar^{n-1})^n$

15. Do same as Example 12. **Ans.** (i) $\frac{1}{32}$ (ii) 6

16. Hint Let the GP consist n terms.

Then, $a_k=k$th term from the beginning $=a_1r^{k-1}$

$a_{n-k+1}=k$th term from the end $=a_n\left(\frac{1}{r}\right)^{k-1}$, where $1<k<n$,

$a_k\, a_{n-k+1}=(a_1r^{k-1})\,a_n\left(\frac{1}{r}\right)^{k-1}=a_1a_n$

for all k satisfying $1<k<n$.

17. Hint Let $\frac{a}{r}$, a, ar be three numbers which are in GP.

According to the question, we get

$$\frac{a}{r}+a+ar=\frac{13}{12} \quad \text{...(i)}$$

and $\frac{a}{r}\times a\times ar=-1 \Rightarrow a=-1$

On substituting $a=-1$ in Eq. (i), we get

$$\frac{-1}{r}+(-1)+(-1)r=\frac{13}{12} \Rightarrow 12r^2+25r+12=0$$

$$\Rightarrow \quad (3r+4)(4r+3)=0 \Rightarrow r=-\frac{4}{3} \text{ or } -\frac{3}{4}$$

Ans. $\frac{3}{4},-1,\frac{4}{3}$ or $\frac{4}{3},-1,\frac{3}{4}$

18. Do same as Q. 17. **Ans.** $\frac{5}{2},1,\frac{2}{5}$ or $\frac{2}{5},1,\frac{5}{2}$

19. Hint Let the terms of given GP be $\frac{a}{r}$, a, ar.

Then, we have product $=1000$

$$\Rightarrow \quad a^3=1000 \Rightarrow a=10$$

Also, it is given that $\frac{a}{r}$, $a+6$, $ar+7$ are in AP

$$\therefore \quad 2(a+6)=\frac{a}{r}+ar+7 \Rightarrow r=2, \frac{1}{2}$$

Ans. 5, 10, 20, … or 20, 10, 5, …

20. Hint Let lengths of the sides of triangle be $\frac{a}{r}$, a, ar, which forms a GP. Given, shortest side $=9$ cm

$$\therefore \quad \frac{a}{r}=9 \Rightarrow a=9r \quad \text{...(i)}$$

Also, given perimeter of the triangle $=37$

$$\therefore \quad \frac{a}{r}+a+ar=37 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get the required values.

Ans. 12 cm, 16 cm

Topic 4
Sum of First n Terms of a GP

If a and r are the first term and common ratio of a GP respectively, then sum of first n terms of this GP is given by

$$\boxed{S_n=\frac{a(1-r^n)}{1-r}} \quad \text{when } r<1$$

and $$\boxed{S_n=\frac{a(r^n-1)}{r-1}} \quad \text{when } r>1\,(r\neq 1)$$

Note

(i) If $r=1$, then $S_n=a+a+\ldots+n$ terms $=na$

(ii) Suppose a GP contains n terms with first term $=a$, common ratio $=r$ and last term $=l$. Then, sum of GP is given by

$$S_n=\frac{a-lr}{1-r}, \text{ when } r<1 \quad \text{and} \quad S_n=\frac{lr-a}{r-1}, \text{ when } r>1.$$

Problems Based on Sum of Terms of a GP

There are so many problems which can be solved by using the sum formula of a GP.

Some types of these problems are given below

Type I Problems Based on Finding the Sum of Given Number of Terms of Finite GP

In this type of problems, a finite GP is given and we have to find sum of given number of terms.

Example 1. Find the sum of the following sequences.

(i) $\frac{1}{2},\frac{3}{2},\frac{9}{2},\ldots$ upto 10 terms

(ii) $2,-\frac{1}{2},\frac{1}{8},\ldots$ upto 12 terms

Sol.

(i) Given sequence is $\frac{1}{2},\frac{3}{2},\frac{9}{2},\ldots$

Clearly, the ratio of any term by its preceding is same. So, the above sequence forms a GP, with first term, $a=\frac{1}{2}$ and common ratio, $r=\frac{3}{2}\div\frac{1}{2}=3>1$

$$\therefore \quad S_{10}=\frac{\frac{1}{2}(3^{10}-1)}{3-1} \quad \left[\because S_n=\frac{a(r^n-1)}{r-1}, \text{ as } r>1\right]$$

$$=\frac{1}{2}\cdot\frac{(3^{10}-1)}{2}=\frac{1}{4}(3^{10}-1)=14762$$

(ii) Given sequence is $2,-\frac{1}{2},\frac{1}{8},\ldots$

Clearly, the ratio of any term by its preceding term is same. So, the above sequence forms a GP with first term, $a=2$ and common ratio, $r=-\frac{1}{2}\div 2=\frac{-1}{4}<1$

$$\therefore \quad S_{12}=\frac{2\left[1-\left(\frac{-1}{4}\right)^{12}\right]}{1-\left(\frac{-1}{4}\right)} \quad \left[\because S_n=\frac{a(1-r^n)}{1-r}, \text{ as } r<1\right]$$

$$=\frac{2\left[1-\frac{1}{(4)^{12}}\right]}{1+\frac{1}{4}}=\frac{2\left[1-\left(\frac{1}{4}\right)^{12}\right]}{\left(\frac{5}{4}\right)}=\frac{8}{5}\left[1-\left(\frac{1}{4}\right)^{12}\right]$$

Example 2. Find the sum of the series
$$2 + 6 + 18 + 54 + \dots + 4374.$$

Sol. Given series is $2 + 6 + 18 + 54 + \dots + 4374$.

Clearly, the ratio of any term by its preceding term is same.

So, the above series forms a GP with first term, $a = 2$, common ratio, $r = \frac{6}{2} = 3 > 1$ and last term, $l = 4374$.

$\therefore$ Required sum $= \frac{(lr - a)}{(r - 1)} = \frac{(4374 \times 3 - 2)}{(3 - 1)}$

$= \frac{13120}{2} = 6560$

Hence, the sum of the given series is 6560.

Example 3. Find the sum to n terms of the sequence given by $a_n = 2^n + 3n,\ n \in N$.

Sol. Let S_n denotes the sum to n terms of the given sequence.

Then, $S_n = a_1 + a_2 + a_3 + \dots + a_n$

$\Rightarrow \quad S_n = (2^1 + 3 \times 1) + (2^2 + 3 \times 2) + (2^3 + 3 \times 3) + \dots + (2^n + 3 \times n)$

$\Rightarrow S_n = (2^1 + 2^2 + 2^3 + \dots + 2^n) + (3 \times 1 + 3 \times 2 + 3 \times 3 + \dots + 3 \times n)$

$= (2^1 + 2^2 + 2^3 + \dots + 2^n) + 3(1 + 2 + 3 + \dots + n)$

$= 2\left(\frac{2^n - 1}{2 - 1}\right) + 3\left\{\frac{n}{2}(1 + n)\right\}$

[$\because 2 + 2^2 + 2^3 + 2^n$ is a GP and $1 + 2 + 3 + \dots + n$ is an AP]

$= 2(2^n - 1) + \frac{3n}{2}(n + 1)$

Example 4. Find the sum of series
$$\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \dots \text{ to } 2n \text{ terms.}$$

Sol. Let $S = \frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \dots$ to $2n$ terms

$= \left[\frac{3}{5} + \frac{3}{5^3} + \dots \text{ to } n \text{ terms}\right] + \left[\frac{4}{5^2} + \frac{4}{5^4} + \dots \text{ to } n \text{ terms}\right]$

$= \frac{\frac{3}{5}\left[1 - \left(\frac{1}{5^2}\right)^n\right]}{\left[1 - \left(\frac{1}{5^2}\right)\right]} + \frac{\frac{4}{5^2}\left[1 - \left(\frac{1}{5^2}\right)^n\right]}{\left[1 - \left(\frac{1}{5^2}\right)\right]}$

$\left[\because S_n = \frac{a(1 - r^n)}{1 - r}, r < 1\right]$

$= \frac{\frac{3}{5}\left(1 - \frac{1}{5^{2n}}\right)}{1 - \frac{1}{25}} + \frac{\frac{4}{25}\left(1 - \frac{1}{5^{2n}}\right)}{1 - \frac{1}{25}}$

$= \frac{\frac{3}{5}\left(1 - \frac{1}{5^{2n}}\right)}{\frac{24}{25}} + \frac{\frac{4}{25}\left(1 - \frac{1}{5^{2n}}\right)}{\frac{24}{25}}$

$= \frac{5}{8}\left(1 - \frac{1}{5^{2n}}\right) + \frac{1}{6}\left(1 - \frac{1}{5^{2n}}\right)$

$= \left(1 - \frac{1}{5^{2n}}\right)\left(\frac{5}{8} + \frac{1}{6}\right) = \left(1 - \frac{1}{5^{2n}}\right)\left(\frac{15 + 4}{24}\right)$

$= \frac{19}{24}\left(1 - \frac{1}{5^{2n}}\right)$

Example 5. Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$.

Sol. Given sequences are 2, 4, 8, 16, 32 ...(i)

and 128, 32, 8, 2, $\frac{1}{2}$ ...(ii)

On multiplying the corresponding terms of Eqs. (i) and (ii), we get a new sequence 256, 128, 64, 32, 16

Let $\quad S = 256 + 128 + 64 + 32 + 16$

Clearly, the ratio of any term by its preceding term is same. So, the above obtained series forms a GP, with first term, $a = 256$ and common ratio, $r = \frac{128}{256} = \frac{1}{2} < 1$

$\therefore$ Required sum, $S_5 = \frac{256\left[1 - \left(\frac{1}{2}\right)^5\right]}{1 - \frac{1}{2}}$

$\left[\because S_n = \frac{a(1 - r^n)}{1 - r}, r < 1\right]$

$= 256 \times 2\left(1 - \frac{1}{2^5}\right)$

$= 512 \times \left(1 - \frac{1}{32}\right) = 512\left(\frac{32 - 1}{32}\right) = 16 \times 31 = 496$

Example 6. A GP consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

Sol. Let the GP be $a, ar, ar^2, ar^3, ar^4, \dots, ar^{2n-2}, ar^{2n-1}$, where $a, ar^2, ar^4, ar^6, \dots$ occupy odd places and $ar, ar^3, ar^5, ar^7, \dots$ occupy even places.

According to the question,

Sum of all terms $= 5 \times$ Sum of terms occupying odd places

$\Rightarrow \quad a + ar + ar^2 + \dots + ar^{2n-1} = 5 \times (a + ar^2 + ar^4 + \dots + ar^{2n-2})$

$\Rightarrow \quad \frac{a(r^{2n} - 1)}{r - 1} = \frac{5a[(r^2)^n - 1]}{r^2 - 1} \quad \left[\because S_n = \frac{a(r^n - 1)}{r - 1}, r > 1\right]$

$\Rightarrow \quad \frac{r^{2n} - 1}{r - 1} = \frac{5(r^{2n} - 1)}{(r - 1)(r + 1)} \Rightarrow 1 = \frac{5}{r + 1}$

$\Rightarrow \quad r + 1 = 5 \Rightarrow \quad r = 4$

Note Students should remember that if a GP has 2n (even terms), then there are n even and n odd terms.

Example 7. The sum of the first three terms of a GP is 16 and the sum of next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the GP.

Sol. Let the GP be $a, ar, ar^2, ar^3, \dots$.

According to the given condition,

Sum of first three terms $= a + ar + ar^2 = 16$ ...(i)

and sum of next three terms

$$= ar^3 + ar^4 + ar^5 = 128 \quad \text{...(ii)}$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{a + ar + ar^2}{ar^3 + ar^4 + ar^5} = \frac{16}{128}$$

$$\Rightarrow \quad \frac{a(1 + r + r^2)}{ar^3(1 + r + r^2)} = \frac{1}{8}$$

$$\Rightarrow \quad \left(\frac{1}{r}\right)^3 = \left(\frac{1}{2}\right)^3$$

On comparing the base of power 3 from both sides, we get

$$\frac{1}{r} = \frac{1}{2} \Rightarrow r = 2$$

On putting $r = 2$ in Eq. (i), we get

$$a + 2a + 4a = 16$$

$$\Rightarrow \quad 7a = 16 \Rightarrow a = \frac{16}{7}$$

Now, sum to n terms, $S_n = \frac{a(r^n - 1)}{r - 1}$ $[\because r = 2 > 1]$

$$= \frac{\frac{16}{7}(2^n - 1)}{2 - 1} = \frac{16}{7}(2^n - 1)$$

Hence, $a = \frac{16}{7}$, $r = 2$ and $S_n = \frac{16}{7}(2^n - 1)$.

Example 8. If S_1, S_2 and S_3 are respectively the sum of n, $2n$ and $3n$ terms of a GP, then prove that

$$S_1(S_3 - S_2) = (S_2 - S_1)^2.$$

Sol. Let a be the first term and r be the common ratio of the given GP. Then,

$$S_1(S_3 - S_2) = \frac{a(1 - r^n)}{(1 - r)} \cdot \left\{\frac{a(1 - r^{3n})}{(1 - r)} - \frac{a(1 - r^{2n})}{(1 - r)}\right\} \quad [\text{let } r < 1]$$

$$= \frac{a(1 - r^n)}{(1 - r)} \cdot \frac{(a - ar^{3n} - a + ar^{2n})}{(1 - r)}$$

$$= \frac{a(1 - r^n)}{(1 - r)} \cdot \frac{ar^{2n}(1 - r^n)}{(1 - r)} = \frac{a^2 r^{2n}(1 - r^n)^2}{(1 - r)^2} \quad \text{...(i)}$$

and $(S_2 - S_1)^2 = \left\{\frac{a(1 - r^{2n})}{(1 - r)} - \frac{a(1 - r^n)}{(1 - r)}\right\}^2$

$$= \frac{(a - ar^{2n} - a + ar^n)^2}{(1 - r)^2}$$

$$= \frac{\{ar^n(1 - r^n)\}^2}{(1 - r)^2} = \frac{a^2 r^{2n}(1 - r^n)^2}{(1 - r)^2} \quad \text{...(ii)}$$

Now, from Eqs. (i) and (ii), we get

$$S_1(S_3 - S_2) = (S_2 - S_1)^2$$

Note Students should remember that above proof can also be by assuming $r > 1$.

Example 9. Let S be the sum, P be the product and R be the sum of reciprocals of n terms in a GP. Prove that $P^2 R^n = S^n$.

Sol. Let the GP be $a, ar, ar^2, ar^3, \dots, ar^{n-1}$.

Given, $S =$ Sum of n terms

$$= a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$$

$$= \frac{a(r^n - 1)}{r - 1} \quad [\text{let } r > 1] \quad \text{...(i)}$$

$R =$ Sum of the reciprocals of n terms

$$= \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \dots + \frac{1}{ar^{n-1}} \quad \left[\because \frac{1}{r} < 1\right]$$

$$= \frac{\frac{1}{a}\left[1 - \left(\frac{1}{r}\right)^n\right]}{1 - \frac{1}{r}} = \frac{1}{a}\left(\frac{1}{1} - \frac{1}{r^n}\right) \times \frac{1}{\frac{r - 1}{r}}$$

$$= \frac{1}{a}\left(\frac{r^n - 1}{r^n}\right) \times \frac{r}{r - 1}$$

$$\Rightarrow \quad R = \frac{(r^n - 1)r}{ar^n(r - 1)} \quad \text{...(ii)}$$

and $P =$ Product of n terms

$$= a \times ar \times ar^2 \times ar^3 \times \dots \times ar^{n-1}$$

$$= a^{1 + 1 + 1 + \dots + n \text{ terms}} \cdot r^{1 + 2 + 3 + \dots + (n - 1) \text{ terms}}$$

$$= a^n r^{\frac{n(n-1)}{2}} \quad \left[\because \Sigma n = \frac{n(n + 1)}{2}\right]$$

$$\Rightarrow \quad P^2 = a^{2n} r^{n(n-1)} \quad \text{...(iii)}$$

Now, we have to prove $P^2 R^n = S^n$

or $P^2 = \frac{S^n}{R^n}$ or $P^2 = \left(\frac{S}{R}\right)^n$

$$\text{RHS} = \left(\frac{S}{R}\right)^n = \left[\frac{a(r^n - 1)}{r - 1} \times \frac{ar^n(r - 1)}{(r^n - 1)r}\right]^n$$

[using Eqs. (i) and (ii)]

$$= [a^2 r^n r^{-1}]^n = (a^2 r^{n-1})^n$$

$$= [a^{2n} r^{n(n-1)}] = P^2 = \text{LHS} \quad [\text{from Eq. (iii)}]$$

Hence proved.

Note Students should remember that, above proof can also be done by assuming $r < 1$.

Type II Problems Based on Finding the Value of Unknown/GP, When Sum of n Terms of a Finite GP is Given

In this type of problems, a GP and sum is given and we have to find r and n. Sometimes GP is not given and we have to find a, r and then GP with the help of given information related to sum.

Example 10. How many terms of GP $3, \frac{3}{2}, \frac{3}{4}, \dots$ are needed to give the sum $\frac{3069}{512}$?

Sol. Given GP is $3, \frac{3}{2}, \frac{3}{4}, \dots$

Here, $a = 3$, $r = \frac{3}{2} \div 3 = \frac{1}{2} < 1$

Let n be the number of terms needed.

Then, $S_n = \frac{3069}{512} \Rightarrow \frac{a(1-r^n)}{1-r} = \frac{3069}{512}$ $[\because r < 1]$

$$\Rightarrow \frac{3\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = \frac{3069}{512}$$

$$\Rightarrow 6\left(1-\frac{1}{2^n}\right) = \frac{3069}{512}$$

$$\Rightarrow 1-\frac{1}{2^n} = \frac{3069}{3072}$$

$$\Rightarrow \frac{1}{2^n} = 1-\frac{3069}{3072} = \frac{3072-3069}{3072}$$

$$\Rightarrow \frac{1}{2^n} = \frac{3}{3072} = \frac{1}{1024}$$

$$\Rightarrow 2^n = 1024$$

$$\Rightarrow 2^n = 2^{10}$$

On comparing the powers from both sides, we get

$$n = 10$$

Hence, 10 terms are needed to give the sum $\frac{3069}{512}$.

Example 11. Find the least value of n for which the sum $1 + 3 + 3^2 + \dots$ to n terms is greater than 7000.

Sol. We have, $S_n = 1 + 3 + 3^2 + \dots$ to n terms

Clearly, the ratio of any terms by its preceding term is same. So, the above series forms a GP, with first term, $a = 1$ and common ratio, $\frac{3}{1} = 3 > 1$.

$$\therefore \quad S_n = 1 \times \left(\frac{3^n-1}{3-1}\right) = \frac{3^n-1}{2} \quad \left[\because S_n = a\left(\frac{r^n-1}{r-1}\right) r > 1\right]$$

Now, $S_n > 7000$

$$\Rightarrow \frac{3^n-1}{2} > 7000$$

$$\Rightarrow 3^n - 1 > 14000$$

$$\Rightarrow 3^n > 14001$$

$$\Rightarrow n \log 3 > \log 14001$$

$$\Rightarrow n > \frac{\log 14001}{\log 3}$$

$$\Rightarrow n > \frac{4.1461}{0.4771} = 8.69$$

Hence, the least value of n is 9.

Applications of GP

Sometimes we can solve some real life problems by using the concept of GP. Some of them are given below

Example 12. At the end of each year, the value of a certain machine has depreciated by 20% of its value at the beginning of that year. If its initial value was ₹ 1250, then find the value at the end of 5 yr.

Sol. Given, every year depreciation in value of machine = 20%

After each year, the value of the machine is 80% (100 − 20)% of its value of the previous year.

So at the end of 1st year value of machine will be 1250 (0.8), at the end of 2nd year value of machine will be 1250 (0.8) (0.8) = 1250 $(0.8)^2$ and so on.

Thus, we have to find the 6th term of the GP whose first term a_1 is 1250 and common ratio r is 0.8.

Hence, value at the end of 5 yr

$$= a_6 = a_1 r^5 = 1250\,(0.8)^5 = 409.6$$

Note In depreciation, the cost value decrease every year.

Example 13. The lengths of three unequal edges of a rectangular solid block are in GP. If the volume of the block is 216 cm^3 and the total surface area is 252 cm^2, then find the length of its edges.

Sol. Let the length, breadth and height of rectangular solid block be $\frac{a}{r}$, a and ar, respectively.

$$\therefore \text{ Volume } = \frac{a}{r} \times a \times ar = 216\,\text{cm}^3$$

[∵ Volume of a cuboid = length × breadth × height]

$$\Rightarrow a^3 = 216 \Rightarrow a^3 = 6^3$$

$$\therefore a = 6\,\text{cm}$$

Now, surface area $= 2\left(\frac{a^2}{r} + a^2 r + a^2\right) = 252$

[∵ surface area of cuboid = 2 $(lb + bh + hl)$]

$$\Rightarrow 2a^2\left(\frac{1}{r} + r + 1\right) = 252$$

$$\Rightarrow 2 \times 36\left(\frac{1+r^2+r}{r}\right) = 252$$

$$\Rightarrow \frac{1+r^2+r}{r} = \frac{252}{2 \times 36} \Rightarrow 1 + r^2 + r = \frac{126}{36} r$$

$$\Rightarrow 1 + r^2 + r = \frac{21}{6} r \Rightarrow 6 + 6r^2 + 6r = 21r$$

$$\Rightarrow 6r^2 - 15r + 6 = 0 \Rightarrow 2r^2 - 5r + 2 = 0$$

$$\Rightarrow (2r-1)(r-2) = 0 \Rightarrow r = \frac{1}{2} \text{ or } 2$$

For $r = \frac{1}{2}$, length $= \frac{a}{r} = \frac{6 \times 2}{1} = 12$ cm breadth $= a = 6$ cm

and height $= ar = 6 \times \frac{1}{2} = 3$ cm

For $r = 2$, length $= \frac{a}{r} = \frac{6}{2} = 3$, breadth $= a = 6$ cm

and height $= ar = 6 \times 2 = 12$ cm

Topic Exercise 4

1. Find the sum of following geometric progressions.

(i) $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \dots n$ terms

(ii) $1, -a, a^2, -a^3, \dots n$ terms (if $a \neq -1$)

(iii) 2, 6, 18, … upto 7 terms

(iv) $1, \sqrt{3}, 3, 3\sqrt{3}, \dots$ upto 10 terms

(v) $0.15, 0.015, 0.0015, \dots$ 20 terms

2. The sum of some terms of GP is 315, whose first term and common ratio are 5 and 2, respectively. Find the last term and number of terms.

3. The first term of a GP is 27 and its 8th term is 1/81. Find the sum of its first 10 terms.

4. Find the sum of first 12 term of a GP whose 8th term is 192 and the common ratio is 2.

5. The 2nd and 5th terms of a GP are $\frac{-1}{2}$ and $\frac{1}{16}$, respectively. Find the sum of the GP upto 8 terms.

6. How many terms of the series $1 + 3 + 3^2 + 3^3 + \dots$ must be taken to make 3280?

7. How many terms of the sequence $\sqrt{3}, 3, 3\sqrt{3}, \dots$ must be taken to make the sum $39 + 13\sqrt{3}$?

8. Given, a GP with $a = 729$ and 7th term is 64, determine S_7.

9. In a GP, if $T_1 = 3$, $T_n = 96$ and $S_n = 189$, then find n.

10. The ratio of the sum of first three terms to that of first 6 terms of a GP is 125 : 152. Find the common ratio.

11. Show that the ratio of the sum of the first n terms of a GP to the sum of terms from $(n+1)$th to $(2n)$th term is $1/r^n$.

12. A person has 2 parents, 4 grandparents and so on. Find the numbers of his ancestors during the ten generations preceding his own.

13. The inventor of the chessboard suggested a reward of one grain of wheat for the first square; 2 grains for the second; 4 grains for the third and so on, doubling the number of grains for subsequent squares. How many grains would have to be given to the inventor? (Note that there are 64 squares in the chessboard.)

14. A man writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with the instruction that they move the chain similarly. Assuming that the chain is not broken and it costs ₹ 2 to mail one letter, find the amount spent on postage when 8th set of letters is mailed.

15. Let a_n be the nth term of the GP of positive *numbers. Let* $\sum_{n=1}^{100} a_{2n} = \alpha$ *and* $\sum_{n=1}^{100} a_{2n-1} = \beta$, *such that* $\alpha \neq \beta$. *Prove that the common ratio of the GP is* $\frac{\alpha}{\beta}$.

Hints and Answers

1. Hint Use the formula, $S_n = \frac{a(r^n - 1)}{r - 1}, r > 1$

or $\frac{a(1 - r^n)}{1 - r}, \ r < 1$

(i) $\frac{\sqrt{7}}{2}(\sqrt{3} + 1)(3^{n/2} - 1)$ (ii) $\frac{1 - (-a)^n}{1 + a}$

(iii) 2186 (iv) $121(\sqrt{3} + 1)$ (v) $\frac{1}{6}[1 - (0.1)^{20}]$

2. Hint Here, $a = 5$ and $r = 2$

$$\because \quad \frac{a(r^n - 1)}{r - 1} = 315 \Rightarrow \frac{5(2^n - 1)}{2 - 1} = 315$$

Ans. Number of terms $= 6$ and last term $= 160$.

3. Hint $\frac{1}{81} = 27(r)^{8-1} \Rightarrow r = \frac{1}{3}$

Ans. $\frac{81}{2}\left(1 - \frac{1}{3^{10}}\right)$

4. Hint $ar^{8-1} = 192 \Rightarrow a \times (2)^7 = 192 \Rightarrow a = \frac{3}{2}$

Ans. $S_{12} = 6142.5$

5. Hint Here, $ar = -\frac{1}{2}$ ...(i)

and $ar^4 = \frac{1}{16}$ ...(ii)

On dividing Eq. (ii) by Eq. (i), we get r

Then, find a using Eq. (i).

Ans. $\frac{85}{128}$

6. Hint Here, $a = 1$, $r = \frac{3}{1} = 3$

$\therefore \quad \frac{1 \cdot (3^n - 1)}{3 - 1} = 3280$ **Ans.** $n = 8$

7. Hint Let the required number of terms be n.

Then, we have

$$\sqrt{3}\frac{[(\sqrt{3})^n - 1]}{[\sqrt{3} - 1]} = 39 + 13\sqrt{3} \quad \left[\because S_n = \frac{a(r^n - 1)}{(r - 1)}, r > 1\right]$$

Ans. $n = 6$

8. Hint $ar^{7-1}=64 \Rightarrow r^6=\left(\frac{2}{3}\right)^6 \Rightarrow r=\frac{2}{3}$

$$\therefore \quad S_7=\frac{729\left[1-\left(\frac{2}{3}\right)^7\right]}{1-2/3}$$

Ans. 2059

9. Hint Here, $T_1=a=3, T_n=96$ and $S_n=189$

Now, find r, using

$$S_n=\frac{a(r^n-1)}{r-1}=\frac{ar^n-a}{r-1}=\frac{r\cdot T_n-a}{r-1}$$

After finding r, use $ar^{n-1}=96$ to find n.

Ans. $n=6$

10. Hint $\frac{a+ar+ar^2}{a+ar+ar^2+ar^3+ar^4+ar^5}=\frac{125}{152}$

$$\Rightarrow \frac{1+r+r^2}{(1+r+r^2)(1+r^3)}=\frac{125}{152}$$

$$\Rightarrow \qquad r^3=\frac{27}{125}$$

Ans. $r=\frac{3}{5}$

11. Hint Let the GP be

$\underbrace{a, ar, ar^2, ar^3, ar^4, ar^5, \ldots, ar^{n-1}}_{n \text{ terms}}, \underbrace{ar^n, ar^{n+1}, \ldots, ar^{2n-1}}_{n \text{ terms}}$

$$\text{Now, required ratio}=\frac{\frac{a(r^n-1)}{r-1}}{\frac{ar^n(r^n-1)}{r-1}}$$

12. Hint Here, $a=2, r=\frac{4}{2}=2$ and $n=10$

$$\because \quad S_{10}=\frac{2(2^{10}-1)}{2-1}=2(2^{10}-1)=2(1024-1)$$

Ans. 2046

13. Hint Required number of grains

$=1+2+2^2+\ldots$ to 64 terms

$=1+(2+2^2+2^3+\ldots+2^{63})=\left\{1+\frac{2(2^{63}-1)}{(2-1)}\right\}$

Ans. $(2^{64}-1)$

14. Hint Successive number of letters are 4, 16, 64, … .

This is a GP with $a=4$ and $r=4$.

Now, the total number of letters, when 8th set of letters is mailed

$$=S_8=\frac{4(4^8-1)}{4-1}=87380$$

Ans. Cost of postage = ₹ (87380×2) = ₹ 174760

15. Hint According to the question,

$$a_2+a_4+\ldots+a_{200}=\alpha$$

and $\quad a_1+a_3+\ldots+a_{199}=\beta$

$\Rightarrow \quad a_r+ar^3+\ldots+ar^{199}=\alpha$

and $\quad a+ar^2+\ldots+ar^{198}=\beta$

$$\Rightarrow \quad ar\left[\frac{1-(r^2)^{100}}{1-r^2}\right]=\alpha \text{ and } a\left\{\frac{1-(r^2)^{100}}{1-r^2}\right\}=\beta$$

Now, dividing the above equations, we get the result.

CHAPTER EXERCISE

a 3 Marks Questions

1. If the numbers $2n-1$, $3n+2$ and $6n-1$ are in AP, then find n and hence find the numbers.
2. If 21, a, b and -3 are in AP, then find the value of $(a+b)$.
3. Determine k, so that k^2+4k+8, $2k^2+3k+6$ and $3k^2+4k+4$ are three consecutive terms of an AP.
4. Which term of the AP 120, 116, 112, ... is first negative term?
5. If the nth term of the two AP 9, 7, 5, ... and 24, 21, 18,... are the same, then find the value of n. Also, find that term.
6. The 4th term of an AP is zero. Prove that the 25th term of the AP is three times its 11th term.
7. If the 2nd term of an AP is 13 and 5th term is 25, then what is its 7th term?
8. The fourth term of an AP is 11. The sum of the fifth and seventh terms of the AP is 24. Find its common difference.
9. The 16th term of an AP is 1 more than twice its 8th term. If the 12th term of an AP is 47, then find its nth term.
10. The eighth term of an AP is half of its second term and the eleventh term exceeds one-third of its fourth term by 1. Find the 15th term.
11. The sum of 5th and 9th terms of an AP is 72 and sum of 7th and 12th terms is 97. Find the AP.
12. How many two-digit numbers are divisible by 6?
13. How many multiples of 9 lie between 10 and 300?
14. Find the number of natural numbers between 101 and 999 which are divisible by both 2 and 5.
15. Split 210 into three parts such that these are in AP and the product of the two greatest parts is 5110.
16. The sum of the first three terms of an AP is 33. If the product of the first and the third terms exceeds the second term by 29, then find the AP.
17. The angles of a triangle are in AP. The greatest angle is thrice the least. Find all the angles of the triangle.
18. If four numbers are in AP such that their sum is 50 and the greatest number is 4 times the least, then find the numbers.
19. Find the sum of the series
$$\frac{3}{\sqrt{5}}+\frac{4}{\sqrt{5}}+\sqrt{5}+\dots \text{ to 25 terms.}$$
20. Find the sum of the first 25 terms of an AP, whose nth term is given by $a_n=7-3n$.
21. If the nth term of an AP is $a_n=2n+1$, find its sum.
22. If an AP has $a=1$, $a_n=20$ and $S_n=399$, then find the value of n.
23. The sum of first, third and seventeenth terms of an AP is 216. Find the sum of the first 13 terms of the AP.
24. Find the sum of first 24 terms of the AP $a_1, a_2, a_3 \dots$, if it is known that $a_1+a_5+a_{10}+a_{15}+a_{20}+a_{24}=225$.
25. If a, b and c are the 1st, 3rd and nth terms respectively of an AP, then prove that the sum to n terms is $\frac{c+a}{2}+\frac{c^2-a^2}{b-a}$.
26. If the number of terms of an AP is $2n+1$, then find the ratio of sum of the odd terms of sum of even terms.
27. Find the sum of all multiple of 7 lying between 500 and 900.
28. Find the sum of all three-digit natural numbers, which are multiples of 11.
29. Find the sum of all two-digit numbers greater than 50 which when divided by 7 leaves remainder 4.
30. The fourth term of a GP is the square of its second term and the first term is -3. Determine its 7th term.
31. If 5th and 8th terms of a GP are 48 and 384, respectively. Find the GP, if terms of GP are real numbers.
32. If $a^x=b^y=c^z$ and x, y, z are in GP, then show that $\log_b a=\log_c b$.
33. Find the sum of n terms of the series
$$(a+b)+(a^2+2b)+(a^3+3b)+\dots.$$

b 4 Marks Questions

34. If 12th term of an AP is 213 and the sum of its four terms is 24, then what is the sum of its first 10 terms?

35. Find the common difference of an AP whose first term is 5 and the sum of its first 4 terms is half the sum of the next 4 terms.

36. The sum of the first term and the fifth term of an ascending AP is 26 and the product of the second term by the fourth term is 160. Find the sum of the first seven terms of this AP.

37. Sum of the first n terms of an AP is $5n^2 - 3n$. Find the AP and also find its 16th term.

38. Solve the equation $-4 + (-1) + 2 + \ldots + x = 437$.

39. Find the sum of all two-digit odd positive numbers.

40. Jaspal Singh repays his total loan of ₹ 118000 by paying every month starting with the first instalment of ₹ 1000. If he increases the instalment by ₹ 100 every month, then what amount will he paid by him in the 30th instalment? What amount of loan does he still have to pay after 30th instalment?

41. Each year, a tree grows 5 cm less than it did the preceding year. If it grew by 1m in the first year, then in how many years will it have ceased growing?

42. Komal decides to solve two questions of Mathematics first day and doubles the number everyday. In how many days will she able to solve the total questions 2046 in the book?

43. Rohit borrows ₹ 10000 from his friend, Mohit. Mohit asks him to return the money just by repaying ₹ 10 at first day and double the amount in the subsequent days than the each previous day continuously for only ten days. Rohit thanks Mohit for his help. Now, determine whether Mohit was in loss or profit.

44. 150 workers were engaged to finish a job in a certain number of days, 4 workers dropped out on second day, 4 more workers dropped out an third day and so on. It took 8 more days of finish the work. Find the number of days in which the work was completed.

45. A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with the instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 25 paise to mail one letter, determine the amount spent on postage when the 8th set of letters is mailed.

Hints and Answers

1. Hint Here, $a_2 - a_1 = a_3 - a_2$

$\Rightarrow \quad 3n + 2 - 2n + 1 = 6n - 1 - 3n - 2$

Ans. $n = 3$ and the numbers are 5, 11, 17.

2. Hint Here, $a - 21 = b - a$

$\Rightarrow \quad 2a - b = 21 \quad \ldots(i)$

and $\quad -3 - b = b - a \Rightarrow a - 2b = 3 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get $a = 13$ and $b = 5$

Ans. 18

3. Hint Let $k^2 + 4k + 8$, $2k^2 + 3k + 6$ and $3k^2 + 4k + 4$ are in AP. Then, we have $(2k^2 + 3k + 6)$ $- (k^2 + 4k + 8) = (3k^2 + 4k + 4) - (2k^2 + 3k + 6)$

Now, simply to find the required value of k.

Ans. $k = 0$

4. Hint Find n for which $a_n < 0$. **Ans.** 32nd term

5. Hint $9 + (n - 1) \times (-2) = 24 + (n - 1) \times (-3)$

Ans. $n = 16$ and 16th term is -21.

6. Hint $a + 3d = 0$

$\Rightarrow \quad a = -3d \quad \ldots(i)$

To prove $a_{25} = 3a_{11}$

Proof Clearly, $a_{25} = a + 24d$ [from Eq. (i)]

Also, $\quad a_{11} = a + 10d$

$\Rightarrow \quad a_{11} = 7d$ [from Eq. (i)]

7. Hint Here, $a_2 = 13$ and $a_5 = 25$

$a + d = 13$ and $a + 4d = 25$

On solving the above equations, we get $a = 9$ and $d = 4$.

Ans. 33

8. Hint Given, $a_4 = 11 \Rightarrow a + 3d = 11 \quad \ldots(i)$

and $\quad a_5 + a_7 = 24 \Rightarrow a + 5d = 12 \quad \ldots(ii)$

Now, solving Eqs. (i) and (ii) and then get the value of d.

Ans. $d = \frac{1}{2}$

9. Hint Here, $a_{16} = 1 + 2a_8$

$\Rightarrow \quad a - d = -1 \quad \ldots(i)$

and $\quad a + 11d = 47 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$d = 4$ and $a = 3$

Ans. $a_n = 4n - 1$

10. Hint We have, $a_8 = \frac{1}{2} a_2$ and $a_{11} = \frac{1}{3} a_4 + 1$

$\Rightarrow \quad a + 13d = 0$ and $2a + 27d = 3$

Now, solving above equations, we get

$$a = -39 \text{ and } d = 3$$

Ans. 3

11. Hint $a_5 + a_9 = 72$ and $a_7 + a_{12} = 97$

Ans. 6, 11, 16, 21, 26, ...

12. Hint 12, 18, 24, ..., 96

Now, proceed same as Example 10 of Topic 1.

Ans. 15

13. Hint 18, 27, 36, …, 297

Now, proceed same as Example 10 of Topic 1. **Ans.** 32

14. Hint Natural numbers between 101 and 999 divisible by both 2 and 5, i.e. divisible by 10, are 110, 120, 130, …, 990.

Here, common difference,

$$d = 120 - 110 = 130 - 120 = \ldots = 10.$$

Let $\quad l = a_n = a + (n-1)d$

Then, $\quad 990 = 110 + (n-1) \times 10$

Ans. $n = 89$

15. Hint Let the three parts be $a - d$, a, $a + d$, where $d > 0$.

$\therefore \quad a - d + a + a + d = 210 \Rightarrow a = 70$

and product of two greatest parts $= 5110$

$\Rightarrow \quad a(a + d) = 5110$

$\Rightarrow \quad 70(70 + d) = 5110 \qquad [\because a = 70]$

$\Rightarrow \quad d = 3$

Ans. 67, 70, 73

16. Hint Let the three terms in AP be $a-d$, a, $a+d$.

Then, $(a-d)+a+(a+d)=33 \Rightarrow a=11$

By the second condition,

$$(a-d)(a+d) = a+29 \Rightarrow d = \pm 9$$

Ans. 2, 11, 20, … and 20, 11, 2, …

17. Hint Let the angles be $(a-d)°, a°, (a+d)°$.

Then, $(a - d) + a + (a + d) = 180$

and $a + d = 3(a - d) \Rightarrow a = 2d$

Ans. $30°, 60°, 90°$

18. Hint Let four numbers be $a-3d$, $a-d$, $a+d$, $a+3d$ which are in AP.

Then, $a - 3d + a - d + a + d + a + 3d = 50$

$\Rightarrow \quad a = \dfrac{25}{2}$ and $a + 3d = 4(a - 3d)$

$\Rightarrow \quad d = \dfrac{a}{5} = \dfrac{25}{2 \times 5} = \dfrac{5}{2}$ **Ans.** 5, 10, 15 and 20

19. Hint Here, $a = \dfrac{3}{\sqrt{5}}$

and common difference, $d = \dfrac{4}{\sqrt{5}} - \dfrac{3}{\sqrt{5}} = \dfrac{1}{\sqrt{5}}$.

$\therefore \quad S_{25} = \dfrac{25}{2}\left[2 \times \dfrac{3}{\sqrt{5}} + (25-1)\dfrac{1}{\sqrt{5}}\right]$ **Ans.** $75\sqrt{5}$

20. Do same as Example 3 of Topic 2. **Ans.** –800

21. Hint We have, $a_n = 2n + 1$

Clearly, $S_n = \dfrac{n}{2}(a_1 + a_n) = \dfrac{n}{2}[2(1) + 1 + (2n+1)]$

Ans. $n(n+2)$

22. Hint $1 + (n-1)d = 20 \qquad \ldots(i)$

and $\dfrac{n}{2}[2 + (n-1)d] = 399 \Rightarrow \dfrac{n}{2}(1+20) = 399$ [from Eq. (i)]

Ans. 38

23. Hint Here, $a_1 + a_3 + a_{17} = 216$

$\Rightarrow \quad a + 6d = 72 \qquad \ldots(i)$

Now, $S_{13} = \dfrac{13}{2}[2a + (13-1)d] = \dfrac{13}{2}[2(a+6d)]$

Ans. 936

24. Hint $a + (a+4d) + (a+9d) + (a+14d) + (a+19d) + (a+23d) = 225$

$\Rightarrow \quad 2a + 23d = 75 \qquad \ldots(i)$

Now, $\quad S_{24} = \dfrac{24}{2}[2a + (24-1)d] = 12(2a + 23d)$

Ans. 900

25. Hint Let A be the first term and D be the common difference of the given AP.

According to the question,

$$a_1 = A = a \qquad \ldots(i)$$

$$a_3 = A + 2D = b \qquad \ldots(ii)$$

and $\quad a_n = A + (n-1)D = c \qquad \ldots(iii)$

On solving Eqs. (i) and (ii), we get

$$D = \frac{b-a}{2}$$

On putting $A = a$ and $D = \left(\dfrac{b-a}{2}\right)$ in Eq. (iii), we get

$$a + (n-1)\left(\frac{b-a}{2}\right) = c$$

$\Rightarrow \; (n-1) = \dfrac{2(c-a)}{b-a} \Rightarrow n = \dfrac{2c - 3a + b}{b-a}$

Now, $S_n = \dfrac{n}{2}(A + a_n) = \dfrac{(2c - 3a + b)}{2(b-a)}(a+c)$

$$= \frac{1}{2(b-a)}[2ac - 3a^2 + ab + 2c^2 - 3ac + bc]$$

$$= \frac{1}{2(b-a)}[(b-a)(c+a) + 2(c^2 - a^2)]$$

26. Hint Sum of odd terms

$$= \frac{(n+1)}{2}[2a + (n+1-1)(2d)] = (n+1)(a + nd)$$

Sum of even terms

$$= \frac{n}{2}[2(a+d)+(n-1)(2d)] = n(a+nd)$$

$$\therefore \frac{S_{odd}}{S_{even}} = \frac{(n+1)(a+nd)}{n(a+nd)}$$

Ans. $(n+1):n$

27. Hint The multiples of 7 lying between 500 and 900 are 504, 511, 518, ..., 896.

Clearly, it forms an AP.

Here, $a = 504$ and $d = 511 - 504 = 7$

Let there are n terms, i.e. $a_n = 896$

$$\Rightarrow \quad a+(n-1)d = 896 \Rightarrow n = 57$$

Now, $S_{57} = \frac{n}{2}(a + a_n) = \frac{57}{2}(504+896)$

Ans. 39900

28. Hint All three-digit natural numbers, multiple of 11 are 110, 121, 132, …, 990.

Here, common difference,

$$d = 121 - 110 = 132 - 121 = \ldots = 11.$$

Let $l = a_n = a + (n-1)d$

Then, $990 = 110 + (n-1) \times 11 \Rightarrow n = 81$

$\because \quad S_n = \frac{n}{2}(a+l)$

$\therefore \quad S_{81} = \frac{81}{2}(110+990)$ **Ans.** $S_{81} = 44550$

29. Hint All two-digit numbers greater than 50 which when divided by 7 leaves remainder 4, are 53, 60, 67, …, 95.

Now, proceed same as Q. 28.

Ans. 518

30. Hint $a_4 = (a_2)^2$ and $a = -3$

$\because \quad a_4 = (a_2)^2 \Rightarrow ar^3 = (ar)^2$

$\Rightarrow -3r^3 = (-3)^2 r^2 \qquad [\because a = -3]$

$\Rightarrow \quad r = -3$

Now, $a_7 = ar^6 = -3(-3)^6 = -3 \times 729$

Ans. -2187

31. Hint Here, $a_5 = 48 \Rightarrow ar^4 = 48 \qquad \ldots$(i)

and $\quad a_8 = 384 \Rightarrow ar^7 = 384 \qquad \ldots$(ii)

On dividing Eq. (ii) by Eq. (i), we get

$$\frac{ar^7}{ar^4} = \frac{384}{48} \Rightarrow r^3 = 8 \Rightarrow r = 2$$

On putting $r = 2$ in Eq. (i), we get $a = 3$

Ans. 3, 6, 12, … are in GP.

32. Hint Let $a^x = b^y = c^z = k$

$\therefore x = \log_a k,\ y = \log_b k,\ z = \log_c k$

Since, x, y and z are in GP.

$$\therefore \frac{\log_b k}{\log_a k} = \frac{\log_c k}{\log_b k} \Rightarrow \frac{\log_k a}{\log_k b} = \frac{\log_k b}{\log_k c}$$

$\Rightarrow \quad \log_b a = \log_c b$

33. Hint Let $S_n = (a+b)+(a^2+2b)+(a^3+3b) + \ldots$ upto n terms

$= (a + a^2 + a^3 + \ldots$ upto n terms$) + (b + 2b + 3b + \ldots$ upto n terms$)$

$$= a\left(\frac{a^n - 1}{a-1}\right) + \frac{n}{2}[2b + (n-1)b]$$

$$\left[\because S_{n(GP)} = \frac{A(R^n - 1)}{R-1},\ R > 1 \text{ and } S_{n(AP)} = \frac{n}{2}\{2A + (n-1)D\}\right]$$

Ans. $\frac{a(1-a^n)}{1-a} + \frac{bn(n+1)}{2}$

34. Hint $a_{12} = 213 \Rightarrow a + 11d = 213 \qquad \ldots$(i)

and $\quad S_4 = 24 \Rightarrow \frac{4}{2}[2a + (4-1)d] = 24$

$\Rightarrow \quad 2a + 3d = 12 \qquad \ldots$(ii)

On solving Eqs. (i) and (ii), we get

$$a = \frac{-507}{19} \text{ and } d = \frac{414}{19}$$

Now, $S_{10} = \frac{10}{2}\left[2 \times \left(\frac{-507}{19}\right) + (10-1) \times \frac{414}{19}\right]$

Ans. 713.68

35. Hint $S_4 = \frac{1}{2}$ (Sum of a_5, a_6, a_7, a_8) $= \frac{1}{2}(S_8 - S_4)$

$\Rightarrow \quad 2S_4 = S_8 - S_4 \Rightarrow 3S_4 = S_8$

$$\Rightarrow 3\left\{\frac{4}{2}[2a + (4-1)d]\right\} = \frac{8}{2}[2a + (8-1)d]$$

$$\left[\because S_n = \frac{n}{2}[2a + (n-1)d]\right]$$

$\Rightarrow \quad 6[2(5) + 3d] = 4[2(5) + 7d]$

$\Rightarrow \quad 60 + 18d = 40 + 28d$

Ans. $d = 2$

36. Hint Here, $\quad a_1 + a_5 = 26$

$\Rightarrow \quad a + 2d = 13 \qquad \ldots$(i)

Also, we have $a_2 \times a_4 = 160$

$\Rightarrow \quad (a+d) \times (a+3d) = 160$

$\Rightarrow (13 - 2d + d)(13 - 2d + 3d) = 160 \qquad$ [from Eq. (i)]

$\Rightarrow \quad (13)^2 - (d)^2 = 160$

$\therefore \quad d = \pm 3$

When $d = 3$, then from Eq. (i) $a = 7$

and when $d = -3$, then from Eq. (i), $a = 19$

Now, the sum of first seven terms,
when $a = 7$ and $d = 3$

$$S_7 = \frac{7}{2}[2 \times 7 + (7-1)3]$$

Again, sum of first seven terms,
when $a = 19$ and $d = -3$

$$S_7 = \frac{7}{2}[2 \times 19 + (7-1) \times -3]$$

Ans. 112, 70

37. Do same as Example 6 of Topic 2.
Ans. 2, 12, 22, ... and 152

38. **Hint** $-4, -1, 2, \dots, x$ forms an AP with first term, $a = -4$ and common difference

$d = -1 - (-4) = 3$

Also, last term, say $a_n = x$

$\because$ nth term of an AP, $a_n = a + (n-1)d$

$\Rightarrow \quad x = -4 + (n-1)3$

$\Rightarrow \quad n = \dfrac{x+7}{3} \qquad \dots(\text{ii})$

$\because$ Sum of first n terms, $S_n = \dfrac{n}{2}[2a + (n-1)d]$

$\therefore \quad S_{\left(\frac{x+7}{3}\right)} = \dfrac{x+7}{2 \times 3}\left[2(-4) + \left(\dfrac{x+4}{3}\right) \cdot 3\right]$

$\Rightarrow \quad x^2 + 3x - 2650 = 0$

Ans. 50, -53

39. **Hint** 11, 13, 15, ..., 99

$\therefore \quad 99 = 11 + (n-1) \times 2$

$\Rightarrow \quad n = 45$

Now, $S_{45} = \dfrac{45}{2}[2 \times 11 + (45-1) \times 2]$

Ans. 2475

40. **Hint** Here, first term, $a = 1000$
and common difference, $d = 100$.
Now, amount paid in 30th instalment,

$$a_{30} = 1000 + (30-1)100 = ₹\ 3900$$

Amount paid in 30 instalments,

$$S_{30} = \frac{30}{2}[2 \times 1000 + (30-1)100]$$
$$= ₹\ 73500$$

Hence, remaining amount of loan that he has to pay
$= ₹\ 118000 - ₹\ 73500$ **Ans.** ₹ 44500

41. **Hint** Given that, tree grow 5 cm or 0.05 m less than preceding year.
The following sequence can be formed.

$$1, (1-0.05), (1-2 \times 0.05), \dots, 0$$

i.e. $\quad 1, 0.95, 0.90, \dots, 0$ which is an AP.
Here, $\quad a = 1, d = 0.95 - 1 = -0.05$ and $l = 0$
Let $l = a_n = a + (n-1)d$.
Then, $0 = 1 + (n-1)(-0.05)$
Ans. 21 yr

42. **Hint** The sequence of solved questions is given as

$$2 + 4 + 8 + 16 + \dots n \text{ days} = 2046$$

By using, $S_n = \dfrac{a(r^n - 1)}{r-1}$, we have

$$2046 = \frac{2(2^n - 1)}{2-1}$$

Ans. 10 days

43. **Hint** Total amount repaid in ten days

$= (10 + 20 + 40 + 80 + \dots \text{upto to 10 days})$

$= \dfrac{10\{2^{10} - 1\}}{2-1}, r > 1$

$= 10(2^{10} - 1) = 10(1024 - 1) = ₹\ 10230$

Ans. Hence, there is a profit of 230.

44. **Hint** Let the number of days in which the work is completed, be n.
Now, according to the question, 4 workers dropped on everyday. So, number of workers present on successive days are 150, 146, 142, 138, ...
If the workers not dropped, then the work would have finished in $(n-8)$ days with 150 workers working on each day.
Hence, the total number of workers who would have worked for all the n days, is $150(n-8)$.
Clearly, the work done in both conditions is same.

i.e. $\quad 150(n-8) = \dfrac{n}{2}[2 \times 150 + (n-1)(-4)]$

Ans. The work will be completed in 25 days.

45. **Hint** Amout spent on first set = ₹ 1 (4×25 paise)
Amount spent on second set = ₹ 4×1 = ₹ 4
[$\because$ each time the number of letters is multiplied by 4]
$\therefore$ Amount spent on all the eight sets

$= \{1 + 4 \times 1 + 4 \times 4 \times 1 + 4 \times 4 \times 4 \times 1 + \dots \text{ upto 8 terms}\}$

$= \dfrac{1(4^8 - 1)}{4-1} = \left(\dfrac{65536 - 1}{3}\right) = \dfrac{65535}{3}$

Ans. ₹ 21845

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. Five distinct positive integers are in arithmetic progression with a positive common difference. If their sum is 10020, then the smallest possible value of the last term is

(a) 2002 (b) 2004
(c) 2006 (d) 2007

2. The sum of series $1+(1+2)+(1+2+3)+(1+2+3+4)+\ldots+(1+2+3+\ldots+20)$ is

(a) 1470 (b) 1540
(c) 1610 (d) 1370

3. An AP starts with a positive fraction and every alternate term is an integer. If the sum of the first 11 terms is 33, then the fourth term is

(a) 2 (b) 3
(c) 5 (d) 6

4. The sum of the series
$45^2-43^2+44^2-42^2+43^2-41^2+42^2-40^2+\ldots$ upto 30 terms is

(a) 1110 (b) 2220
(c) 3330 (d) 4440

5. Take a point $A(3, 4)$ on the graph and draw two lines from it, one is parallel to X-axis and another parallel to Y-axis. Again, take four points on both lines on both sides of A, such that their x-coordinates and y-coordinates form an AP with common difference 2. Then, the area of circle, passing through these four points, is

(a) 12 sq units
(b) 13 sq units
(c) 12.56 sq units
(d) 13.56 sq units

6. If a, b, c and d are in GP, then $(a^2+b^2+c^2)(b^2+c^2+d^2)$ is equal to

(a) $(ab+bc+cd)$ (b) $(ab+bc+cd)^2$
(c) $a^2+b^2+c^2$ (d) $b^2+c^2+d^2$

7. If a, b and c are respectively the pth, qth and rth terms of a GP, then the value of $(q-r)\log a+(r-p)\log b+(p-q)\log c$ is

(a) $\log abc$ (b) $\log ab$
(c) $\log bc$ (d) 0

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 400.

CHAPTER

10

Reflection

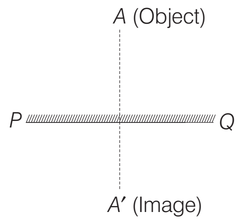
In this chapter, we will study about an important concept of coordinate geometry, i.e. 'reflection'. **Reflection** is an image or duplication of an object that appears identical but in opposite direction. Here, we will find the reflection of a point about the lines and axes and invariant points.

Chapter Objectives

- Reflection of a Point in a Line
- Reflection of a Point in the Origin
- Invariant Point

Reflection of a Point in a Line

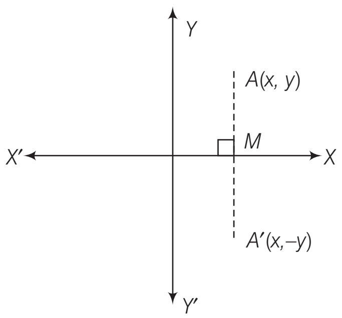
When an object is placed in front of a plane mirror, then the image is formed at the same distance behind the mirror as the object is in front of it and the mirror is perpendicular to the line joining the object and its image.



Definition The reflection (or image) of a point A in a line PQ is a point A' such that PQ is perpendicular bisector of the line segment AA'. Here, the line PQ is called the **axis of reflection** (or mediator).

Reflection of a Point in the X-axis $(y = 0)$

Let $A(x, y)$ be any point in the coordinate plane, then $A'(x, -y)$ is the reflection of $A(x, y)$ in the X-axis, if and only if the perpendicular distance from A to X-axis is equal to the perpendicular distance from A' to X-axis, i.e. $MA = MA'$.



Rule to find the reflection of a point (x, y) in the X-axis $(y = 0)$

(i) Keep the sign of abscissa or x-coordinate unchanged.

(ii) Change the sign of ordinate or y-coordinate.

Example 1. Find the reflection of the following points in the X-axis.

(i) (5, 3) (ii) (– 1, 2)
(iii) (3, 0) (iv) (– 2, – 5)

Sol.

(i) The reflection of the point (5, 3) in the X-axis, is the point (5, –3).

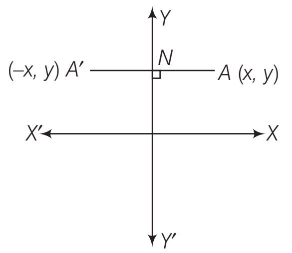
(ii) The reflection of the point (– 1, 2) in the X-axis, is the point (– 1, – 2).

(iii) The reflection of the point (3, 0) in the X-axis, is the point (3, 0).

(iv) The reflection of the point (– 2, – 5) in the X-axis, is the point (– 2, 5).

Reflection of a Point in the Y-axis $(x = 0)$

Let $A(x, y)$ be any point in the coordinate plane, then $A'(-x, y)$ is the reflection of $A(x, y)$ in the Y-axis, if and only if the perpendicular distance from A to Y-axis is equal to the perpendicular distance from A' to Y-axis, i.e. $NA = NA'$.



Rule to find the reflection of a point (x, y) in the Y-axis $(x = 0)$

(i) Keep the sign of ordinate or y-coordinate unchanged.

(ii) Change the sign of abscissa or x-coordinate.

Example 2. Find the reflection of the following points in the Y-axis.

(i) (1, 0) (ii) (– 3, 2.5) (iii) (4.5, – 7)
(iv) (0, – 1.5)

Sol.

(i) The reflection of the point (1, 0) in the Y-axis is (– 1, 0).

(ii) The reflection of the point (– 3, 2.5) in the Y-axis is (3, 2.5).

(iii) The reflection of the point (4.5, – 7) in the Y-axis is (– 4.5, – 7).

(iv) The reflection of the point (0, – 1.5) in the Y-axis is (0, – 1.5).

Example 3. Find the reflection of the following points in the line $x = 0$.

(i) (– 2.5, 3.5) (ii) (0, 7)
(iii) (3, – 1.5) (iv) (– 1, 0)

Sol. The line $x = 0$ means Y-axis, i.e. reflection of points is to found in Y-axis.

(i) The reflection of the point (– 2.5, 3.5) in the line $x = 0$ is (2.5, 3.5).

(ii) The reflection of the point (0, 7) in the line $x = 0$ is (0, 7).

(iii) The reflection of the point (3, – 1.5) in the line $x = 0$ is (–3, –1.5).

(iv) The reflection of the point (– 1, 0) in the line $x = 0$ is (1, 0).

Example 4. Use a graph paper for this question.
(Take 2 cm = 1 unit on both X and Y-axes)

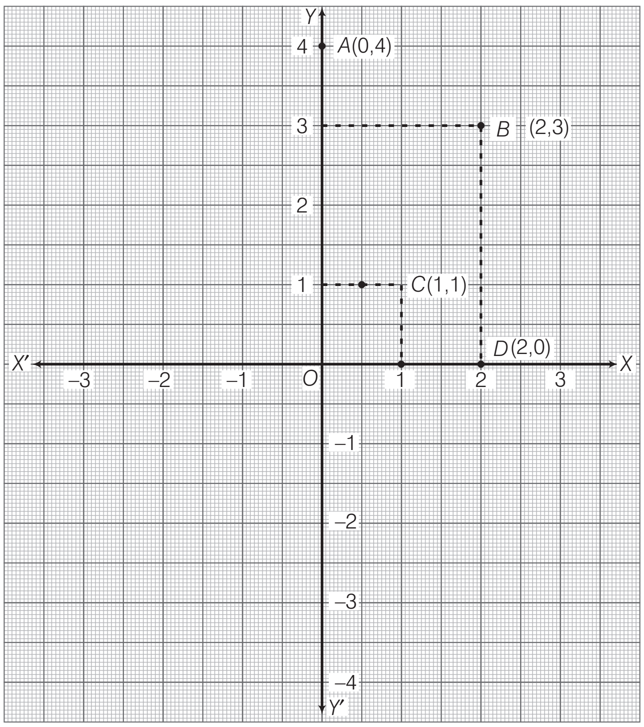
(i) Plot the following points.

$A(0, 4)$, $B(2, 3)$, $C(1, 1)$ and $D(2, 0)$

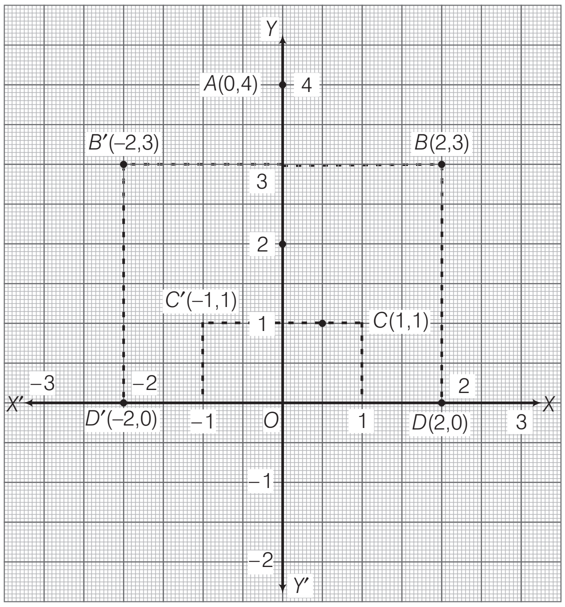
(ii) Reflect points B, C, D in the Y-axis and write down their coordinates.
Name the images as B', C' and D', respectively. ***[2017]***

Sol.

(i) We can plot the points $A(0, 4)$, $B(2, 3)$, $C(1, 1)$ and D (2, 0) on graph paper as follows



(ii) We know that, the image of point (x, y) on Y-axis has the coordinates $(-x, y)$.
Image of $B(2, 3)$ on Y-axis has the coordinates B' (– 2, 3).
Image of C (1, 1) on Y-axis has the coordinates C'(– 1, 1).
Image of $D(2, 0)$ on Y-axis has the coordinates D' (–2, 0).

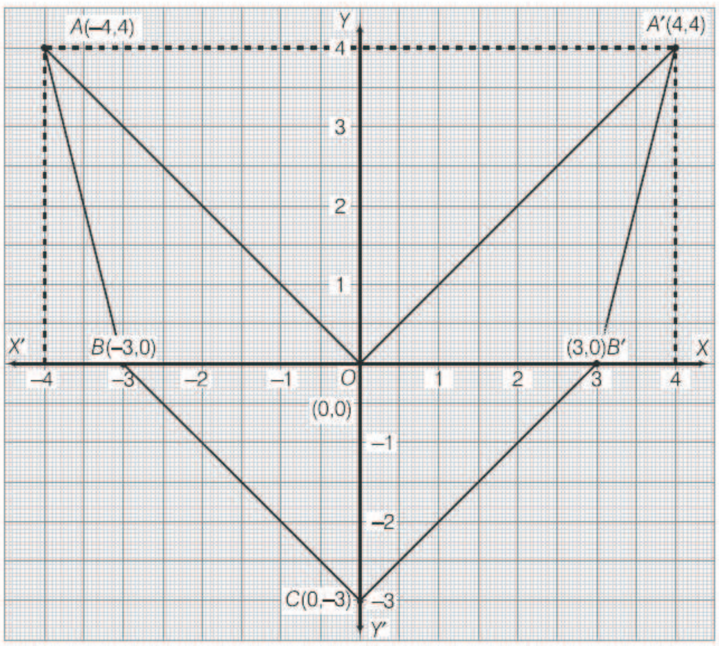


Example 5. Use graph paper for this question. (Take 2 cm = 1 unit along both X and Y-axes).
Plot the points $O(0, 0)$, $A(-4, 4)$, $B(-3, 0)$ and $C(0, -3)$

(i) Reflect points A and B on the Y-axis and name them A' and B', respectively. Write down their coordinates.

(ii) Name the figure $OABCB'A'$. ***[2016]***

Sol. We can plot the given points on the graph paper as follows.

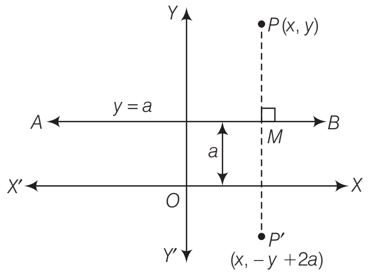


(i) When points $A(-4, 4)$ and $B(-3, 0)$ are reflected in Y-axis. Then, the image of points $A(-4, 4)$ and $B(-3, 0)$ are $A'(4, 4)$ and $B'(3, 0)$, respectively.

(ii) $OABCB'A'$ is a hexagon.

Reflection of a Point in a Line Parallel to X-axis

Let $P(x, y)$ be any point in the coordinate plane and $y = a$ be a line parallel to X-axis at a distance 'a' from it as shown in the following figure.



Let $P'(x_1, y_1)$ be the reflection of point P in the line $y = a$. Then, mid-point M of PP' lies on the line $y = a$ and PP' is perpendicular to $y = a$.

$\therefore$ Coordinates of M $= \left(\dfrac{x + x_1}{2}, \dfrac{y + y_1}{2}\right)$

Since, M lies on the line $y = a$.

$\therefore \quad \dfrac{y + y_1}{2} = a \Rightarrow y + y_1 = 2a \Rightarrow y_1 = -y + 2a$

Also, x-coordinate of P', M and P are same.

$\therefore \quad x_1 = x$

So, the coordinates of $P' = (x, -y + 2a)$

Thus, the reflection of the point $P(x, y)$ in the line $y = a$ is the point $P'(x, -y + 2a)$.

Example 6. Find the reflection of the point (4, 5) in the line $y = 2$.

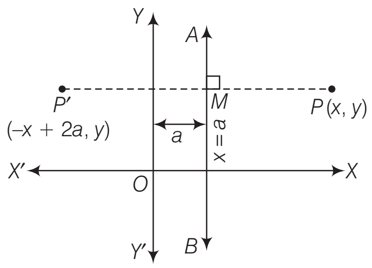
Sol. We know that, the reflection of a point $P(x, y)$ in the line $y = a$ is $P'(x, -y + 2a)$.

$\therefore$ The reflection of the point (4, 5) in the line $y = 2$ is the point $(4, -5 + 2 \times 2)$, i.e. $(4, -1)$.

[$\because x = 4$, $y = 5$ and $a = 2$]

Reflection of a Point in a Line Parallel to Y-axis

Let $P(x, y)$ be any point and $x = a$ be a line parallel to Y-axis at a distance 'a' from it as shown in the following figure. Let $P'(x_1, y_1)$ be the reflection of P in the line $x = a$.



Then, the mid-point M of PP' lies on the line $x = a$ and PP' is perpendicular to $x = a$.

$\therefore$ Coordinates of M $= \left(\dfrac{x + x_1}{2}, \dfrac{y + y_1}{2} \right)$

Clearly, M lies on the line $x = a$.

$\therefore \quad \dfrac{x + x_1}{2} = a \Rightarrow x + x_1 = 2a \Rightarrow x_1 = -x + 2a$

Also, y-coordinate of P', M and P are same.

$\therefore \quad y_1 = y$

So, the coordinates of $P' = (-x + 2a, y)$.

Thus, the coordinates of the reflection of the point $P(x, y)$ in the line $x = a$, is $(-x + 2a, y)$.

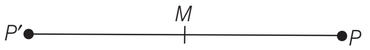
Example 7. Find the reflection of the point (3, 2) in the line $x = 2$.

Sol. We know that, the reflection of a point $P(x, y)$ in the line $x = a$ is $(-x + 2a, y)$.

$\therefore$ The reflection of the point (3, 2) in the line $x = 2$ is $(-3 + 2 \times 2, 2)$, i.e. (1, 2). $[\because x = 3, y = 2$ and $a = 2]$

Reflection of a Point in a Given Point

The reflection of a point P in a given point M, is the point P' such that M is the mid-point of the line segment PP'.



Example 8. Find the reflection of the point (– 1, 5) in the given point (2, 3).

Sol. Let $P'(x, y)$ be the reflection of the point $P(-1, 5)$ in the given point $M(2, 3)$. Then, M is the mid-point of the line segment PP'.

$\therefore$ Coordinates of $M = \left(\dfrac{-1 + x}{2}, \dfrac{5 + y}{2} \right)$

But coordinates of $M = (2, 3)$ [given]

$\therefore \quad (2, 3) = \left(\dfrac{-1 + x}{2}, \dfrac{5 + y}{2} \right)$

On comparing both sides, we get

$\dfrac{-1 + x}{2} = 2$ and $\dfrac{5 + y}{2} = 3 \Rightarrow -1 + x = 4$ and $5 + y = 6$

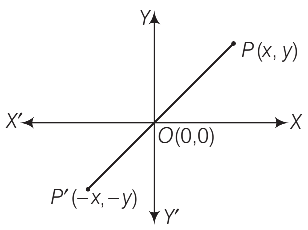
$\Rightarrow \quad x = 4 + 1$ and $y = 6 - 5 \Rightarrow x = 5$ and $y = 1$

$\therefore$ The coordinates of $P' = (5, 1)$

Thus, the reflection of the point (– 1, 5) in the given point (2, 3) is (5, 1).

Reflection of a Point in the Origin

Let $P(x, y)$ be any point in the coordinate plane as shown in the following figure.



In order to find the reflection of P in the origin O, let us join OP. Clearly, point P' is the reflection of point P in the origin O. Let (x_1, y_1) be the coordinates of P'.

Then, coordinates of the mid-point of PP'

$$= \left(\frac{x + x_1}{2}, \frac{y + y_1}{2} \right)$$

But origin is the mid-point of PP'.

$\therefore \quad \dfrac{x + x_1}{2} = 0 \Rightarrow \dfrac{y + y_1}{2} = 0$

$\Rightarrow \quad x_1 = -x, y_1 = -y$

So, the coordinates of $P' = (-x, -y)$. **Thus, the reflection of point $P(x, y)$ in the origin, is $P'(-x, -y)$.**

Example 9. Find the reflection of (2, 3) in the origin.

Sol. We know that, the reflection of a point $P(x, y)$ in the origin $O(0, 0)$ is $P'(-x, -y)$.

$\therefore$ The reflection of the point (2, 3) in the origin, is the point (– 2, – 3).

Some Important Points about Reflection

(i) If three points A, B and C are clockwise, then their images A', B' and C' are anti-clockwise.

(ii) The image of a figure under reflection is congruent to the original figure.

(iii) Reflection in the X-axis, followed by the reflection in the Y-axis is equivalent to reflection in the origin.

Invariant Point

Any point that remains unaltered under a given transformation is called an **invariant point**.

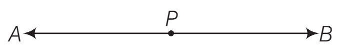
or

Reflection of a point in the point itself is called invariant point.

or

A point is called an invariant point with respect to a given line if and only if it lies on the reflection line.

e.g. If AB is a reflection line and a point P lies on it, then P is the invariant point, as the image of P is P itself.



Example 10. The point $P(a, b)$ is invariant, when the reflected in the origin. Find the values of a and b.

Sol. The image of point P in the origin is $P'(-a, -b)$.

Now, as P is invariant

$\therefore (a, b) = (-a, -b)$

$\Rightarrow \quad a = -a$ and $b = -b \Rightarrow 2a = 0$ and $2b = 0$

$\Rightarrow \quad a = 0$ and $b = 0$

CHAPTER EXERCISE

a 3 Marks Questions

1. Find the coordinates of the image of each of the following points under the reflection in X-axis.
(i) $(5, 2)$ (ii) $(-2, 3)$
(iii) $(4, -6)$ (iv) $(0, -1)$
(v) $(7, 0)$ (vi) $(-5, -3)$

2. Find the coordinates of the image of each of the following points under the reflection in the line $y = 0$.
(i) $(3, -6)$ (ii) $(1, -2)$
(iii) $(-3, -1)$ (iv) $(0, -2)$
(v) $(3, 0)$ (vi) $\left(\frac{1}{2}, \frac{3}{4}\right)$

3. Find the coordinates of the image of each of the following points under the reflection in the line $x = 0$.
(i) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (ii) $(2, -1)$
(iii) $(0, -3)$ (iv) $(-7, 0)$
(v) $(-5, -3)$ (vi) $(2, 8)$

4. Find the coordinates of the image of each of the following points under the reflection in Y-axis.
(i) $\left(-\frac{1}{2}, \frac{4}{5}\right)$ (ii) $(3, 7)$
(iii) $(3, 0)$ (iv) $(-3, -8)$
(v) $\left(0, \frac{7}{2}\right)$ (vi) $(-6, 7)$

5. (i) The point $T(-3, 5)$ is reflected in the line $y = 3$ to the point T'. Find the coordinates of the point T'.
(ii) Find the image of the point $S(1, 4)$ in the line $y = -3$.

6. (i) The point $P(3, 1)$ is reflected in the line $x = 2$ to the point P'. Find the coordinates of the point P'.
(ii) Find the image of the point $Q(1, -2)$ in the line $x = -1$.

7. Find the coordinates of the image of each of the following points under the reflection in the origin.
(i) $(2, -5)$ (ii) $(0, -1)$
(iii) $(0, 0)$ (iv) $(7, 8)$
(v) $\left(\frac{1}{2}, \frac{3}{2}\right)$ (vi) $\left(\frac{1}{5}, 0\right)$

8. A point S is reflected in the origin and the coordinates of its image are $(2, -5)$.
(i) Find the coordinates of S.
(ii) Find the coordinates of the image of S under the reflection in the X-axis.
(iii) Find the coordinates of the image of S under the reflection in the Y-axis.

9. A point P is reflected in the $x = 0$ and the coordinates of its image are $(-1, 2)$.
(i) Find the coordinates of P.
(ii) Find the coordinates of the image of P under the reflection in the line $y = 0$.
(iii) Find the coordinates of the image of P under the reflection in the origin.

10. The point $A(-4, -5)$ on reflection in Y-axis is mapped on A'. The point A' on reflection in the origin is mapped on A''. Find the coordinates of A' and A''. Write down a single transformation that maps A and A''.

11. The point $B(1, 7)$ on the reflection in the line $y = 0$ is mapped on B'. The point B' on reflection in the origin is mapped on B''. Find the coordinates of B' and B''. Also, write down a single transformation that maps B and B''.

12. A point $P(h, k)$ is reflected in X-axis to $P'(2, -3)$, write down the values of h and k. P'' is the image of P, when reflected in the Y-axis. Write down the coordinates of P''. Find the coordinates of P''', when P is reflected in the origin.

13. (i) A point $Q(a, b)$ is reflected in the line $x = 0$ to $Q'(-5, 2)$. Write down the values of a and b.
(ii) Q'' is the image of Q, when reflected in the line $y = 0$. Write down the coordinates of Q''.
(iii) Find the coordinates of Q''', when Q is reflected in the origin.

14. Points P and Q have coordinates $(2, 5)$ and $(0, 3)$. Find
(i) the image P' of P under the reflection in the X-axis.
(ii) the image Q' of Q under the reflection in the line PP'.

15. Points A and B have coordinates $(1, -3)$ and $(2, 4)$. Find the following.

(i) The image A' of A under the reflection in the line $x = 0$.
(ii) The image B' of B under the reflection of the line AA'.

16. Use a graph paper for this question. (Take 10 small divisions = 1 unit on both axes). P and Q have coordinates (5, 0) and (– 2, 4).
(i) P is invariant when reflected in an axis. Name the axis.
(ii) Find the image of Q on reflection in the axis found in (i).
(iii) $(0, k)$ on reflection in the origin is invariant. Write the value of k.
(iv) Write the coordinates of the image of Q, obtained by reflecting it in the origin followed by reflection in X-axis. ***[2005]***

17. Use a graph paper for this question.
(Take 1 box = 1 unit on both the axes)
S and R have coordinates (0, – 3) and (– 1, 6), respectively.
(i) S is invariant when reflected in an axis. Name the axis.
(ii) $(a, 0)$ on the reflection in the origin is invariant. Write the value of a.
(iii) Write the coordinates of the image of R under the reflection in the line $x = 0$.

18. The points $A(2, 3)$, $B(4, 5)$ and $C(7, 2)$ are the vertices of ΔABC.
(i) Write down the coordinates of A_1, B_1, C_1, if $\Delta A_1B_1C_1$ is the image of ΔABC when reflected in the origin.
(ii) Write down the coordinates of A_2, B_2, C_2, if $\Delta A_2B_2C_2$ is the image of ΔABC when reflected in the X-axis.
(iii) Assign the special name of the quadrilateral BCC_2B_2 and find its area. ***[2006]***

19. The ΔEFG, where E is (1, 2), F (– 3, 5) and G is (4, 7), is reflected in the line $x = 0$ to $\Delta E'F'G'$. $\Delta E'F'G'$ is then reflected in the origin to $\Delta E''F''G''$.
(i) Write down the coordinates of $E''F''G''$
(ii) Write down a single transformation that maps ΔEFG onto $\Delta E''F''G''$.

20. Use graph paper for this question (Take 10 small divisions = 1 unit on both axes).
Plot points $P(3, 2)$ and $Q(-3, -2)$. From P and Q, draw perpendiculars PM and QN on the X-axis.
(i) Name the image of P on reflection in the origin.
(ii) Assign the special name to the geometrical figure $PMQN$ and find its area.
(iii) Write the coordinates of the point to which M is mapped on reflection in
(a) X-axis (b) Y-axis (c) origin.

b 4 Marks Questions

21. Use graph paper for this question.
(Take 2 cm = 1 unit along both X and Y-axes)
$ABCD$ is a quadrilateral whose vertices are $A(2, 2)$, $B(2, -2)$, $C(0, -1)$, and $D(0, 1)$.
(i) Reflect quadrilateral $ABCD$ on the Y-axis and name it as $A'B'CD$.
(ii) Write down the coordinates of A' and B'.
(iii) Name two points which are invariant under the above reflection.
(iv) Name the polygon $A'B'CD$. ***[2018]***

22. Use graph paper for this question.
$A(1, 1)$, $B(5, 1)$, $C(4, 2)$ and $D(2, 2)$ are the vertices of a quadrilateral. Name the quadrilateral $ABCD$. A, B, C and D are reflected in the origin onto A', B', C' and D', respectively. Locate A', B', C' and D' on the graph sheet and write their coordinates. Are D, A, A' and D' collinear points? ***[2004]***

23. Use graph paper for this question.
(i) The point $P(2, -4)$ is reflected about the line $x = 0$ to get the image Q. Find the coordinates of Q.
(ii) Point Q is reflected about the line $y = 0$ to get the image R. Find the coordinates of R.
(iii) Name the figure PQR.
(iv) Find the area of the figure PQR. ***[2007]***

24. Using graph paper and taking 1 cm = 1 unit along with X-axis and Y-axis.
(i) Plot the point $A(-4, 4)$ and $B(2, 2)$.
(ii) Reflect A and B in the origin to get the images A' and B', respectively.
(iii) Write down the coordinates of A' and B'.
(iv) Give the geometrical name for the figure $ABA'B'$. ***[2012]***

25. Use a graph paper to answer the following questions. (Take 1 cm = 1 unit on both axes)
(i) Plot $A(4, 4)$, $B(4, -6)$ and $C(8, 0)$, the vertices of a ΔABC.
(ii) Reflect ABC on the Y-axis and name it as $A'B'C'$.
(iii) Write the coordinates of the images A', B' and C'.
(iv) Give a geometrical name for the figure $AA'C'B'BC$. ***[2011]***

26. Use a graph paper for this question.

(Taking 1cm = 1 unit along both the X and Y-axes)

(i) Plot the points A (0, 5), B (2, 5), C (5, 2), D (5, − 2), E (2, − 5) and F (0, − 5).

(ii) Reflect the points B, C, D and E on the Y-axis and name them respectively as B', C', D' and E'.

(iii) Write the coordinates of B', C', D' and E'.

(iv) Name the figure formed by $BCDEE'D'C'B'$.

[2015]

27. Using a graph paper, plot the points $A(6, 4)$ and $B(0, 4)$.

(i) Reflect A and B in the origin to get the images A' and B'.

(ii) Write the coordinates of A' and B'.

(iii) State the geometrical name for the figure $ABA'B'$.

(iv) Find its perimeter. **[2013]**

28. Using graph paper to answer the following questions. (Take 2 cm = 1 unit on both axes)

(i) Plot the points $A(-4, 2)$ and $B(2, 4)$.

(ii) A' is the image of A, when reflected in the Y-axis. Plot it on the graph paper and write the coordinates of A'.

(iii) B' is the image of B, when reflected in the line AA'. Write the coordinates of B'.

(iv) Write the geometric name of the figure $ABA'B'$.

[2014]

Hints and Answers

1. Hint The image of a point $P(x, y)$ under the reflection of X-axis is $P'(x, -y)$, i.e. only the sign of y-coordinate will be changed.

Ans. (i) (5, −2) (ii) (−2, −3) (iii) (4, 6) (iv) (0, 1) (v) (7, 0) (vi) (−5, 3)

2. Do same as Q. 1.

Ans. (i) (3, 6) (ii) (1, 2) (iii) (−3, 1) (iv) (0, 2) (v) (3, 0) (vi) $\left(\frac{1}{2}, \frac{-3}{4}\right)$

3. Hint The image of a point $P(x, y)$ under the reflection of line $x = 0$ (i.e. Y-axis) is $P'(-x, y)$, i.e. only the sign of x-coordinate will be changed.

Ans (i) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ (ii) (−2, −1) (iii) (0, −3) (iv) (7, 0) (v) (5, −3) (vi) (−2, 8)

4. Do same as Q. 3.

Ans. (i) $\left(\frac{1}{2}, \frac{4}{5}\right)$ (ii) (−3, 7) (iii) (−3, 0) (iv) (3, −8) (v) $\left(0, \frac{7}{2}\right)$ (vi) (6, 7)

5. Hint We know that the reflection of the point (x, y) in the line $y = a$ is the point $(x, -y + 2a)$.

Ans. (i) (−3, 1) (ii) (1, −10)

6. Hint We know that the reflection of the point (x, y) in the line $x = a$ is the point $(-x + 2a, y)$.

Ans. (i) (1, 1) (ii) (−3, −2)

7. Hint The image of a point $P(x, y)$ under the reflection of origin is $P'(-x, -y)$, i.e. the sign of both coordinates will be changed.

Ans (i) (−2, 5) (ii) (0, 1) (iii) (0, 0) (iv) (−7, −8) (v) $\left(-\frac{1}{2}, -\frac{3}{2}\right)$ (vi) $\left(-\frac{1}{5}, 0\right)$

8. (i) **Hint** The sign of the both coordinates will be changed.

Ans. (−2, 5)

(ii) Do same as Example 1. **Ans.** (−2, −5)

(iii) Do same as Example 2. **Ans.** (2, 5)

9. (i) Do same as Q. 3. **Ans.** (1, 2)

(ii) Do same as Q. 2. **Ans.** (1, −2)

(iii) Do same as Q. 5. **Ans.** (−1, −2)

10. Hint

(i) The reflection of a point (x, y) in the Y-axis is $(-x, y)$.

(ii) The reflection of a point (x, y) in the origin is $(-x, -y)$.

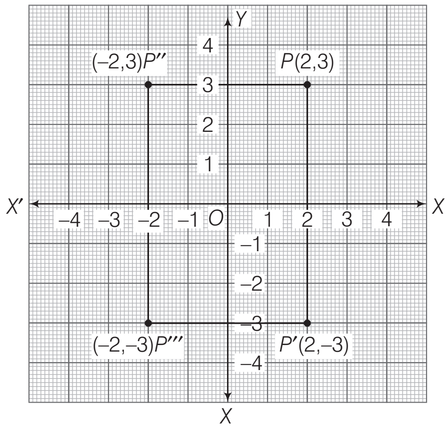
Ans. $A'(4, -5)$, $A''(-4, 5)$, Reflection of A in X-axis.

11. (i) The reflection of a point (x, y) in the line $y = 0$ is the point $(x, -y)$.

(ii) The reflection of a point (x, y) in the origin is $(-x, -y)$.

Ans. $B'(1, -7)$, $B''(-1, 7)$, Reflection of B in Y-axis.

12. **Hint**



Ans. $h = 2, k = 3, P'' \equiv (-2, 3), P''' = (-2, -3)$

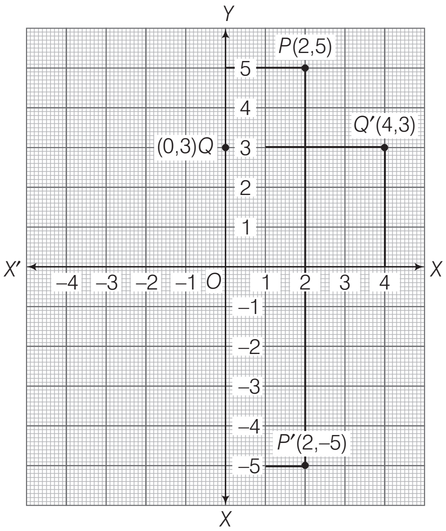
13. (i) Do same as Q. 3. (ii) Do same as Q. 2.

(iii) Do same as Q. 7.

Ans. (i) $(a = 5, b = 2)$ (ii) $(5, -2)$ (iii) $(-5, -2)$

14. **Hint** (i) Do same as Q. 1.

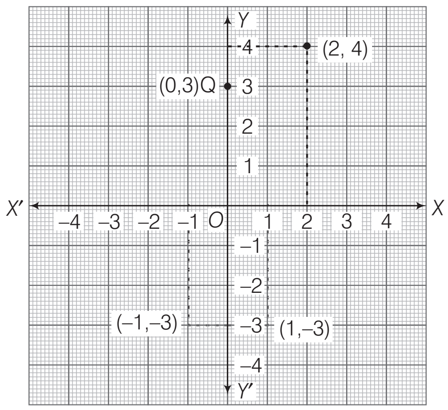
(ii) Do same as Example 7. (here $x = 2$)



Ans. (i) $(2, -5)$ (ii) $(4, 3)$

15. **Hint** (i) Do same as Q. 3.

(ii) Do same as Example 6. (here, $y = -3$)



Ans. (i) $(-1, -3)$ (ii) $(2, -10)$

16. (i) **Hint** We know that a point is invariant when line of reflection passing through it.

Ans. Y-axis.

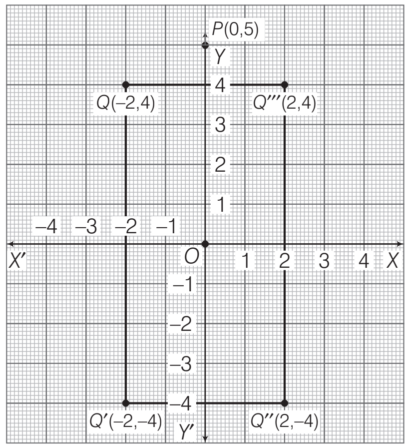
(ii) **Hint** Reflect the point $(-2, 4)$ in Y-axis.

Ans. $(2, 4)$

(iii) **Hint** The reflection of $(0, k)$ in the origin has coordinates $(0, -k)$. But, it is given that $(0, k)$ on reflection in the origin is invariant.

$\therefore \quad (0, k) = (0, -k)$ **Ans.** $k = 0$

(iv) **Hint** The reflection of Q in the X-axis is $Q'(-2, -4)$. Now, the reflection of Q'' in the origin is $Q'''(2, 4)$ as shown in the graph below



Ans. $(2, 4)$

17. **Hint** Do same as Q. 16.

Ans. (i) Y-axis (ii) $a = 0$ (iii) $(1, 6)$

18. (i) **Hint** Image of any point (x, y) in the origin is $(-x, -y)$.

Ans. $A_1(-2, -3), B_1(-4, -5), C_1(-7, -2)$

(ii) **Hint** The reflection of the point (x, y) in the X-axis is $(x, -y)$.

Ans. $A_2(2, -3), B_2(4, -5), C_2(7, -2)$

(iii) **Hint** Plot the points B, C, C_2 and B_2 on a graph paper.

$\therefore$ Area of trapezium $= \frac{1}{2}$ (Sum of parallel sides)

$\times$ (Perpendicular distance between parallel sides)

$$= \frac{1}{2}(4 + 10) \times 3$$

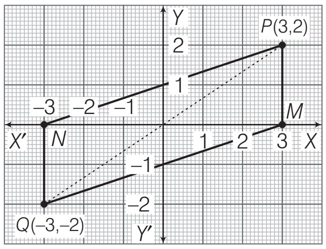
Ans. Trapezium; 21 sq units

19. Do same as Q. 18.

Ans. (i) $(1, -2), (-3, -5), (4, -7)$

(ii) Reflection in the X-axis.

20. (i) **Hint**



Ans. Point Q

(ii) **Hint** Clearly, $PMQN$ is a parallelogram.

Area of parallelogram $PMQN$

$$= 2 \text{ (Area of } \Delta NMP)$$
$$= 2 \times \frac{1}{2}(NM \times MP)$$

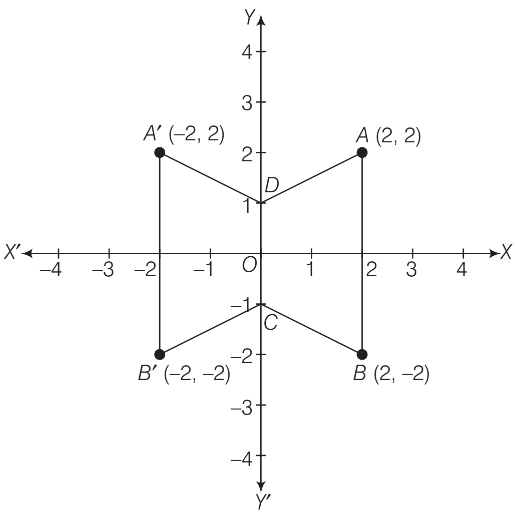
Ans. 12 sq units

(iii) **Hint** Reflect the point (3, 0) in X-axis, Y-axis and origin, respectively.

Ans. (a) (3, 0) (b) (− 3, 0)

(c) (− 3, 0).

21. (i)

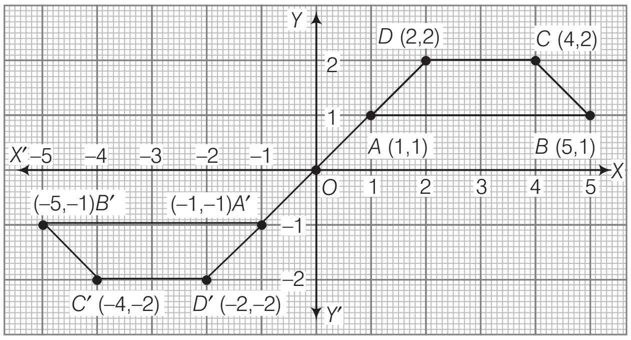


(ii) $A' = (-2, 2)$ and $B' = (-2, -2)$

(iii) C and D are invariant under the reflection.

(iv) $A'B'CD$ is a trapezium.

22. **Hint** $A'(-1, -1)$, $B'(-5, -1)$, $C'(-4, -2)$, $D'(-2, -2)$



Ans. Trapezium, Yes

23. Let us take 10 small divisions = 1 unit on both the axes.

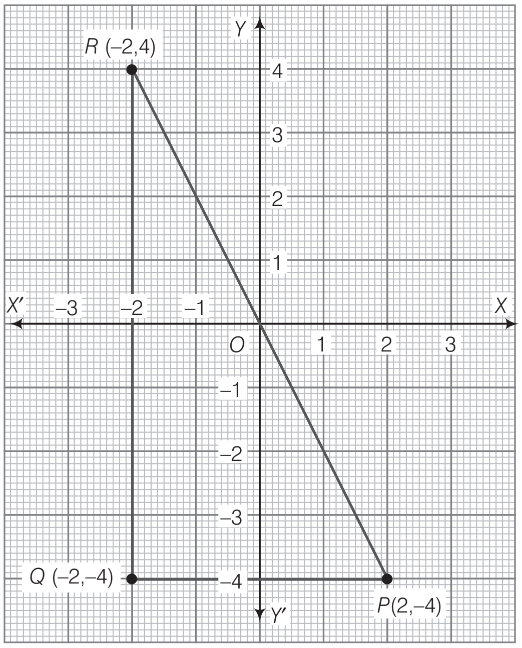
(i) Do same as Example 3.

Ans. (− 2, − 4)

(ii) Do same as Q. 2.

Ans. (− 2, 4)

(iii) **Hint**



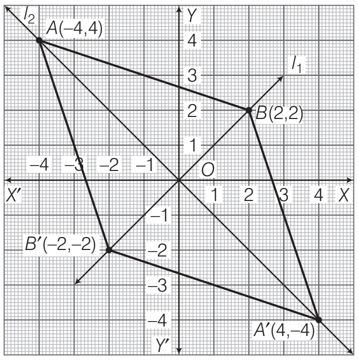
Ans. Right angled triangle

(iv) **Hint** We have, $PQ = 2 + 2 = 4$ units and $QR = 4 + 4 = 8$ units

$\therefore$ Area of $\Delta PQR = \frac{1}{2}(PQ \times QR) = \frac{1}{2} \times 4 \times 8$

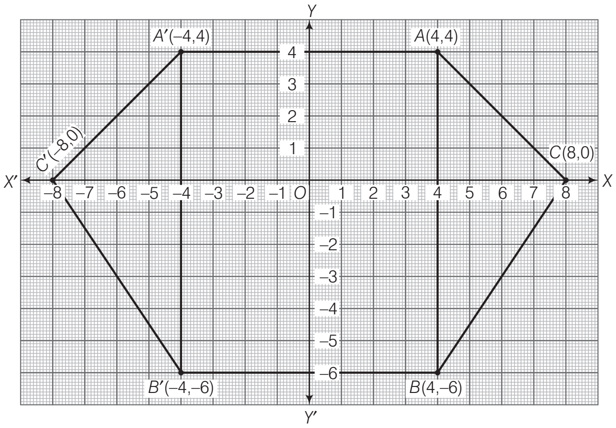
Ans. 16 sq units

24. Hint



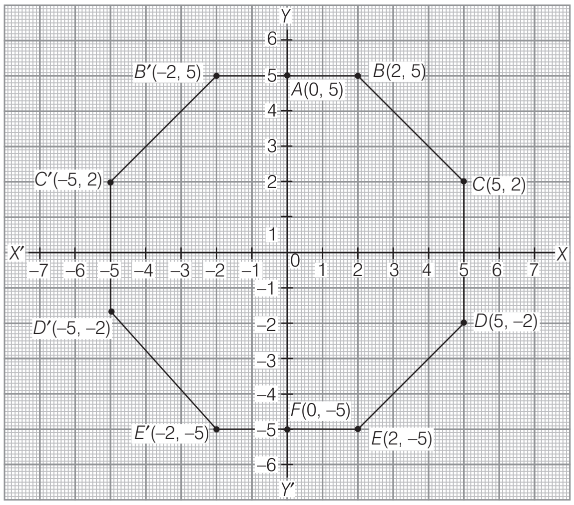
Ans. (iii) $A'(4, -4)$ and $B'(-2, -2)$ (iv) Rhombus

25. Hint



Ans. (iii) $A'(-4, 4)$, $B'(-4, -6)$ and $C'(-8, 0)$ (iv) Hexagon

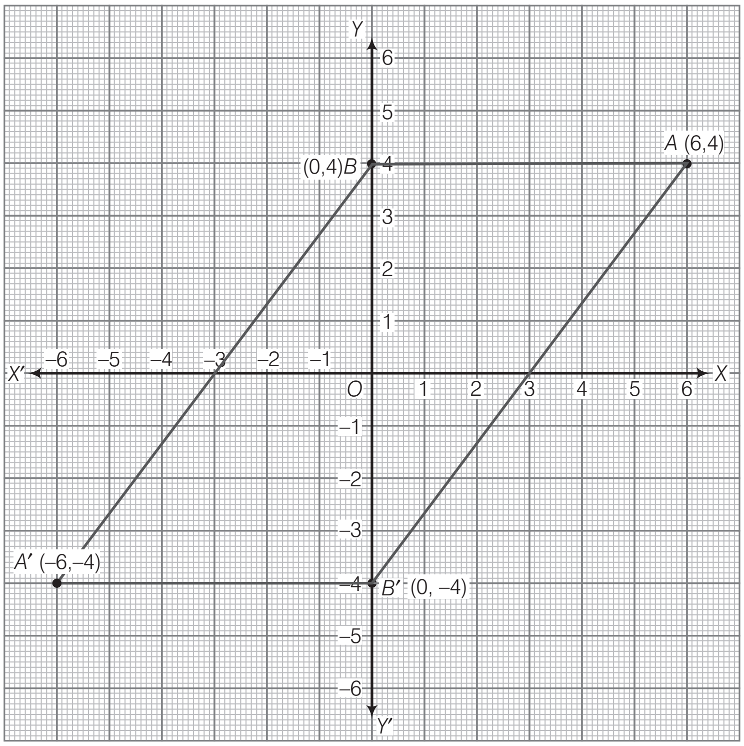
26. Hint (i)



(ii) Reflection of a point (x, y) in the Y-axis is $(-x, y)$.

Ans. (iii) $B'(-2, 5)$, $C'(-5, 2)$, $D'(-5, -2)$ and $E'(-2, -5)$ (iv) Octagon

27. **Hint**



(ii) $A'(-6,-4)$ and $B'(0,-4)$

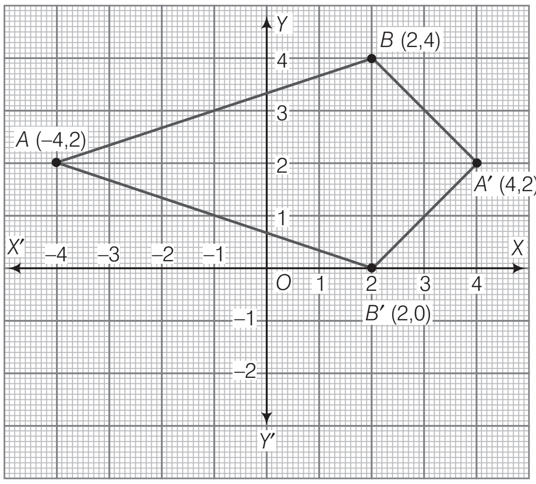
(iii) Parallelogram

(iv) **Hint** Perimeter of a parallelogram $ABA'B'$

$$= 2(AB + AB') = 2(6+10) \qquad [\because \text{ in right angled } \Delta ABB', AB' = \sqrt{AB^2 + BB'^2} = \sqrt{6^2+8^2} = \sqrt{100} = 10]$$

Ans. 32 units

28. **Hint** (i)



(ii) Since, B' is the image of B, when reflected in the line $y = 2$. **Ans.** (4, 2)

(iii) (2, 0) (iv) Kite

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1 Use graph paper for this question.
(Take 2 cm = 1 unit along both X and Y-axes)
$ABCD$ is a quadrilateral whose vertices are $A(2, 2)$, $B(2, -2)$, $C(0, -1)$, and $D(0, 1)$.
(i) Reflect quadrilateral $ABCD$ on the Y-axis and name it as $A'B'CD$.
(ii) Write down the coordinates of A' and B'.
(iii) Name two points which are invariant under the above reflection.
(iv) Name the polygon $A'B'CD$.

2017

2 Use a graph paper for this question (Take 2 cm = 1 unit on both X and Y-axes)
(i) Plot the points $A(0, 4)$, $B(2, 3)$, $C(1, 1)$ and $D(2, 0)$.
(ii) Reflect points B, C, D in the Y-axis and write down their coordinates. Name the images as B', C' and D', respectively.

2016

3 Use graph paper for this question. (Take 2 cm = 1 unit along both X and Y-axes)
Plot the points $O(0, 0)$, $A(-4, 4)$, $B(-3, 0)$ and $C(0, -3)$.
(i) Reflect points A and B on the Y-axis and name them A' and B', respectively. Write down their coordinates.
(ii) Name the figure $OABCB'A'$.

2015

4 Use a graph paper for this question.
(Take 1 cm = 1 unit along both the X and Y-axes)
(i) Plot the points $A(0, 5)$, $B(2, 5)$, $C(5, 2)$, $D(5, -2)$, $E(2, -5)$ and $F(0, -5)$.
(ii) Reflect the points B, C, D and E on the Y-axis and name them respectively as B', C', D' and E'.
(iii) Write the coordinates of B', C', D' and E'.
(iv) Name the figure formed by $BCDEE'D'C'B'$.

2014

5 Using a graph paper to answer the following questions.
(Take 2 cm = 1 unit on both axes)
(i) Plot the points $A(-4, 2)$ and $B(2, 4)$.
(ii) A' is the image of A, when reflected in the Y- axis. Plot it on the graph paper and write the coordinates of A'.
(iii) B' is the image of B, when reflected in the line AA'. Write the coordinates of B'.
(iv) Write the geometric name of the figure $ABA'B'$.

2013

6 Using a graph paper, plot the points $A(6, 4)$ and $B(0, 4)$.
(i) Reflect A and B in the origin to get the images A' and B'.
(ii) Write the coordinates of A' and B'.
(iii) State the geometrical name for the figure $ABA'B'$.
(iv) Find its perimeter.

2012

7 Using a graph paper and taking 1 cm = 1 unit along with X and Y-axes.
(i) Plot the points $A(-4, 4)$ and $B(2, 2)$.
(ii) Reflect A and B in the origin to get the images A' and B', respectively.
(iii) Write down the coordinates of A' and B'.
(iv) Give the geometrical name for the figure $ABA'B'$.

2011

8 Use a graph paper to answer the following questions.
(Take 1 cm = 1 unit on both the axes)
(i) Plot $A(4, 4)$, $B(4, -6)$ and $C(8, 0)$, the vertices of a ΔABC.
(ii) Reflect ABC on the Y-axis and name it as $A'B'C'$.
(iii) Write the coordinates of the images A', B' and C'.
(iv) Give a geometrical name for the figure $AA'C'B'BC$.

* All these questions are completely covered in chapter either as solved examples or in chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. If the image of the point P under the reflection in the X-axis is $(-3, 2)$, then the coordinates of the point P are

(a) $(3, 2)$ (b) $(-3, -2)$ (c) $(3, -2)$ (d) $(-3, 0)$

2. If $(4, 3)$ and $(-4, -3)$ are opposite two vertices of a rectangle, then other two vertices are

(a) $(4, -3)$ and $(-4, 3)$ (b) $(4, -3)$ and $(-3, 4)$ (c) $(-4, 4)$ and $(-3, 4)$ (d) None of these

3. A point M is reflected in X-axis to $M'(4, -5)$. M'' is the image of M, when reflected in the Y-axis. The coordinates of M''' when M'' is reflected in the origin, is

(a) $(4, 5)$ (b) $(-4, -5)$ (c) $(4, -5)$ (d) None of these

4. The point $P(h, k)$ is reflected in the X-axis, then it is reflected in the origin to P'. If P' has coordinate $(-8, 5)$, then the value of (h, k) is

(a) $(8, 5)$ (b) $(5, 8)$ (c) $(-5, -8)$ (d) $(-8, -5)$

5. Which of the following points is invariant with respect to the line $y = -2$?

(a) $(3, 2)$ (b) $(3, -2)$ (c) $(2, 3)$ (d) $(-2, 3)$

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 401.

CHAPTER

11

Section and Mid-Point Formulae

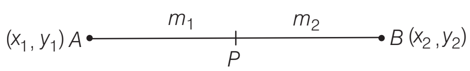
Chapter Objectives

- Section Formula for Internal Division
- Mid-point Formula
- Centroid of a Triangle

In class IX, we used the concepts of coordinate geometry to find the distance between two points whose coordinates are given. Here, we will extend our study and will find the coordinates of a point which divides the line segment joining two points in a given ratio. Alongwith it we will also learn to find the coordinates of mid-point of a line segment and the coordinates of centroid of a triangle whose vertices are given.

Section Formula for Internal Division

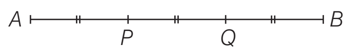
Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two points and $P(x, y)$ is a point on the line segment joining A and B such that $AP : BP = m_1 : m_2$, then the point P is said to divide the line segment AB internally
in the ratio $m_1 : m_2$.



Then, the coordinates of point P are $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$.

This formula is known as **section formula** for **internal division.**

Note Trisection means a line divided into three equal parts

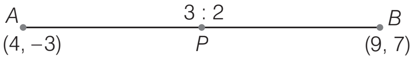


i.e. AP = PQ = QB

Here, P and Q trisect AB in the ratio of 1 : 2 and 2 : 1, respectively.

Example 1. Find the coordinates of the point which divides the line segment joining the points (4, – 3) and (9, 7) internally in the ratio 3 : 2.

Sol. Let $P(x, y)$ divides the line segment joining the points $A(4, -3)$ and $B(9, 7)$ in the ratio of 3 : 2.



By section formula, we get

$$\text{Coordinates of } P = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

$$= \left(\frac{3\times 9 + 2\times 4}{3+2}, \frac{3\times 7 + 2\times(-3)}{3+2}\right)$$

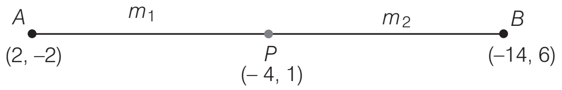
$$[\because (x_1, y_1) \equiv (4, -3), (x_2, y_2) \equiv (9, 7) \text{ and } m_1 = 3, m_2 = 2]$$

$$= \left(\frac{27+8}{5}, \frac{21-6}{5}\right) = P\left(\frac{35}{5}, \frac{15}{5}\right) = P\ (7, 3)$$

Therefore, (7, 3) is the required point.

Example 2. In what ratio, does the point $P(-4, 1)$ divide the line segment joining the points $A(2, -2)$ and $B(-14, 6)$? Also, verify the ratio.

Sol. Let point $P(-4, 1)$ divides the line segment joining the points $A(2, -2)$ and $B(-14, 6)$ in the ratio $m_1 : m_2$.



By section formula, we get

$$\text{Coordinates of } P = \left(\frac{-14m_1 + 2m_2}{m_1 + m_2}, \frac{6m_1 - 2m_2}{m_1 + m_2}\right)$$

$$[\because (x_1, y_1) \equiv (2, -2) \text{ and } (x_2, y_2) \equiv (-14, 6)]$$

But coordinates of P is given, i.e. (– 4, 1).

$$\therefore \quad (-4, 1) = \left(\frac{-14m_1 + 2m_2}{m_1 + m_2}, \frac{6m_1 - 2m_2}{m_1 + m_2}\right)$$

On equating the x-coordinates both sides of Eq. (i), we get

$$-4 = \frac{-14m_1 + 2m_2}{m_1 + m_2} \Rightarrow -4m_1 - 4m_2 = -14m_1 + 2m_2$$

$$\Rightarrow \quad -4m_1 + 14m_1 = 4m_2 + 2m_2 \Rightarrow 10m_1 = 6m_2$$

$$\Rightarrow \quad \frac{m_1}{m_2} = \frac{6}{10} \Rightarrow \frac{m_1}{m_2} = \frac{3}{5}$$

or $\quad m_1 : m_2 = 3 : 5$

On equating the y-coordinates both sides of Eq. (i), we get

$$1 = \frac{6m_1 - 2m_2}{m_1 + m_2} \Rightarrow 1 = \frac{6\frac{m_1}{m_2} - 2}{\frac{m_1}{m_2} + 1}$$

[dividing the numerator and denominator of RHS by m_2]

$$\Rightarrow \quad 1 = \frac{6\left(\frac{3}{5}\right) - 2}{\frac{3}{5} + 1} \qquad \left[\because \frac{m_1}{m_2} = \frac{3}{5}\right]$$

$$\Rightarrow \quad 1 = \frac{\frac{18}{5} - 2}{\frac{3+5}{5}} = \frac{18-10}{8} \Rightarrow 1 = \frac{8}{8} \Rightarrow 1 = 1$$

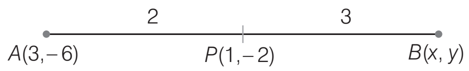
Hence, the point (– 4, 1) divides the line segment joining the points $A(2, -2)$ and $B(-14, 6)$ in the ratio 3 : 5.

Example 3. $P(1, -2)$ is a point on the line segment joining $A(3, -6)$ and $B(x, y)$ such that $AP : PB$ is equal to 2 : 3. Find the coordinates of B. ***[2017]***

Sol. Given a line segment joining $A(3, -6)$ and $B(x, y)$ such that

$$AP : PB = 2 : 3$$

i.e. P divides AB in the ratio 2 : 3 internally.



By section formula, we get

Coordinates of P

$$= \left[\frac{(2x) + (3\times 3)}{2+3}, \frac{2y + 3\times(-6)}{2+3}\right] = \left(\frac{2x+9}{5}, \frac{2y-18}{5}\right)$$

$$[\because (x_1, y_1) \equiv (3, -6), (x_2, y_2) \equiv (x, y) \text{ and } m_1 = 2, m_2 = 3]$$

But given coordinates of $P \equiv (1, -2)$

$$\therefore \quad (1, -2) = \left(\frac{2x+9}{5}, \frac{2y-18}{5}\right)$$

On equating x and y-coordinates both sides, we get

$$0\frac{2x+9}{5} = 1 \text{ and } \frac{2y-18}{5} = -2$$

$$\Rightarrow \quad 2x + 9 = 5 \text{ and } 2y - 18 = -10$$

$$\Rightarrow \quad 2x = 5 - 9 \text{ and } 2y = -10 + 18$$

$$\Rightarrow \quad 2x = -4 \text{ and } 2y = 8$$

$$\Rightarrow \quad x = -2 \text{ and } y = 4$$

Thus, coordinates of $B \equiv (-2, 4)$.

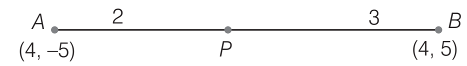
Example 4. If the line joining the points $A(4, -5)$ and $B(4, 5)$ is divided by the point P, such that $\frac{AP}{AB} = \frac{2}{5}$, then find the coordinates of P. ***[2007]***

Sol. Given a line segment joining $A(4, -5)$ and $B(4, 5)$ such that $\frac{AP}{AB} = \frac{2}{5} \Rightarrow \frac{AB}{AP} = \frac{5}{2} \Rightarrow \frac{AB}{AP} - 1 = \frac{5}{2} - 1$

[subtracting 1 from both sides]

$$\Rightarrow \quad \frac{AB - AP}{AP} = \frac{5-2}{2} \Rightarrow \frac{PB}{AP} = \frac{3}{2}, \text{ i.e. } m_1 : m_2 = 2 : 3$$

i.e. P divides AB is the ratio 2 : 3 internally.



By section formula, we get

$$\text{Coordinates of } P = \left(\frac{2\times 4 + 3\times 4}{2+3}, \frac{2\times 5 + 3\times(-5)}{2+3}\right)$$

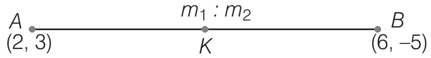
$$[\because (x_1, y_1) \equiv (4, -5), (x_2, y_2) \equiv (4, 5) \text{ and } m_1 = 2, m_2 = 3]$$

$$= \left(\frac{8+12}{2+3}, \frac{10-15}{2+3}\right) = \left(\frac{20}{5}, \frac{-5}{5}\right) = (4, -1)$$

Hence, the coordinates of P are (4, – 1).

Example 5. The line segment joining $A(2, 3)$ and $B(6, -5)$ is intersected by the X-axis at the point K. Write the coordinates of the point K. Hence, find the ratio in which K divides AB. ***[2006]***

Sol. Let K divides the line segment joining the points A (2, 3) and B (6, – 5) in the ratio $m_1 : m_2$.



By section formula, we get

$\therefore$ Coordinates of K are $\left(\dfrac{6m_1+2m_2}{m_1+m_2}, \dfrac{-5m_1+3m_2}{m_1+m_2}\right)$

$[\because (x_1, y_1) \equiv (2, 3) \text{ and } (x_2, y_2) \equiv (6, -5)]$

But line segment AB intersected at point K by the X-axis, so coordinates of K will be $(x, 0)$

$$\therefore \quad (x, 0) = \left(\frac{6m_1+2m_2}{m_1+m_2}, \frac{-5m_1+3m_2}{m_1+m_2}\right) \quad \ldots\text{(i)}$$

On equating y-coordinates both sides of Eq. (i), we get

$$\frac{-5m_1+3m_2}{m_1+m_2}=0 \Rightarrow -5m_1+3m_2=0 \Rightarrow 5m_1=3m_2 \Rightarrow \frac{m_1}{m_2}=\frac{3}{5}$$

Hence, the required ratio is 3 : 5.

On equating x-coordinates both sides, we get

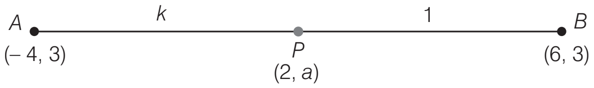
$$x=\frac{6m_1+2m_2}{m_1+m_2}=\frac{6\cdot\frac{m_1}{m_2}+2}{\frac{m_1}{m_2}+1}=\frac{6\times\frac{3}{5}+2}{\frac{3}{5}+1}$$

$$=\frac{18+10}{8}=\frac{28}{8}=\frac{7}{2} \qquad \left[\because \frac{m_1}{m_2}=\frac{3}{5}\right]$$

So, coordinates of point K is $\left(\dfrac{7}{2}, 0\right)$.

Example 6. Find the ratio in which the point $(2, a)$ divides the joining of $(-4, 3)$ and (6, 3). Hence, find a.

Sol. Let the point $P(2, a)$ divides the line joining the points $A(-4, 3)$ and $B(6, 3)$ in the ratio $k : 1$.



By section formula, we get

Coordinates of $P=\left(\dfrac{6\times k+1\times -4}{k+1}, \dfrac{3\times k+1\times 3}{k+1}\right)$

$[\because (x_1, y_1) \equiv (-4, 3), (x_2, y_2) \equiv (6, 3) \text{ and } m_1 = k, m_2 = 1]$

But coordinates of P is given, i.e. $(2, a)$.

$$\therefore \quad (2, a)=\left(\frac{6k-4}{k+1}, \frac{3k+3}{k+1}\right) \quad \ldots\text{(i)}$$

On equating x-coordinates both sides of Eq. (i), we get

$$2=\frac{6k-4}{k+1} \Rightarrow 2k+2=6k-4 \Rightarrow 4k=6 \Rightarrow k=\frac{6}{4} \Rightarrow k=\frac{3}{2}$$

Hence, the point P divides the line joining the points $(-4, 3)$ and $(6, 3)$ in the ratio 3 : 2.

On equating y-coordinates both sides of Eq. (i), we get

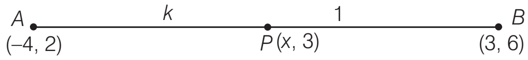
$$a=\frac{3k+3}{k+1}=\frac{3\times\frac{3}{2}+3}{\frac{3}{2}+1}=\frac{\frac{9}{2}+3}{\frac{3}{2}+1}$$

$$=\frac{9+6}{3+2}=\frac{15}{5}=3 \qquad \left[\because k=\frac{3}{2}\right]$$

Example 7. Calculate the ratio, in which the line joining $A(-4, 2)$ and $B(3, 6)$ is divided by $P(x, 3)$. Also, find

(i) x. (ii) length of AP. ***[2014]***

Sol. Let $P(x, 3)$ divides the line segment joining the points $A(-4, 2)$ and $B(3, 6)$ in the ratio $k : 1$.



By section formula, we get

Coordinates of P

$$=\left[\frac{3\times k+1\times -4}{k+1}, \frac{k\times 6+1\times 2}{k+1}\right]=\left(\frac{3k-4}{k+1}, \frac{6k+2}{k+1}\right)$$

$[\because (x_1, y_1) \equiv (-4, 2), (x_2, y_2) \equiv (3, 6) \text{ and } m_1 = k, m_2 = 1]$

But coordinates of P is given, i.e. $(x, 3)$.

$$\therefore \quad (x, 3)=\left(\frac{3k-4}{k+1}, \frac{6k+2}{k+1}\right) \quad \ldots\text{(i)}$$

On equating y-coordinates both sides of Eq. (i), we get

$$3=\frac{6k+2}{k+1} \Rightarrow 3(k+1)=6k+2$$

$$\Rightarrow \quad 3k+3=6k+2 \Rightarrow 6k-3k=3-2$$

$$\Rightarrow \quad 3k=1 \Rightarrow k=\frac{1}{3}$$

Hence, the required ratio is $\dfrac{1}{3}$.

(i) On equating x-coordinates both sides of Eq. (i), we get

$$x=\frac{3k-4}{k+1}$$

$$\Rightarrow \quad x=\frac{3\left(\frac{1}{3}\right)-4}{\frac{1}{3}+1}=\frac{\frac{3-12}{3}}{\frac{1+3}{3}}=\frac{-9}{4} \qquad \left[\because k=\frac{1}{3}\right]$$

(ii) Length of $AP=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$$\left[\because \text{distance}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\right]$$

$$=\sqrt{\left(-\frac{9}{4}+4\right)^2+(3-2)^2}$$

$$\left[\because (x_1, y_1) \equiv (-4, 2) \text{ and } (x_2, y_2) \equiv \left(\frac{-9}{4}, 3\right)\right]$$

$$=\sqrt{\left(\frac{-9+16}{4}\right)^2+(1)^2}=\sqrt{\left(\frac{7}{4}\right)^2+1^2}$$

$$=\sqrt{\frac{49}{16}+1}=\sqrt{\frac{49+16}{16}}=\frac{\sqrt{65}}{4} \text{ units}$$

Example 8. Given a line segment AB joining the points $A(-4, 6)$ and $B(8, -3)$. Find

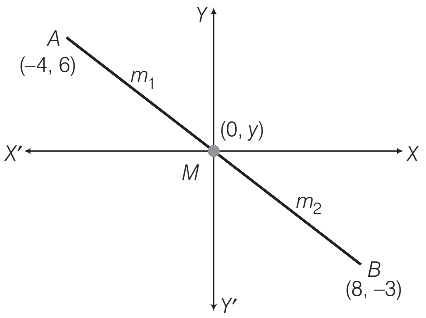
(i) the ratio in which AB is divided by the Y-axis.

(ii) the coordinates of the point of intersection.

(iii) the length of AB. ***[2012]***

Sol. Given coordinates of A and B are $(-4, 6)$ and $(8, -3)$, respectively.

(i) Let AB is divided by the Y-axis at point $M(0, y)$ in the ratio $m_1 : m_2$.



Then, coordinates of $M = \left(\dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}, \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$

$$= \left(\frac{m_1 \times 8 + m_2 \times (-4)}{m_1 + m_2}, \frac{m_1 \times (-3) + m_2 \times 6}{m_1 + m_2}\right)$$

$[\because (x_1, y_1) \equiv (-4, 6) \text{ and } (x_2, y_2) \equiv (8, -3)]$

But x-coordinate of M is 0.

$\therefore \quad 0 = \dfrac{m_1 \times 8 + m_2 \times (-4)}{m_1 + m_2}$

$\Rightarrow \quad 0 = 8m_1 - 4m_2$

$\Rightarrow \quad 4m_2 = 8m_1$

$\Rightarrow \quad \dfrac{4}{8} = \dfrac{m_1}{m_2} \Rightarrow \dfrac{m_1}{m_2} = \dfrac{1}{2}$

(ii) At the point of intersection, $x = 0$

and $\quad y = \dfrac{-3m_1 + 6m_2}{m_1 + m_2}$

$$= \frac{-3\dfrac{m_1}{m_2} + 6}{\dfrac{m_1}{m_2} + 1} = \frac{-3\left(\dfrac{1}{2}\right) + 6}{\dfrac{1}{2} + 1} \qquad \left[\because \frac{m_1}{m_2} = \frac{1}{2}\right]$$

$$= \frac{-3 + 12}{3} = \frac{9}{3} = 3$$

Hence, the coordinates of M are $(0, 3)$.

(iii) Length of $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(8 + 4)^2 + (-3 - 6)^2}$$

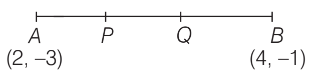
$[\because (x_1, y_1) = (-4, 6) \text{ and } (x_2, y_2) = (8, -3)]$

$$= \sqrt{(12)^2 + (-9)^2}$$

$$= \sqrt{144 + 81} = \sqrt{225} = 15 \text{ units}$$

Example 9. Find the coordinates of the points of trisection of the line segment joining $(2, -3)$ and $(4, -1)$.

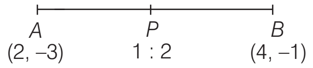
Sol. Let P and Q be the points of trisection as shown below



Then, $AP : PB = 1 : 2$ and $AQ : QB = 2 : 1$

(i) Let P divides AB in the ratio 1:2, then $\dfrac{m_1}{m_2} = \dfrac{1}{2}$.

Here, $A(x_1, y_1) \equiv (2, -3)$ and $B(x_2, y_2) \equiv (4, -1)$



Now, $\quad P(x, y) \equiv P\left(\dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}, \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$

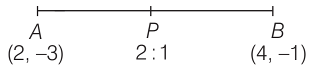
[by section formula]

$$= P\left[\frac{1 \times 4 + 2 \times 2}{1 + 2}, \frac{1 \times (-1) + 2 \times (-3)}{1 + 2}\right]$$

$$= P\left(\frac{4 + 4}{3}, \frac{-1 - 6}{3}\right) = P\left(\frac{8}{3}, \frac{-7}{3}\right)$$

(ii) Let Q divide AB in the ratio 2:1, then $\dfrac{m_1}{m_2} = \dfrac{2}{1}$.

Here, $A(x_1, y_1) \equiv (2, -3)$ and $B(x_2, y_2) \equiv (4, -1)$



Now, $\quad Q(x, y) \equiv Q\left(\dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}, \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$

$$= Q\left[\frac{2 \times 4 + 1 \times 2}{2 + 1}, \frac{2 \times (-1) + 1 \times (-3)}{2 + 1}\right]$$

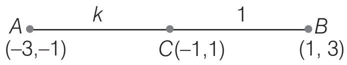
$$= Q\left(\frac{8 + 2}{3}, \frac{-2 - 3}{3}\right) = Q\left(\frac{10}{3}, \frac{-5}{3}\right)$$

To Show Collinearity using Section Formula

To show collinearity of points, say A, B and C by section formula, first we assume that C divides AB in the ratio of $k : 1$. Then, write the coordinates of C by section formula and then equate x and y-coordinates both sides separately. If value of k from both equations are same, then A, B and C are collinear, otherwise not.

Example 10. Using section formula, show that the points $A(-3, -1)$, $B(1, 3)$ and $C(-1, 1)$ are collinear.

Sol. Let $C(-1, 1)$ divides AB in the ratio $k : 1$.



Then, by section formula, we get

$$C(-1, 1) = C\left(\frac{k - 3}{k + 1}, \frac{3k - 1}{k + 1}\right)$$

On equating x-coordinates both sides, we get

$$-1 = \frac{k - 3}{k + 1} \Rightarrow -k - 1 = k - 3$$

$\Rightarrow \quad -2k = -3 + 1 \Rightarrow -2k = -2 \Rightarrow k = 1$

On equating y-coordinates both sides, we get

$$1 = \frac{3k - 1}{k + 1} \Rightarrow k + 1 = 3k - 1 \Rightarrow 2k = 2 \Rightarrow k = 1$$

Since, in both cases value of k is same. So, C divides AB in the ratio 1 : 1.

Hence, A, B and C are collinear.

Mid-point Formula

If a point divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio 1:1, then we get mid–point of this line segment which bisect the line joining two points.

Thus, the coordinates of the mid-point P of the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Example 11. Find the mid-point of the line segment joining the points (3, – 1) and (7, 9).

Sol. Let $C(x, y)$ be the mid-point of the line joining the points $A(3, -1)$ and $B(7, 9)$.

$\therefore$ Coordinates of mid-point of AB, i.e.

$$C(x, y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Here, $x_1 = 3, x_2 = 7$ and $y_1 = -1, y_2 = 9$.

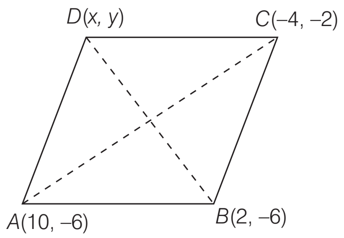
$$\therefore \quad C(x, y) = \left(\frac{3+7}{2}, \frac{-1+9}{2}\right) = \left(\frac{10}{2}, \frac{8}{2}\right) = (5, 4)$$

Hence, the mid-points of line segment joining (3, – 1) and (7, 9) is (5, 4).

Example 12. Find the fourth vertex of a parallelogram $ABCD$, if the three consecutive vertices are $A(10, -6)$, $B(2, -6)$ and $C(-4, -2)$.

Sol. Given, in parallelogram $ABCD$, the three consecutive vertices are $A(10, -6)$, $B(2, -6)$ and $C(-4, -2)$.

Let the coordinates of fourth vertex be $D(x, y)$.



We know that the diagonals of a parallelogram bisect each other.

$\therefore$ Coordinates of mid-point of diagonal AC

= coordinates of mid-point of diagonal BD

$$\Rightarrow \left(\frac{10+(-4)}{2}, \frac{-6+(-2)}{2}\right) = \left(\frac{2+x}{2}, \frac{-6+y}{2}\right)$$

$$\left[\because \text{mid-point } = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\right]$$

$$\Rightarrow \left(\frac{6}{2}, -\frac{8}{2}\right) = \left(\frac{2+x}{2}, \frac{-6+y}{2}\right)$$

$$\Rightarrow (3, -4) = \left(\frac{2+x}{2}, \frac{-6+y}{2}\right)$$

On equating x and y-coordinates both sides, we get

$$\Rightarrow \frac{2+x}{2} = 3 \text{ and } \frac{-6+y}{2} = -4$$

$$\Rightarrow 2+x = 6 \text{ and } -6+y = -8$$

$$\Rightarrow x = 4 \text{ and } y = -2$$

Hence, the fourth vertex of the parallelogram $ABCD$ is $D(4, -2)$.

Example 13. If the points $A(6, 1)$, $B(8, 2)$, $C(9, 4)$ and $D(p, 3)$ are the vertices of a parallelogram, taken in order, then find the value of p.

Sol. Given, vertices of a parallelogram are $A(6, 1)$, $B(8, 2)$, $C(9, 4)$ and $D(p, 3)$.

We know that diagonals of a parallelogram bisect each other.

$\therefore$ Coordinates of mid-point of diagonal AC

= Coordinates of mid-point of diagonal BD

$$\Rightarrow \left(\frac{6+9}{2}, \frac{1+4}{2}\right) = \left(\frac{8+p}{2}, \frac{2+3}{2}\right)$$

$$\left[\because \text{mid-point } = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\right]$$

$$\Rightarrow \left(\frac{15}{2}, \frac{5}{2}\right) = \left(\frac{8+p}{2}, \frac{5}{2}\right)$$

On equating x-coordinates both sides, we get

$$\frac{15}{2} = \frac{8+p}{2} \Rightarrow 15 = 8+p \Rightarrow p = 15-8 \Rightarrow p = 7$$

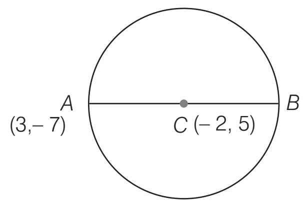
Hence, the required value of p is 7.

Example 14. AB is a diameter of a circle with centre $C(-2, 5)$ and $A(3, -7)$. Find

(i) the length of radius AC.

(ii) the coordinates of B. ***[2013]***

Sol. (i) Let $A(3, -7) = (x_1, y_1)$ and $C(-2, 5) = (x_2, y_2)$



$$\text{Length of radius, } AC = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$
$$= \sqrt{(-2-3)^2 + (5+7)^2}$$
$$= \sqrt{(-5)^2 + (12)^2}$$
$$= \sqrt{25+144} = \sqrt{169} = 13 \text{ units}$$

(ii) Let the coordinates of B be (x_3, y_3).

Since, C is the centre of circle. Then, C is the mid-point of diameter AB.

$$\Rightarrow \text{Coordinates of } C = \left(\frac{3+x_3}{2}, \frac{-7+y_3}{2}\right)$$

$$\left[\because \text{mid-point} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\right]$$

But coordinates of C is given, i.e. $(-2, 5)$.

$$\therefore \quad (-2, 5) = \left(\frac{3+x_3}{2}, \frac{-7+y_3}{2}\right)$$

On equating x and y-coordinates both sides, we get

$$-2 = \frac{3+x_3}{2} \text{ and } 5 = \frac{-7+y_3}{2}$$

$\Rightarrow \quad -2\times 2 = 3 + x_3$ and $5\times 2 = -7 + y_3$

$\Rightarrow \quad x_3 = -4 - 3$ and $y_3 = 10 + 7$

$\Rightarrow \quad x_3 = -7$ and $y_3 = 17$

Hence, the coordinates of B are $(-7, 17)$.

Example 15. If $A(10,5)$, $B(6,-3)$ and $C(2,1)$ are the vertices of a ΔABC. L is the mid-point of AB and M is the mid-point of AC. Write down the coordinates of L and M. Show that $LM = \frac{1}{2}BC$. ***[2001]***

Sol. Given vertices of a triangle are $A(10,5)$, $B(6,-3)$ and $C(2,1)$.

$\therefore$ Coordinates of L = Mid-point of $AB = \left(\frac{10+6}{2}, \frac{5-3}{2}\right)$

$$= \left(\frac{16}{2}, \frac{2}{2}\right) = (8,1)$$

$$\left[\because \text{coordinates of mid-point} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\right]$$

Coordinates of M = Mid-point of AC

$$= \left(\frac{10+2}{2}, \frac{5+1}{2}\right) = \left(\frac{12}{2}, \frac{6}{2}\right) = (6,3)$$

Now, length of $LM = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$$= \sqrt{(6-8)^2 + (3-1)^2}$$

$$[\because (x_1, y_1) = (8,1) \text{ and } (x_2, y_2) = (6,3)]$$

$$= \sqrt{(-2)^2 + (2)^2} = \sqrt{4+4}$$

$$= \sqrt{8} = 2\sqrt{2} \text{ units} \quad \ldots\text{(i)}$$

and length of $BC = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$$= \sqrt{(2-6)^2 + (1+3)^2}$$

$$[\because (x_1, y_1) = (6,-3) \text{ and } (x_2, y_2) = (2,1)]$$

$$= \sqrt{(-4)^2 + (4)^2} = \sqrt{16+16} = 4\sqrt{2} \text{ units}$$

Now, $\quad BC = 4\sqrt{2}$

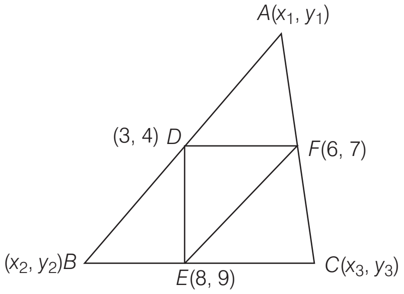
$\Rightarrow \quad \frac{1}{2}BC = 2\sqrt{2}$ [dividing both sides by 2]

$\Rightarrow \quad \frac{1}{2}BC = LM$ [from Eq. (i)]

Hence proved.

Example 16. The mid-points D, E and F of the sides AB, BC and CA of a triangle are (3, 4), (8, 9) and (6, 7), respectively. Find the coordinates of the vertices of the triangle.

Sol. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the ΔABC such that $D(3, 4)$, $E(8, 9)$ and $F(6, 7)$ are mid-points of the sides AB, BC and CA, respectively.



Since, $D(3, 4)$ is the mid-point of AB.

$\therefore \quad (3, 4) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

On equating x and y-coordinates both sides, we get

$$\frac{x_1+x_2}{2} = 3 \text{ and } \frac{y_1+y_2}{2} = 4$$

$\Rightarrow \quad x_1 + x_2 = 6 \quad \ldots$(i)

and $\quad y_1 + y_2 = 8 \quad \ldots$(ii)

As $E(8, 9)$ is the mid-point of BC.

$\therefore \quad \frac{x_2+x_3}{2} = 8$ and $\frac{y_2+y_3}{2} = 9$

$\Rightarrow \quad x_2 + x_3 = 16 \quad \ldots$(iii)

and $\quad y_2 + y_3 = 18 \quad \ldots$(iv)

Also, $F(6, 7)$ is the mid-point of AC.

$\therefore \quad \frac{x_3+x_1}{2} = 6$ and $\frac{y_3+y_1}{2} = 7$

$\Rightarrow \quad x_3 + x_1 = 12 \quad \ldots$(v)

and $\quad y_3 + y_1 = 14 \quad \ldots$(vi)

On adding Eqs. (i), (iii) and (v), we get

$$2(x_1 + x_2 + x_3) = 34 \Rightarrow x_1 + x_2 + x_3 = 17 \quad \ldots\text{(vii)}$$

On subtracting one-by-one Eqs. (i), (iii) and (v) from Eq. (vii), we get

$$x_3 = 11, x_1 = 1 \text{ and } x_2 = 5$$

On adding Eqs. (ii), (iv) and (vi), we get

$$2(y_1 + y_2 + y_3) = 40$$

$\Rightarrow \quad y_1 + y_2 + y_3 = 20 \quad \ldots$(viii)

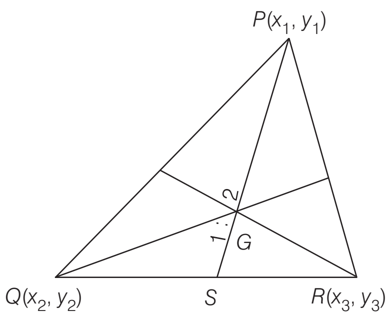
On subtracting one-by-one Eqs. (ii), (iv) and (vi) from Eq. (viii), we get

$$y_3 = 12, y_1 = 2 \text{ and } y_2 = 6$$

Hence, the vertices of the triangle are $A(1, 2)$, $B(5, 6)$ and $C(11, 12)$.

Centroid of a Triangle

The point where all the medians of a triangle meet is called the **centroid of the triangle**.



Let $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ be the given vertices of a ΔPQR and let G be the centroid of ΔPQR, then coordinates of G are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

Example 17. Find the coordinates of the centroid of a triangle with vertices (3, 5), (2, – 1) and (6, 12).

Sol. Let the given coordinates be $A(3, 5)$, $B(2, -1)$ and $C(6, 12)$.

Now, we know that coordinates of centroid of a triangle are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

Here, $x_1 = 3$, $x_2 = 2$, $x_3 = 6$, $y_1 = 5$, $y_2 = -1$ and $y_3 = 12$

$$\text{Coordinates of centroid} = \left[\frac{3+2+6}{3}, \frac{5+(-1)+12}{3}\right] = \left(\frac{11}{3}, \frac{16}{3}\right)$$

Example 18. The centroid of ΔABC is the origin. If the coordinates of A and B are respectively, (1,3) and (2, – 4), then find the coordinates of C.

Sol. Let the centroid and the third vertex of the triangle be $G(0,0)$ and $C(x_3, y_3)$, respectively.

Also, let $A(x_1, y_1) = (1,3)$ and $B(x_2, y_2) = (2, -4)$

$\because$ Coordinates of the centroid of a triangle

$$= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

$$\therefore \quad (0,0) = \left(\frac{1+2+x_3}{3}, \frac{3-4+y_3}{3}\right)$$

On equating x and y-coordinates both sides, we get

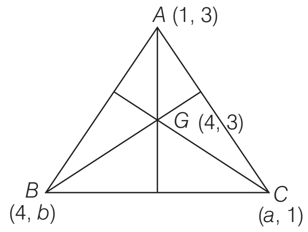
$$0 = \frac{3+x_3}{3} \text{ and } 0 = \frac{-1+y_3}{3}$$

$\Rightarrow \quad x_3 = -3$ and $y_3 = 1$

Hence, the coordinates of third vertex be $C(-3, 1)$.

Example 19. ABC is a triangle and $G(4, 3)$ is the centroid of the triangle. If $A(1, 3)$, $B(4, b)$ and $C(a, 1)$, then find a and b. Find the length of side BC. ***[2011]***

Sol. Given vertices and centroid of a triangle are $A(1, 3)$, $B(4, b)$, $C(a, 1)$ and $G(4, 3)$.



We know that,

Coordinates of centroid of the triangle,

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Here, $(x_1, y_1) = (1, 3)$, $(x_2, y_2) = (4, b)$ and $(x_3, y_3) = (a, 1)$

$\therefore$ Centroid of the triangle,

$$G = \left(\frac{1+4+a}{3}, \frac{3+b+1}{3}\right) = \left(\frac{5+a}{3}, \frac{4+b}{3}\right)$$

But centroid is given by $G(4, 3)$.

$$\therefore \quad (4, 3) = \left(\frac{5+a}{3}, \frac{4+b}{3}\right)$$

On equating x and y-coordinates both sides, we get

$$4 = \frac{5+a}{3} \text{ and } 3 = \frac{4+b}{3}$$

$\Rightarrow \quad 4 \times 3 = 5 + a$ and $3 \times 3 = 4 + b$

$\Rightarrow \quad 12 = 5 + a$ and $9 = 4 + b$

$\Rightarrow \quad a = 12 - 5$ and $b = 9 - 4$

$\Rightarrow \quad a = 7$ and $b = 5$

$\therefore$ Coordinates of B and C are (4, 5) and (7, 1), respectively.

Now, the length of side BC

$$= \sqrt{(7-4)^2 + (1-5)^2}$$

$$\left[\because \text{ distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right]$$

$$= \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \text{ units}$$

CHAPTER EXERCISE

3 Marks Questions

1. Find the coordinates of the point which divides the line segment joining of $(-1, 7)$ and $(4, -3)$ in the ratio $2 : 3$.

2. Find the ratio, in which the line segment joining the points $(-3, 10)$ and $(6, -8)$ is divided by $(-1, 6)$.

3. In what ratio, does the point $(5, 4)$ divide the line segment joining the points $(2, 1)$ and $(7, 6)$?

4. In what ratio does the point $(-2/5, 6)$ divide the join of $(-4, 3)$ and $(2, 8)$.

5. P divides the distance between $A(-2, 1)$ and $B(1, 4)$ in the ratio $2 : 1$. Calculate the coordinates of the point P.

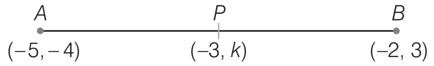
6. If the line joining the points $A(4, -5)$ and $B(4, 5)$ is divided by the point C such that $\frac{AC}{AB} = \frac{3}{5}$, then find the coordinates of C.

7. Find the ratio, in which Y-axis divides the line joining $(1, -2)$ and $(-3, -4)$.

8. What will be the value of y, if the point $\left(\frac{23}{5}, y\right)$ divides the line segment joining the points $(5, 7)$ and $(4, 5)$ in the ratio $2 : 3$ internally?

9. Find the ratio in which Y-axis divides the line segment joining the points $A(5, -6)$ and $B(-1, -4)$. Also, find the coordinates of the point of division.

10. Find the ratio in which the point $(-3, k)$ divides the line segment joining the points $(-5, -4)$ and $(-2, 3)$. Also, find the value of k.



11. A point P divides the line joining $A(-4, 1)$ and $B(17, 10)$ in the ratio $1 : 2$.

(i) Calculate the distance OP, where O is the origin.

(ii) In what ratio, does the Y-axis divide the line AB?

12. Find the coordinates of the points of trisection of the line segment joining $(4, -1)$ and $(-2, -3)$.

13. Let P and Q be the points of trisection of the line segment joining the points $A(2, -2)$ and $B(-7, 4)$ such that P is nearer to A. Find the coordinates of P and Q.

14. Find the points of trisection of the line segment joining the points $(5, -6)$ and $(-7, 5)$.

15. Find the coordinates of the points of trisection of the line segment joining the points $(5, -3)$ and $(2, -9)$.

16. Find the value of p, if $(p, -2)$, $(-5, 6)$ and $(1, 2)$ are collinear.

17. The line segment joining $A(-3, 1)$ and $B(5, -4)$ is a diameter of a circle, whose centre is C. Find the coordinates of the point C.

18. Find the third vertex of a triangle, if its two vertices are $(-1, 4)$ and $(5, 2)$, and mid-point of one side is $(0, 3)$.

19. Show that $A(-1, 0)$, $B(3, 1)$, $C(2, 2)$ and $D(-2, 1)$ are the vertices of a parallelogram $ABCD$.

20. If the vertices of ΔABC are $A(5, -1)$, $B(-3, -2)$, $C(-1, 8)$, find the length of median through A.

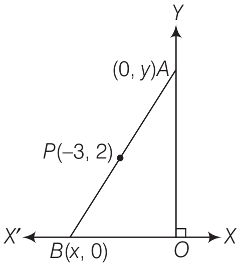
21. Points $(-5, 2)$, $(3, -6)$ and $(7, 4)$ are the vertices of a triangle. Find the length of its median from $(3, -6)$.

22. The mid-point of the line joining $(3a, 4)$ and $(-2, 2b)$ is $(2, 2a + 2)$. Find the values of a and b.

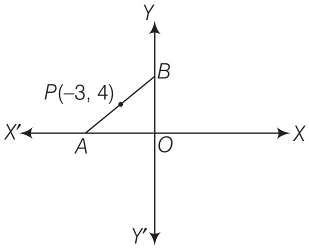
23. The mid-point of the line segment joining $(2a, 4)$ and $(-2, 2b)$ is $(1, 2a + 1)$. Find the values of a and b. **[2007]**

24. If two adjacent vertices of a parallelogram are $(3, 2)$ and $(-1, 0)$, and the diagonals intersect at $(2, -5)$, then find the other two vertices.

25. $P(-3, 2)$ is the mid-point of the line AB as shown in the given figure. Find the coordinates of points A and B.



26. In the figure given below, the line segment AB meets X-axis at A and Y-axis at B. The point $P(-3, 4)$ on AB divides it in the ratio $2 : 3$. Find the coordinates of A and B. **[2013]**



27. Find the centroid of a triangle, whose vertices are $(3, -5)$, $(-7, 4)$ and $(10, -2)$.

28. Two vertices of a triangle are $(-1, 4)$ and $(5, 2)$. If the centroid is $(0, -3)$, then find the third vertex.

29. The centroid of a triangle is the point $(6, -1)$. If two vertices are $(3, 4)$ and $(-2, 5)$, then find third vertex.

30. Two vertices of a triangle are $(3, -5)$ and $(-7, 4)$. If its centroid is $(2, -1)$, then find the third vertex.

b 4 Marks Questions

31. (i) Write down the coordinates of the point P, that divides the line joining $A(2, 1)$ and $B(-7, 10)$ in the ratio 2 : 1.
(ii) Calculate the distance OP, where O is the origin.
(iii) In what ratio, does the Y-axis divide the line AB?

32. The line segment joining $A(-1, 3)$ and $B(6, 2)$ is intersected by the Y-axis at point K. Write down the coordinates of K. Also, find the ratio in which K divides.

33. Find the ratio in which the line segment joining the points $A(3, -3)$ and $B(-2, 7)$ is divided by X-axis. Also, find the coordinates of the point of division.

34. If $R(x, y)$ is a point on the line segment joining the points $P(a, b)$ and $Q(b, a)$, then prove that $x + y = a + b$.

35. Find the ratio in which the line joining points $(a + b, b + a)$ and $(a - b, b - a)$ is divided by the point (a, b).

36. If $P(9a - 2, -b)$ divides the line segment joining $A(3a + 1, -3)$ and $B(8a, 5)$ in the ratio 3 : 1. Find the values of a and b.

37. If $A(-3, 4)$, $B(3, -1)$ and $C(-2, 4)$ are the vertices of ΔABC. Find the length of line segment AP, where P lies inside BC such that $BP : PC = 2 : 3$.

38. The line segment joining $A\left(-1, \frac{5}{3}\right)$ and $B(a, 5)$ is divided in the ratio 1 : 3 at P, the point where the line segment AB intersects Y-axis. Calculate
(i) the value of a. (ii) the coordinates of P.

39. The line segment joining $A(-2, 3)$ and $B(m, 4)$ is divided in the ratio 1 : 4 at P, the point where the line segment AB intersects Y-axis. Calculate
(i) the value of m.
(ii) the coordinates of P.

40. If $A(-4, 3)$ and $B(8, -6)$.
(i) Find the length of AB.
(ii) In what ratio, is the line joining AB divided by the X-axis? *[2008]*

41. The line joining $P(-4, 5)$ and $Q(3, 2)$ intersect the Y-axis at R. PM and QN are perpendicular from P and Q on the X-axis. Find
(i) the ratio $PR : PQ$.
(ii) the coordinates of R.
(iii) the area of the quadrilateral $PMNQ$.

42. The centre O of a circle has the coordinates $(4, 5)$ and one point on the circumference is $(8, 10)$. Find the coordinates of the other end of the diameter of the circle through this point.

43. If the coordinates of the mid-points of the sides of a triangle are $(3, -2)$, $(-3, 1)$ and $(4 - 3)$, then find the coordinates of its vertices.

44. If two vertices of a parallelogram are $(2, 3)$ and $(0, -1)$ and the diagonals cut at $(-5, 2)$, then find the other vertices of the parallelogram.

45. If a vertex of a triangle be $(1, 1)$ and the mid-points of the sides through it be $(-2, 3)$ and $(5, 2)$. Find the other vertices.

46. The coordinates of the vertices of ΔABC are $A(7, 2)$, $B(9, 10)$ and $C(1, 4)$. If E and F are the mid-points of AB and AC respectively, prove that $EF = \frac{1}{2}BC$.

47. $A(2, 5)$, $B(-1, 2)$ and $C(5, 8)$ are the vertices of a ΔABC. P and Q are points on AB and AC respectively, such that $AP : PB = AQ : QC = 1 : 2$.
(i) Find the coordinates of P and Q.
(ii) Show that $PQ = \frac{1}{3}BC$.

48. Find coordinates of vertices of the triangle, the middle points of whose sides are $\left(0, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 0\right)$.

49. Prove that the diagonals of a rectangle of $ABCD$, with vertices $A(2, -1)$, $B(5, -1)$, $C(5, 6)$ and $D(2, 6)$ are equal and bisect each other.

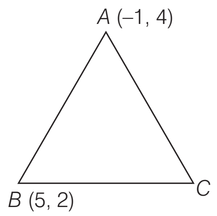
50. The vertices of ΔABC are $A(6, -2)$, $B(0, -6)$ and $C(4, 8)$. Find the coordinates of mid-points of AB, BC and AC.

51. If (a, b) is the mid-point of the segment joining the points $A(10, -6)$, $B(22, 4)$ and $a - 2b = 18$, find (a, b) and the distance AB.

52. Let $A(4, 2)$, $B(6, 5)$ and $C(1, 4)$ be the vertices of ΔABC.
(i) If the median from A meets BC at D, then find the coordinates of the point D.
(ii) Find the coordinates of the point P on AD such that $AP : PD = 2:1$.
(iii) Find the coordinates of points Q and R on medians BE and CF respectively, such that $BQ : QE = 2:1$ and $CR : RF = 2:1$.
(iv) What do you observe?

Hints and Answers

1. Do same as Example 1. **Ans.** (1, 3)
2. Do same as Example 2. **Ans.** (2 : 3)
3. Do same as Example 2. **Ans.** 3 : 2
4. Do same as Example 2. **Ans.** 3 : 2
5. Do same as Example 3. **Ans.** (0, 3)
6. Do same as Example 4. **Ans.** (4, 1)
7. **Hint** Let the coordinates of a point on Y-axis be $(0, y)$.
Ans. 1 : 3.
8. **Hint** By section formula, y-coordinate of internally division, $y = \dfrac{2 \times 5 + 3 \times 7}{2+3}$. **Ans.** $\dfrac{31}{5}$
9. **Hint** In section formula, put the x-coordinate equals to zero. **Ans.** $5:1, \left(0, -\dfrac{13}{2}\right)$
10. **Hint** $(-3, k) = \left(\dfrac{-2m_1 - 5m_2}{m_1 + m_2}, \dfrac{3m_1 - 4m_2}{m_1 + m_2}\right)$
Ans. $2 : 1, k = \dfrac{2}{3}$
11. **Hint** First, use section formula
i.e. $\left(\dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}, \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$
(i) Using distance formula,
$OP = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **Ans.** 5 units
(ii) In section formula, put the x-coordinate equals to zero.
Ans. 4 : 17
12. Do same as Example 9. **Ans.** $\left[\left(2, -\dfrac{5}{3}\right) \text{and} \left(0, -\dfrac{7}{3}\right)\right]$
13. Do same as Example 9. **Ans.** (– 1, 0), (–4, 2)
14. Do same as Example 9. **Ans.** $\left(1, \dfrac{-7}{3}\right), \left(-3, \dfrac{4}{3}\right)$
15. Do same as Example 9. **Ans.** $(4, -5), (3, 7)$
16. **Hint** Let the given points be $A(p, -2)$, $B(-5, 6)$ and $C(1, 2)$. Since, the given points are collinear, therefore Coordinates of mid-point AB = Coordinates of C
Ans. $p = 7$
17. **Hint** Use the mid-point formula. **Ans.** $\left(1, \dfrac{-3}{2}\right)$
18. **Hint** There are following two cases arise
Case (i) Suppose mid-point (0, 3) between A and C.

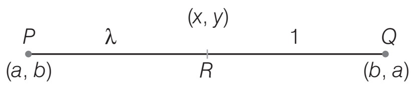


Case (ii) Suppose mid-point (0, 3) between B and C.
Ans. (–5, 4) or (1, 2)
19. **Hint** Show that $AB = CD$, $BC = DA$ and mid-point of AC and BD coincide.
20. **Hint** First, find coordinates of D, which is the mid-point of BC. Then, use the distance formula,
$AD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. **Ans.** $\sqrt{65}$ unit
21. Do same as Q. 20. **Ans.** $1 - \sqrt{85}$.
22. **Hint** $(2, 2a + 2) = \left(\dfrac{3a-2}{2}, \dfrac{4+2b}{2}\right)$ **Ans.** $a = 2, b = 4$
23. Do same as Q. 22. **Ans.** $a = 2, b = 3$
24. **Hint** Diagonals of a parallelogram bisects each other.
Ans. $(1, -12)$ and $(5, -10)$
25. **Hint** Since, coordinate of P = coordinate of mid-point of line segment AB $(-3, 2) = \left(\dfrac{x}{2}, \dfrac{y}{2}\right)$
Ans. (0, 4), (– 6, 0)
26. Do same as Q. 25. **Ans.** $A = (-5, 0)$, $B = (0, 10)$
27. Do same as Example 17. **Ans.** $(2, -1)$
28. Do same as Example 18. **Ans.** $(-4, -15)$
29. Do same as Example 18. **Ans.** $(17, -12)$
30. Do same as Example 18. **Ans.** $(10, -2)$
31. Do same as Q. 11.
Ans. (i) $(-4, 7)$ (ii) $\sqrt{65}$ unit (iii) 2 : 7
32. **Hint** Let the coordinates of point K be $(0, y)$ and let the point K divides AB in the ratio $p:1$.
Ans. $\left(0, \dfrac{20}{7}\right)$, 1 : 6

33. Do same as Example 5.

Ans. $3:7, \left(\frac{3}{2}, 0\right)$

34. **Hint** Let point R divides the line joining P and Q in the ratio $\lambda : 1$.



$$x = \frac{\lambda b + a}{\lambda + 1} \text{ and } y = \frac{\lambda a + b}{\lambda + 1}$$

On adding, we get $x + y = \dfrac{\lambda b + a + \lambda a + b}{\lambda + 1}$

35. **Hint** Let $A(a+b, b+a)$, $B(a-b, b-a)$ and $P(a, b)$ and P divides AB in $k:1$, then

$$a = \frac{k(a-b) + 1(a+b)}{k+1}$$

Ans. $1:1$

36. **Hint**

$$(9a-2, -b) = \left[\frac{3 \times 8a + 1(3a+1)}{3+1}, \frac{3 \times 5 + 1 \times (-3)}{3+1}\right]$$

Equating the x-coordinate and y-coordinate and further simplify the values of a and b.

Ans. $a = 1, b = -3$

37. **Hint** First using section formula to find the coordinates of P. Further, using distance formula,

$$AP = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Ans. 5 units

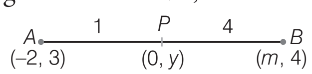
38. **Hint**

(i) First, put the x-coordinate of internally division is equal to zero. Then, determine the value of a.

(ii) Using section formula.

Ans. (i) $a = 3$ (ii) $\left(0, \frac{5}{2}\right)$

39. **Hint** Using section formula,



Ans. (i) 8 (ii) $\left(0, \frac{16}{5}\right)$

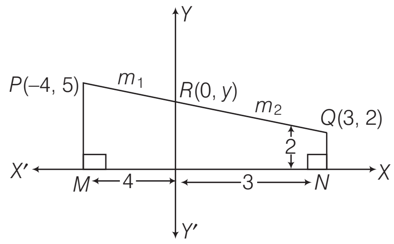
40. **Hint** (i) Using distance formula,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(ii) In section formula, put the y-coordinate equals to zero.

Ans. (i) 15 units (ii) 1 : 2

41. **Hint**



Ans. (i) 4 : 7 (ii) $\left(0, \frac{23}{7}\right)$ (iii) 24.5 sq units

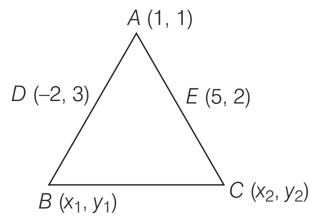
42. **Hint** Use mid-point formula. **Ans.** (0, 0)

43. Do same as Example 16. **Ans.** $(10, -6), (-4, 2), (-2, 0)$

44. **Hint** Diagonals of a parallelogram bisect each other.

Ans. $(-12, 1), (-10, 5)$

45. **Hint** Using mid-point formula in AB and AC.



Ans. $(-5, 5)$ and $(9, 3)$

46. Do same as Example 15.

47. **Hint**

(i) To find the coordinates of P and Q using section formula.

(ii) Use distance formula.

Ans. (i) $P = (1, -1), Q = (3, 6)$

48. Do same as Example 16. **Ans.** (0, 0), (1, 0) and (0, 1)

49. **Hint** Use distance and mid-point formulae to prove that the diagonals of a rectangle are equal and bisect each other, respectively.

50. **Hint** Using mid-point formula in AB, BC, CA.

Ans. $(3, -4), (2, 1), (5, 3)$

51. **Hint** First use mid-point formula and then distance formula for AB. **Ans.** $(16, -1), 2\sqrt{61}$ units

52. **Hint**

(i) Use mid-point formula.

(ii) Use internal section formula.

(iii) BE is the median of ΔABC. So, E is the mid-point of AC and its coordinate are $E\left(\frac{5}{2}, 3\right)$.

Similarly for F.

Ans. (i) $\left(\frac{7}{2}, \frac{9}{2}\right)$ (ii) $\left(\frac{11}{3}, \frac{11}{3}\right)$ (iii) $\left(\frac{11}{3}, \frac{11}{3}\right)$

(iv) We observe that the points P, Q and R coincide at the point $\left(\frac{11}{3}, \frac{11}{3}\right)$.

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2017

1 $P(1, -2)$ is a point on the line segment joining $A(3, -6)$ and $B(x, y)$ such that $AP : PB$ is equal to $2 : 3$. Find the coordinates of B.

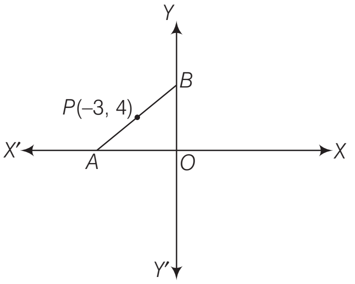
2014

2 Calculate the ratio, in which the line joining $A(-4, 2)$ and $B(3, 6)$ is divided by $P(x, 3)$. Also, find

(i) x. (ii) length of AP.

2013

3 In the figure given below, the line segment AB meets X-axis at A and Y-axis at B. The point $P(-3, 4)$ on AB divides it in the ratio $2 : 3$. Find the coordinates of A and B.



4 AB is a diameter of a circle with centre $C(-2, 5)$ and $A(3, -7)$. Find

(i) the length of radius AC. (ii) the coordinates of B.

2012

5 Given a line segment AB joining the points $A(-4, 6)$ and $B(8, -3)$. Find

(i) the ratio in which AB is divided by the Y-axis.

(ii) the coordinates of the point of intersection.

(iii) the length of AB.

2011

6 ABC is a triangle and $G(4, 3)$ is the centroid of the triangle. If $A(1, 3)$, $B(4, b)$ and $C(a, 1)$, then find a and b. Find the length of side BC.

* All these questions are completely covered in chapter either as solved examples or in chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

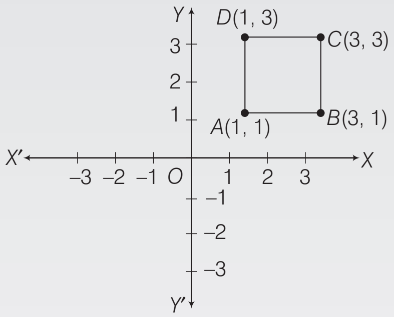
1. Join two points $P(2, 2)$ and $Q(4, 2)$ in a plane. Fixed the point P and rotate the line PQ in anti-clockwise direction at an angle of 270°. Then, the new coordinates of point Q and the area formed by this figure will be

(a) (4, 2); 4.92 sq units (b) (2, 4); 9.42 sq units
(c) (4, 2); 9.42 sq units (d) (2, 4); 2.94 sq units

2. If G be the centroid of a ΔABC and P be any other point in the plane, then $GA^2 + GB^2 + GC^2 + 3GP^2$ is equal to

(a) $PA^2 + PC^2 - PB^2$ (b) $PB^2 + PC^2 - PA^2$
(c) $PA^2 + PB^2 - PC^2$ (d) $PA^2 + PB^2 + PC^2$

3. A figure is shown below



If we rotate this graph about O at an angle of 180° in anti-clockwise direction, then the intersection point of diagonals will be

(a) (3, 2) (b) (2, 2) (c) (2, 3) (d) (2, 1)

4. If $(-4, 3)$ and $(4, 3)$ are two vertices of an equilateral triangle and the origin lies in the interior of the triangle, then the coordinates of the third vertex will be

(a) (0, 1) (b) (1, 0)
(c) $(0, 3 - 4\sqrt{3})$ (d) $(1, 3 - 4\sqrt{3})$

5. Suppose there are four points $A(2, 4)$, $B(6, 4)$, $C(6, 6)$ and $D(2, 6)$, which lie in the first quadrant.
If we rotate only the axes at an angle of 90° in anti-clockwise direction, then what will be the new coordinates of the point C and what will be the name of the figure, when we join adjacent points.

(a) (–6, 6); square
(b) (6, – 6); rectangle
(c) (6, 4); square
(d) (2, – 6); rectangle

6. Suppose PQ be a pole, whose coordinates are $P(1, 3)$ and $Q(3, 3)$ and A be the position of a man whose coordinates are (1, 1).

(i) If a pole makes an angle of elevation to the point A, then the angle θ is
(ii) Also, if we shift the origin at (1, 1), then the angle θ is

(a) 45°, 45° (b) 45°, 60°
(c) 45°, 90 (d) 75, 45°

7. The sum of the squares of the distances of a moving point (x, y) from two fixed points $(a, 0)$ and $(-a, 0)$ is equal to a constant quantity $2b^2$. The value of $x^2 + y^2 + a^2$ is equal to

(a) ab (b) $-b^2$
(c) $-a^2$ (d) b^2

8. A point (x, y) moves, so that the sum of its distances from $(ae, 0)$ and $(-ae, 0)$ is $2a$. If $b^2 = a^2(1 - e^2)$, then $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ is equal to

(a) 0 (b) –1
(c) 1 (d) 2

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 401.

CHAPTER

12

Equation of a Straight Line

In earlier classes, we studied about coordinate axes, coordinate plane, plotting of points in a plane, distance between two points, etc., all these are the basics concepts of coordinate geometry. In this chapter we will study about straight line and its slope (gradient) along with equation of a straight line in different forms and the conditions for two lines to be parallel or perpendicular.

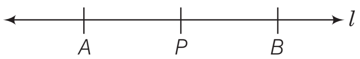
Chapter Objectives

- Straight Line and Its Equation in Different Forms
- Conditions for Two Lines to be Parallel and Perpendicular

Topic 1
Straight Line and Its Equation in Different Forms

Straight Line

A **straight line** is a curve, such that all the points on the line segment joining any two points of it lies on it.



Every first degree equation in x and y represents a straight line, i.e. equation of the form $ax + by + c = 0$ represent a straight line.

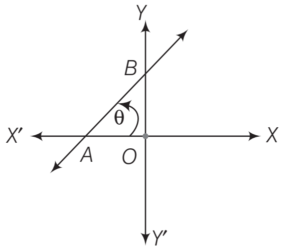
e.g. $x + y = 0$, $3x - y + 2 = 0$ are straight lines.

Inclination (angle of inclination) of a Line

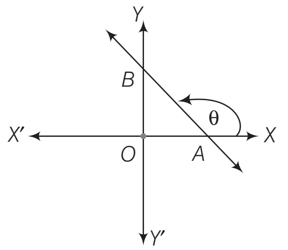
The angle made by the line with positive X–axis in anti–clockwise direction, is called inclination (angle of inclination) of a line. The inclination is usually denoted by θ.

Thus, $0° \leq \theta < 180°$.

An inclination of a line is either acute or obtuse, as shown in the following figures.



(i) Acute inclination



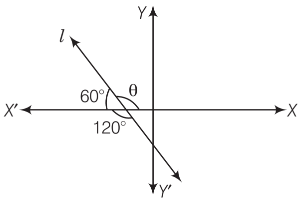
(ii) Obtuse inclination

Inclination of Axes

(i) The inclination of X–axis and lines parallel to it, is 0°.

(ii) The inclination of Y–axis and lines parallel to it, is 90°.

Example 1. Find the inclination of a line l in the following figure.



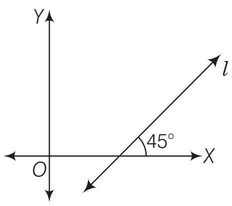
Sol. From the figure, we see that $\theta = 120°$ [vertically opposite angles]

Hence, inclination of a line l is 120°.

Slope (or Gradient) of a Line

If θ is the angle of inclination of a line l, then $\tan\theta$ is called the slope or gradient of the line l and it is denoted by m, i.e. $m = \tan\theta$.

Thus, slope of a line is the tangent of the angle made by the line in the anti–clockwise direction with the positive X–axis.



e.g. From the figure, slope of line $l = m = \tan 45° = 1$

(i) The slope of X-axis is $m = \tan 0° = 0$

(ii) The slope of Y-axis is not defined (as $\tan 90°$ is not defined).

(iii) The slope of a line is positive, if it makes an acute angle in the anti-clockwise direction with X-axis.

(iv) The slope of a line is negative, if it makes an obtuse angle in the anti-clockwise direction with the X-axis.

Example 2. Find the slope of a line, whose inclination is 60°.

Sol. Let θ be the inclination of a line, then its slope $= \tan\theta$.

At $\theta = 60°$, slope of a line $= \tan 60° = \sqrt{3}$

Slope of a Line Passing through Two Points

The slope of a line passing through two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y_1 - y_2}{x_1 - x_2}.$$

Example 3. Find the slope of a line joining following two points.

(i) $A(1, 2)$ and $B(3, -4)$ (ii) $(3, -2)$ and $(7, -2)$.

Sol. We know that slope of a line joining two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

(i) Given, $A(1, 2) \equiv (x_1, y_1)$ and $B(3, -4) \equiv (x_2, y_2)$

$\therefore$ Slope of line joining the points A and B is

$$m = \frac{-4-2}{3-1} = \frac{-6}{2} = -3$$

(ii) Slope of line joining the points $(3, -2)$ and $(7, -2)$ is

$$m = \frac{-2-(-2)}{7-3} = \frac{0}{4} = 0$$

Example 4. The slope of a line joining $P(6, k)$ and $Q(1-3k, 3)$ is 1/2. Find

(i) k. (ii) mid-point of PQ, using the value of 'k' found in part (i).

Sol. Given points are $P(6, k)$ and $Q(1-3k, 3)$. ***[2016]***

(i) Slope of line joining these two points $= \dfrac{1}{2}$

$$\Rightarrow \quad \frac{3-k}{1-3k-6} = \frac{1}{2} \quad \left[\because \text{slope} = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

$$\Rightarrow \quad \frac{3-k}{-3k-5} = \frac{1}{2}$$

$$\Rightarrow \quad 6 - 2k = -3k - 5$$

$$\Rightarrow \quad 3k - 2k = -5 - 6$$

$$\therefore \quad k = -11$$

(ii) When $k = -11$, the coordinates of points P and Q are respectively $(6, -11)$ and $(34, 3)$.

$$\therefore \text{Mid-point of } PQ = \left(\frac{6+34}{2}, \frac{-11+3}{2}\right) \quad \left[\because \text{mid-point} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\right]$$

$$= \left(\frac{40}{2}, \frac{-8}{2}\right)$$

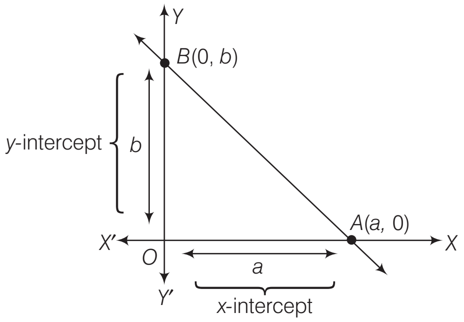
$$= (20, -4)$$

Hence, the mid-point of PQ is $(20, -4)$.

Intercept Made by a Line on the Axes

If a line intersect X–axis at a distance of a units from the origin and Y–axis at a distance of b units from the origin, then a is called the **x-intercept** and b is called the **y-intercept**.

Consider the figure, in which OA is called x–intercept and OB is called y–intercept.



Convention for the Signs of Intercepts

(i) We considered x–intercept as positive, if it is measured to the right of origin; and negative, if it is measured to the left of origin.

(ii) We considered y–intercept as positive, if it is measured above the origin; and negative, if it is measured below the origin.

Example 5. Find the intercepts made by a line passing through the following points.

(i) $A\,(2, 0)$ and $B(0, 3)$ (ii) $P(5, 0)$ and $Q(0, -3)$

Sol.

(i) Given points are $A(2, 0)$ and $B(0, 3)$.
In coordinates $A(2, 0)$, x-coordinate is $+\,2$ and y-coordinate is 0, so x-intercept is 2.
In coordinates $B(0, 3)$, x-coordinate is 0 and y-coordinate is $+\,3$, so y-intercept is 3.
Hence, the intercepts made by a line passing through A and B are 2 and 3, respectively.

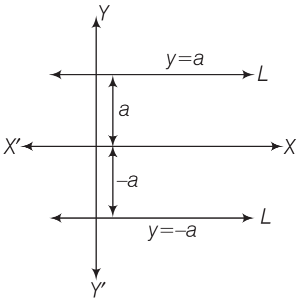
(ii) Given points are $P(5, 0)$ and $Q\,(0, -3)$.
In coordinates $P(5, 0)$, x-coordinate is $+\,5$ and y-coordinate is 0, so x-intercept is 5.
In coordinates $Q\,(0, -3)$, x-coordinate is 0 and y-coordinate is $-\,3$, so y-intercept is $-\,3$.
Hence, the intercepts made by a line passing through P and Q are 5 and $-\,3$, respectively.

Equation of a Straight Line in Different Forms

The different forms of the equation of a line under different conditions are given below

Equation of a Line Parallel to X-axis

Line parallel to X–axis is also called as **horizontal line**. If a line L is parallel to X–axis at a distance of a units from the X–axis, then ordinate of every point lying on the line is either a or $-\,a$.



$\therefore$ **Equation of a line L is either $y = a$ or $y = -\,a$.**

Example 6. Find the equation of a straight line parallel to X-axis and passing through the point (3,– 5).

Sol. Let the equation of a line parallel to X-axis is

$$y = a \qquad \text{...(i)}$$

Since, it is passing through a point $(3, -\,5)$.

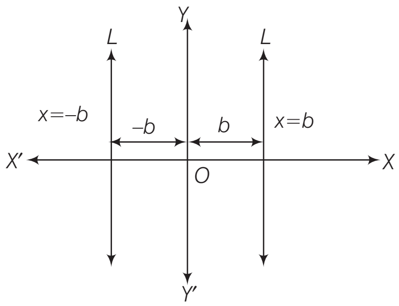
$\therefore \qquad -\,5 = a \Rightarrow a = -\,5$

On putting $a = -\,5$ in Eq. (i), we get $y = -\,5$

Hence, the required equation of a straight line is $y + 5 = 0$.

Equation of a Line Parallel to Y-axis

Line parallel to Y–axis is also called as **vertical line.** If a line L is parallel to Y–axis at a distance of b units from the Y–axis, then abscissa of every point lying on this line is either b or $-\,b$.



$\therefore$ **Equation of a line L is either $x = b$ or $x = -\,b$.**

Note (i) Equation of X-axis is y = 0. (ii) Equation of Y-axis is x = 0.

Example 7. Find the equation of a straight line parallel to Y-axis and passing through the point $(-\,4, 7)$.

Sol. Let the equation of a line parallel to Y-axis is

$$x = b \qquad \text{...(i)}$$

Since, it is passing through a point $(-\,4, 7)$.

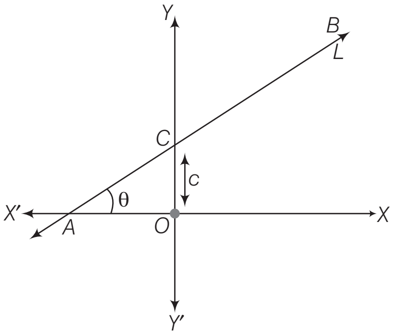
$\therefore \qquad -\,4 = b$

Now, putting $b = -\,4$ in Eq. (i), we get $x = -\,4$

Hence, the required equation of line parallel to Y-axis is $x + 4 = 0$.

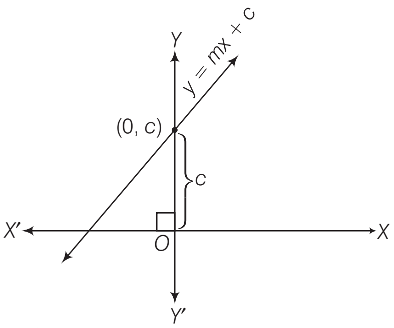
Equation of a Line in Slope-Intercept Form

Suppose the line L with slope m, makes an intercept c on Y–axis, i.e. the line L with the slope m passes through the point $(0, c)$. Then, the equation of the line L is given by



$$\boxed{y = mx + c}$$

Geometrical Understanding of c As per the equation $y = mx + c$, the constant c is called the y-intercept. It is the ordinate of the point, where the line intercepts the Y-axis. Also, it is the point on the line where $x = 0$.



(i) If the line passes through the origin, then $0 = m(0) + c \Rightarrow c = 0$.
Thus, the equation of line passing through the origin is $y = mx$, where m is the slope of the line.

(ii) If the line is parallel to X-axis, then $m = 0$. Therefore, the equation of a line parallel to X-axis is $y = c$.

Example 8. Find the equation of a line, whose inclination is 30° and y-intercept is 2.

Sol. Given, inclination, $\theta = 30°$ and y-intercept, $c = 2$

Now, slope of a line, $m = \tan\theta = \tan 30° = \dfrac{1}{\sqrt{3}}$ $[\because \theta = 30°]$

$\therefore$ The equation of line having slope $m = \dfrac{1}{\sqrt{3}}$ and y-intercept $c = 2$ is $y = mx + c$

$$\Rightarrow \quad y = \frac{1}{\sqrt{3}}x + 2 \Rightarrow \sqrt{3}\, y = x + 2\sqrt{3}$$

which is the required equation of a line.

Example 9. Find the equation of the line through $(1, 3)$ and making an intercept of 5 on the Y-axis.

Sol. Since, y-intercept of the line, $c = 5$, therefore its equation is

$$y = mx + 5 \qquad [\because y = mx + c] \quad \ldots\text{(i)}$$

where, m is unknown constant.

As the line (i) passes through the point (1, 3), we get

$$3 = m \cdot 1 + 5$$

$$\Rightarrow \quad m = -2$$

On substituting $m = -2$ in Eq. (i), we get

$$y = -2x + 5,$$

which is the required equation.

Example 10. Find the slope and y-intercept of the lines $\dfrac{x}{4} + \dfrac{y}{5} = 1$.

Sol. Given equation of the line is $\dfrac{x}{4} + \dfrac{y}{5} = 1$

It can be rewritten as $\dfrac{y}{5} = 1 - \dfrac{x}{4}$

$$\Rightarrow \quad y = -\frac{5x}{4} + 5 \qquad \ldots\text{(i)}$$

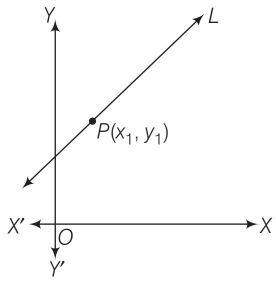
On comparing Eq. (i) with $y = mx + c$, we get

$$m = -5/4 \text{ and } c = 5$$

Hence, the slope of a line is $-5/4$ and y-intercept is 5.

Equation of a Line in Point-Slope Form

The equation of a straight line having slope m and passes through the point $Q(x_1, y_1)$ is given by



$$\boxed{y - y_1 = m(x - x_1)}$$

Example 11. Find the equation of the line passing through the point (3, 8) and having slope 2.

Sol. We know that equation of line passing through the point (x_1, y_1) and having slope m is given by

$$y - y_1 = m(x - x_1)$$

$\therefore$ Equation of line passing through the point (3, 8) and having slope 2 is

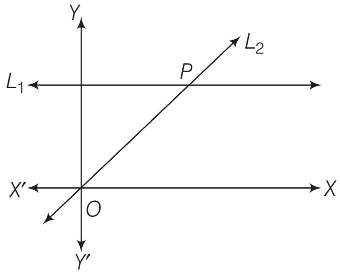
$$y - 8 = 2(x - 3) \quad [\because m = 2, (x_1, y_1) = (3, 8)]$$

$$\Rightarrow \quad y - 8 = 2x - 6$$

$$\Rightarrow \quad 2x - y + 2 = 0$$

which is the required equation.

Example 12. Given equation of line L_1 is $y = 4$.



(i) Write the slope of line L_2, if L_2 is bisector of $\angle O$.

(ii) Write the coordinates of point P.

(iii) Find the equation of L_2. ***[2011]***

Sol.

(i) Given, L_2 is the bisector of $\angle O$.

Since, $\angle XOY = 90°$

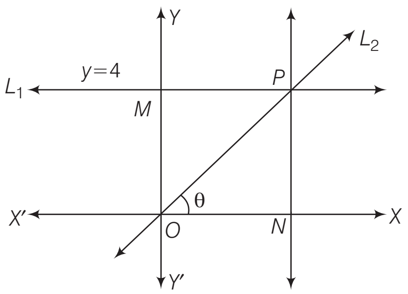
$\therefore$ $\angle POX = \angle POY$

$= \dfrac{90°}{2} = 45° = \theta$ (say)

Now, slope of $L_2 = \tan 45° = 1$ $[\because \theta = 45°]$

(ii) Let line L_1 intersects Y-axis at M. Through P, draw a line parallel to Y-axis which intersect X-axis at N.

Then, clearly $PN = OM = 4$



In ΔONP, $\angle PON + \angle ONP + \angle NPO = 180°$

$\Rightarrow$ $45° + 90° + \angle NPO = 180°$

$\Rightarrow$ $\angle NPO = 180° - 90° - 45° = 45°$

$\Rightarrow$ $ON = NP$

[$\because$ sides opposite to equal angles are equal]

So, the coordinates of N are (4, 0) and the coordinates of M are (0, 4).

Hence, the coordinates of P are (4, 4).

(iii) Equation of line L_2, which is passing through origin (0, 0) and having slope 1, is

$y - 0 = 1(x - 0)$ $[\because y - y_1 = m(x - x_1)]$

$\Rightarrow$ $y = x$

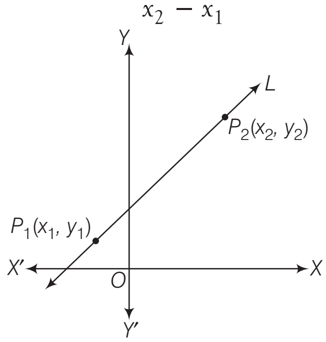
Hence, the required equation of L_2 is $y = x$.

Equation of a Line in Two Points Form

The equation of a line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Here, the slope of a line is $\dfrac{y_2 - y_1}{x_2 - x_1}$.



Example 13. Find the equation of line, which is passing through the points (1, 1) and (2, 4).

Sol. We know that equation of a line passing through the two points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Here, $(x_1, y_1) \equiv (1, 1)$ and $(x_2, y_2) \equiv (2, 4)$

$\therefore$ Equation of a line is $y - 1 = \dfrac{4-1}{2-1}(x - 1)$

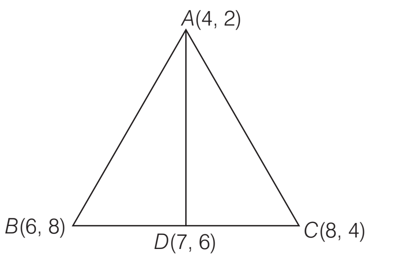
$\Rightarrow$ $y - 1 = 3(x - 1) \Rightarrow 3x - y - 2 = 0$

Example 14. Suppose $A(4, 2)$, $B(6, 8)$ and $C(8, 4)$ are the vertices of a ΔABC. Write down the equation of the median of the triangle through A.

Sol. Mark a point D on side BC, which is mid-point of BC.

Then, coordinates of mid-point D

$$= \left(\frac{6+8}{2}, \frac{8+4}{2}\right) = \left(\frac{14}{2}, \frac{12}{2}\right) = (7, 6)$$



$\therefore$ Equation of a line passing through A and D is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \Rightarrow y - 2 = \frac{6-2}{7-4}(x - 4)$$

$[\because A(x_1, y_1) \equiv A(4, 2), D(x_2, y_2) \equiv D(7, 6)]$

$\Rightarrow$ $y - 2 = \dfrac{4}{3}(x - 4)$

$\Rightarrow$ $3y - 6 = 4x - 16 \Rightarrow 4x - 3y - 10 = 0$

Example 15. Three vertices of a parallelogram $ABCD$ taken in order are $A(3, 6)$, $B(5, 10)$ and $C(3, 2)$. Find

(i) the coordinates of the fourth vertex D.

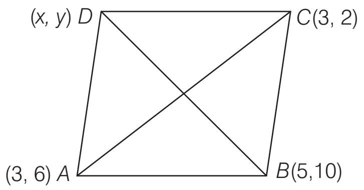
(ii) the length of diagonal BD.

(iii) the equation of side AB of the parallelogram $ABCD$.

[2015]

Sol.

(i) Let the coordinates of vertex D be (x, y).
Since, $ABCD$ is a parallelogram and we know that diagonals bisect each other in a parallelogram. So, the mid-point of AC and mid-point of BD are same.



Now, the mid-point of $AC = \left(\frac{3+3}{2}, \frac{6+2}{2}\right) = (3, 4)$

and the mid-point of $BD = \left(\frac{5+x}{2}, \frac{10+y}{2}\right)$

$\therefore \quad \left(\frac{5+x}{2}, \frac{10+y}{2}\right) = (3, 4)$

On comparing both sides, we get

$$\frac{5+x}{2} = 3 \text{ and } \frac{10+y}{2} = 4 \Rightarrow 5 + x = 6 \text{ and } 10 + y = 8$$

$\therefore \quad x = 1 \text{ and } y = -2$

Hence, the coordinates of D are $(1, -2)$.

(ii) Length of diagonal $BD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ [by distance formula]

$= \sqrt{(5-1)^2 + (10+2)^2}$ [$\because x_1 = 1, x_2 = 5, y_1 = -2$ and $y_2 = 10$]

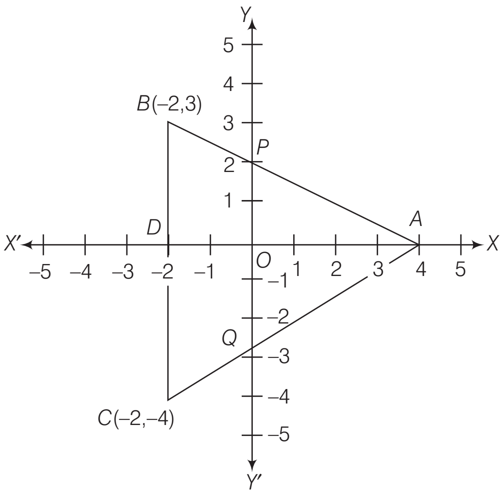
$= \sqrt{(4)^2 + (12)^2} = \sqrt{16 + 144} = \sqrt{160} = 4\sqrt{10}$ units

(iii) Equation of side AB having points $(x_1, y_1) = (3, 6)$ and $(x_2, y_2) = (5, 10)$ is given by $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$\therefore \quad y - 6 = \frac{10-6}{5-3}(x - 3) \Rightarrow y - 6 = \frac{4}{2}(x - 3)$

$\Rightarrow \quad y - 6 = 2(x - 3) \Rightarrow y - 6 = 2x - 6 \Rightarrow 2x - y = 0$

Example 16. In the given figure, ABC is a triangle and BC is parallel to Y-axis. AB and AC intersect the Y-axis at P and Q, respectively.



(i) Write the coordinates of A.
(ii) Find the length of AB and AC.
(iii) Find the ratio, in which Q divides AC.
(iv) Find the equation of the line AC. ***[2015]***

Sol. Given points are $B(-2, 3)$ and $C(-2, -4)$.

(i) Coordinates of $A = (4, 0)$ [from figure]

(ii) Length of $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ [by distance formula]

$= \sqrt{(-2-4)^2 + (3-0)^2}$

$= \sqrt{(-6)^2 + (3)^2} = \sqrt{36 + 9} = \sqrt{45}$ units

Length of $AC = \sqrt{(-2-4)^2 + (-4-0)^2}$

$= \sqrt{(-6)^2 + (-4)^2} = \sqrt{36 + 16} = \sqrt{52}$ units

(iii) Let Q divides AC in $\lambda : 1$.
Using section formula,

Coordinates of $Q = \left(\frac{-2\lambda + 4}{\lambda + 1}, \frac{-4\lambda + 0}{\lambda + 1}\right) = \left(\frac{4 - 2\lambda}{\lambda + 1}, \frac{-4\lambda}{\lambda + 1}\right)$

Since, Q lies on Y-axis, therefore x-coordinate will be 0.

$\therefore \quad \frac{4 - 2\lambda}{\lambda + 1} = 0 \Rightarrow 2\lambda = 4 \Rightarrow \lambda = 2$

So, $\quad \lambda : 1 = 2 : 1$

Hence, the required ratio is $2 : 1$.

(iv) Equation of AC having points $A(4, 0)$ and $C(-2, -4)$, is given by

$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \Rightarrow y - 0 = \frac{-4 - 0}{-2 - 4}(x - 4)$

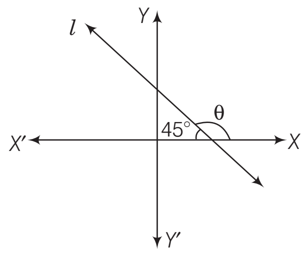
$\Rightarrow \quad y = \frac{2}{3}(x - 4) \Rightarrow 3y = 2x - 8$

$\Rightarrow \quad 2x - 3y - 8 = 0$

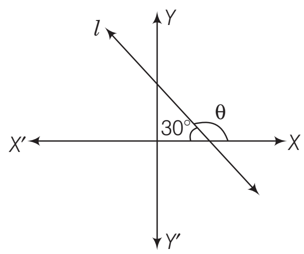
Topic Exercise 1

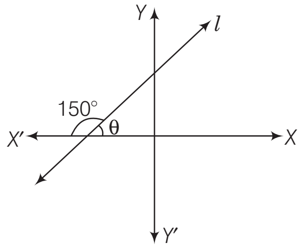
1. Find the inclination of a line l in the following figure.

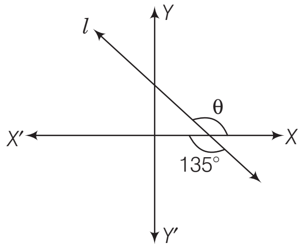
(i)



(ii)



(iii) 

(iv) 

2. Find the slope of line, whose inclination is
(i) 0° (ii) 90° (iii) 30°

3. Find the slope of the line which passes through the points $A(4,-6)$ and $B(-2,-5)$.

4. Determine x so that 2 is the slope of the line joining points (2, 5) and $(x, 3)$.

5. Find the intercept made by a line passing through the following points.
(i) $A(3,0)$ and $B(0,3)$
(ii) $P(-2,0)$ and $P(0,-3)$
(iii) $A(-5,0)$ and $B(1,0)$

6. Find the equation of a straight line parallel to X-axis and passing through the point $(-2,7)$.

7. Find the equation of the line parallel to the Y-axis and 3 units to the right of it.

8. Find the equation of a line, which is equidistant from the lines $x=-6$ and $x=12$.

9. Find the equation of a straight line whose inclination is 60° and y-intercept is -3.

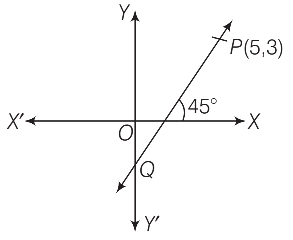
10. Find the equation of a line, whose gradient is $1/\sqrt{3}$ and y-intercept is 3.

11. Find the equation of a line passing through the point $(3,-1)$ and making an intercept of -4 on Y-axis.

12. Find the slope and y-intercept of the following lines.
(i) $4x-5y-6=0$ (ii) $\frac{x}{2}+\frac{y}{7}=1$

13. The equation of a line AB is $4y-4x+8=0$.
(i) Write down the slope of the line AB.
(ii) Calculate the angle that the line AB makes with the positive direction of X-axis.

14. The line through $P(5,3)$ intersects Y-axis at Q.



(i) Write the slope of the line.
(ii) Write the equation of the line.
(iii) Find the coordinates of Q. ***[2012]***

15. Find the equation of the line passing through the point (2, 3) and having slope 3.

16. Find the equation of a line passing through (4, 8) and equally inclined to the coordinate axes in the first quadrant.

17. Find the equation of a line passing through the point (2, 4) and intersecting the line $2x+3y+6=0$ on the X-axis.

18. In ΔABC, if $A(3,5)$, $B(7,8)$ and $C(1,-10)$, then find the equation of the median through A. ***[2013]***

19. Suppose $P(3,4)$, $Q(7,-2)$ and $R(-2,-1)$ are the vertices of ΔPQR. Write down the equation of the median of the triangle through R. ***[2004]***

20. $ABCD$ is a parallelogram, where $A(x,y)$, $B(5,8)$, $C(4,7)$ and $D(2,-4)$. Find
(i) the coordinates of A.
(ii) the equation of diagonal BD. ***[2011]***

Hints and Answers

1. (i) **Hint** Use linear pair property of angles. **Ans.** 45°
(ii) Do same as part (i). **Ans.** 150°
(iii) Do same as part (i). **Ans.** 30°
(iv) Do same as Example 1. **Ans.** 135°

2. Do same as Example 2.
Ans. (i) 0 (ii) not defined (iii) $\frac{1}{\sqrt{3}}$

3. Do same as Example 3. **Ans.** $-\frac{1}{6}$

4. Do same as Example 4(i). **Ans.** 1

5. Do same as Example 5.
Ans. (i) 3, 3 (ii) $-2,-3$ (iii) $-5,1$

6. Do same as Example 6. **Ans.** $y-7=0$

7. **Hint** Let the equation of line parallel to Y-axis is $x=b$.
Now, as it given that line is 3 units to the right of Y-axis, therefore $b=3$. **Ans.** $x=3$

8. Hint Since, the lines $x = 12$ and $x = -6$ cut X-axis at $A(12, 0)$ and $B(-6, 0)$, respectively. Therefore, a line equidistant from these two lines will pass through the mid-point of AB.

Now, coordinates of the mid-point of AB

$$= \left(\frac{-6+12}{2}, \frac{0+0}{2}\right) = (3, 0)$$

Ans. $x = 3$

9. Do same as Example 8. **Ans.** $\sqrt{3}x - y - 3 = 0$

10. Hint Put $m = 1/\sqrt{3}$ and $c = 3$ in the equation $y = mx + c$. **Ans.** $\sqrt{3}y = x + 3\sqrt{3}$

11. Do same as Example 9. **Ans.** $y = x - 4$

12. Do same as Example 10.

Ans. (i) $\frac{4}{5}, -\frac{6}{5}$ (ii) $-\frac{7}{2}, 7$

13. (i) Do same as Example 10. **Ans.** 1

(ii) **Hint** Slope of a line = $\tan\theta$. **Ans.** 45°.

14. (i) **Hint** From the given figure, $\theta = 45°$.

$\therefore$ Slope of the line PQ is $m = \tan\theta$. **Ans.** 1

(ii) **Hint** The equation of line passing through $P(5, 3)$ and having slope 1, is $y - 3 = 1(x - 5)$.

Ans. $y = x - 2$

(iii) **Hint** Since, the coordinates of Q lie on Y-axis, so put $x = 0$ in the equation obtained in part (ii).

Ans. $(0, -2)$

15. Do same as Example 11. **Ans.** $3x - y - 3 = 0$

16. Hint We know that a line equally inclined to the coordinate axes in the first quadrant makes angle 45° with the positive direction of X-axis.

$\therefore$ m = Slope of the line = $\tan 45° = 1$

Ans. $x - y + 4 = 0$

17. Hint Since, the line $2x + 3y + 6 = 0$ intersects the X-axis, put $y = 0$, we get $x = -3$

$\therefore$ The point of intersection on X-axis is $(-3, 0)$.

The equation of line passes through (2, 4) and $(-3, 0)$ is

$$y - 4 = \frac{0-4}{-3-2}(x - 2)$$

Ans. $4x - 5y + 12 = 0$

18. Do same as Example 14. **Ans.** $6x + y - 23 = 0$

19. Do same as Example 14. **Ans.** $2x - 7y = 3$

20. Do same as Example 15.

Ans. (i) $(3, -3)$ (ii) $4x - y - 12 = 0$

Topic 2

Conditions for Two Lines to be Parallel and Perpendicular

Conditions for Two Lines to be Parallel

Two (non–vertical) lines are parallel if and only if (iff) their slopes are equal.

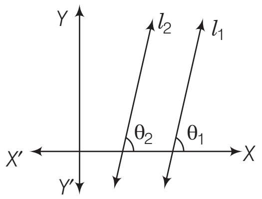
Proof Let l_1, l_2 be two (non–vertical) lines; m_1, m_2 be their slopes and θ_1, θ_2 be the inclinations of these lines.

(i) First, consider lines l_1 and l_2 are parallel. Then, their corresponding angles are equal.

i.e. $\theta_1 = \theta_2$

On taking tangent both sides, we get

$$\tan\theta_1 = \tan\theta_2$$



$\Rightarrow$ $m_1 = m_2$ [$\because m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$]

Thus, if **two lines are parallel, then their slopes are equal**.

(ii) Second, consider lines l_1 and l_2 have equal slopes,

i.e. $m_1 = m_2 \Rightarrow \tan\theta_1 = \tan\theta_2 \Rightarrow \theta_1 = \theta_2$

[converse of part (i)]

Thus, lines l_1 and l_2 are parallel.

Hence, two lines are parallel if and only if their slopes are equal (i.e. $m_1 = m_2$).

Note The slope of every horizontal line is same and it is equal to zero.

Example 1. Check whether the pair of lines $2x + 3y + 1 = 0$ and $4x + 6y + 7 = 0$ is parallel or not.

Sol. Given lines are $2x + 3y + 1 = 0$ and $4x + 6y + 7 = 0$

$\Rightarrow$ $3y = -2x - 1$ and $6y = -4x - 7$

$\Rightarrow$ $y = -\frac{2}{3}x - \frac{1}{3}$ and $y = -\frac{4}{6}x - \frac{7}{6}$

$\Rightarrow$ $y = -\frac{2}{3}x - \frac{1}{3}$ and $y = -\frac{2}{3}x - \frac{7}{6}$

On comparing with $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively, we get

$$m_1 = -\frac{2}{3} \text{ and } m_2 = -\frac{2}{3}$$

$\because \quad m_1 = m_2$

Hence, given pair of lines is parallel.

Example 2. If $3x - 5y + 7 = 0$ and $px + 4y + 6 = 0$ are parallel lines, then find the value of p.

Sol. Given lines are

$$3x - 5y + 7 = 0 \text{ and } px + 4y + 6 = 0$$

$$\Rightarrow \quad 5y = 3x + 7 \text{ and } 4y = -px - 6$$

$$\Rightarrow \quad y = \frac{3}{5}x + \frac{7}{5} \text{ and } y = -\frac{p}{4}x - \frac{6}{4}$$

On comparing with $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively, we get $m_1 = \frac{3}{5}$ and $m_2 = -\frac{p}{4}$

Since, the lines are parallel.

$$\therefore \quad m_1 = m_2 \Rightarrow \frac{3}{5} = -\frac{p}{4} \Rightarrow p = \frac{-4 \times 3}{5} = -\frac{12}{5}$$

Example 3. Find the equation of the line through the point $P(-5, 1)$ and parallel to the line joining the points $A(7, -1)$ and $B(0, 3)$.

Sol. Slope of the line joining the points $A(7, -1)$ and $B(0, 3)$ is

$$m_1 = \frac{3-(-1)}{0-7} = -\frac{4}{7} \quad \left[\because m = \frac{y_2 - y_1}{x_2 - x_1}\right] \quad \ldots(i)$$

$\therefore$ Slope of a line parallel to line $AB = -\frac{4}{7}$ $\quad [\because m_1 = m_2]$

Now, the equation of line passing through $P(-5, 1)$ and having slope $-\frac{4}{7}$, is

$$y - 1 = -\frac{4}{7}(x - (-5)) \quad [\because y - y_0 = m(x - x_0)]$$

$$\Rightarrow \quad 7y - 7 = -4x - 20 \Rightarrow 4x + 7y + 13 = 0$$

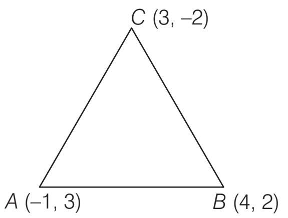
Example 4. $A(-1, 3)$, $B(4, 2)$ and $C(3, -2)$ are the vertices of a triangle.

(i) Find the coordinates of the centroid G of the triangle.

(ii) Find the equation of the line through G and parallel to AC. ***[2017]***

Sol. Given $A(-1, 3)$, $B(4, 2)$ and $C(3, -2)$ are the vertices of the triangle.

(i) We know that coordinates of centroid of a triangle, whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3), is given by $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.



$\therefore$ Coordinates of centroid G of

$$\Delta ABC = \left(\frac{-1+3+4}{3}, \frac{3-2+2}{3}\right) = (2, 1)$$

(ii) We know that the slope of line passing through points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$

$\therefore$ Slope of line $AC = \frac{-2-3}{3-(-1)} = -\frac{5}{4}$

We know that slope of parallel lines are equal.

$\therefore$ Slope of line parallel to $AC = -\frac{5}{4}$

Now, equation of the line through G and parallel to AC is

$$y - 1 = -\frac{5}{4}(x - 2)$$

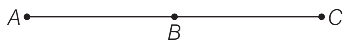
[$\because$ equation of line passing through (x_1, y_1) having slope m is $y - y_1 = m(x - x_1)$]

$$\Rightarrow \quad 4y - 4 = -5x + 10 \Rightarrow 5x + 4y - 14 = 0$$

Collinearity of Three Points in a Plane

Three points are said to be collinear, if they lie on a straight line.

Suppose A, B and C are three points in a plane, then these three points are said to be collinear if and only if AB is



parallel to BC, i.e. slope (Gradient) of

$AB =$ slope (Gradient) of BC.

Example 5. Show that the points $A(-6, -1)$, $B(0, 2)$ and $C(8, 6)$ are collinear.

Sol. Given points are $A(-6, -1)$, $B(0, 2)$ and $C(8, 6)$.

Now, slope of AB is $m_1 = \frac{2+1}{0+6} = \frac{3}{6} = \frac{1}{2}$

and slope of BC is

$$m_2 = \frac{6-2}{8-0} = \frac{4}{8} = \frac{1}{2} \quad \left[\because \text{slope of a line} = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

Here, $m_1 = m_2$ and B is common point.

Hence, points A, B and C are collinear.

Example 6. The points $P(2, 3)$, $Q(0, -5)$ and $R(-2, k)$ are collinear, then find the value of k.

Sol. Given, $P(2, 3)$, $Q(0, -5)$ and $R(-2, k)$ are collinear points.

$\therefore$ Slope of PQ = Slope of QR

$$\Rightarrow \quad \frac{-5-3}{0-2} = \frac{k+5}{-2-0} \quad \left[\because \text{slope of the line} = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

$$\Rightarrow \quad \frac{-8}{-2} = \frac{k+5}{-2}$$

$$\Rightarrow \quad 4 = \frac{k+5}{-2}$$

$$\Rightarrow \quad -8 = k + 5$$

$$\Rightarrow \quad k = -8 - 5 = -13$$

Hence, the required value of k is -13.

Example 7. Find the value of a, for which the points $A(a, 3)$, $B(2, 1)$ and $C(5, a)$ are collinear. Also, find the equation of the line. ***[2014]***

Sol. Given points $A(a, 3)$, $B(2, 1)$ and $C(5, a)$ are collinear.

$\therefore$ Slope of AB = Slope of BC

$$\Rightarrow \quad \frac{1-3}{2-a} = \frac{a-1}{5-2} \qquad \left[\because \text{slope} = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

$$\Rightarrow \quad \frac{-2}{2-a} = \frac{a-1}{3}$$

$$\Rightarrow \quad -2 \times 3 = (a-1)(2-a)$$

$$\Rightarrow \quad -6 = 2a - a^2 - 2 + a$$

$$\Rightarrow \quad a^2 - 3a - 4 = 0$$

$$\Rightarrow \quad a^2 - 4a + a - 4 = 0 \qquad \text{[splitting the middle term]}$$

$$\Rightarrow \quad a(a-4) + 1(a-4) = 0$$

$$\Rightarrow \quad (a+1)(a-4) = 0$$

$$\therefore \quad a = -1, 4$$

Now, the given points become $A(-1, 3)$, $B(2, 1)$ and $C(5, -1)$ or $A(4, 3)$, $B(2, 1)$ and $C(5, 4)$.

Now, we have to find equation of a line.

***Case* I When collinear points are $A(-1, 3)$, $B(2, 1)$ and $C(5, -1)$.**

Then, the equation of a line passing through the points $A(-1, 3)$ and $B(2, 1)$ is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow \quad y - 3 = \frac{1-3}{2+1}(x+1)$$

$$\Rightarrow \quad y - 3 = -\frac{2}{3}(x+1)$$

$$\Rightarrow \quad 3(y-3) = -2(x+1)$$

$$\Rightarrow \quad 3y - 9 = -2x - 2$$

$$\Rightarrow \quad 2x + 3y - 7 = 0$$

***Case* II When collinear points are $A(4, 3)$, $B(2, 1)$ and $C(5, 4)$.**

Then, the equation of a line passing through the points $A(4, 3)$ and $B(2, 1)$ is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow \quad y - 3 = \frac{1-3}{2-4}(x-4) \Rightarrow y - 3 = \frac{-2}{-2}(x-4)$$

$$\Rightarrow \quad y - 3 = 1(x-4)$$

$$\Rightarrow \quad y - 3 = x - 4$$

$$\Rightarrow \quad x - y - 1 = 0$$

Hence, the required equations of the line are

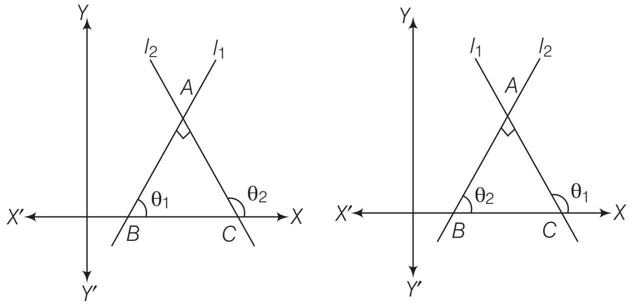
$$2x + 3y - 7 = 0 \text{ and } x - y - 1 = 0.$$

Conditions for Two Lines to be Perpendicular

Two (non-vertical) lines are perpendicular if and only if (iff) the product of their slopes is -1.

Proof Let l_1, l_2 be two (non-vertical) lines; m_1, m_2 be their slopes and θ_1, θ_2 be the inclinations of these lines.

(i) First, consider lines l_1 and l_2 are perpendicular.



Then, $\theta_2 = \theta_1 + 90°$ or $\theta_1 = 90° + \theta_2$

[$\because$ exterior angle of a ΔABC is equal to the sum of opposite interior angles]

On taking tangent on both sides, we get

$$\tan\theta_2 = \tan(90° + \theta_1)$$

or $$\tan\theta_1 = \tan(90° + \theta_2)$$

$$\Rightarrow \quad \tan\theta_2 = -\cot\theta_1$$

or $$\tan\theta_1 = -\cot\theta_2$$

$$\Rightarrow \quad \tan\theta_2 = -\frac{1}{\tan\theta_1}$$

or $$\tan\theta_1 = -\frac{1}{\tan\theta_2}$$

$$\Rightarrow \tan\theta_1 \tan\theta_2 = -1 \text{ or } \tan\theta_1 \tan\theta_2 = -1$$

Therefore, in either cases, $m_1 \cdot m_2 = -1$

(ii) Second, consider the lines l_1 and l_2 such that the product of their slopes is -1,

i.e. $\quad m_1 \cdot m_2 = -1 \qquad$ [converse of part (i)]

$$\Rightarrow \quad \tan\theta_1 \tan\theta_2 = -1$$

$$\Rightarrow \quad \tan\theta_1 = -\frac{1}{\tan\theta_2}$$

$$\Rightarrow \quad \tan\theta_1 = -\cot\theta_2$$

$$\Rightarrow \quad \tan\theta_1 = \tan(90° + \theta_2)$$

$$\Rightarrow \quad \theta_1 = 90° + \theta_2$$

Thus, lines l_1 and l_2 are perpendicular.

Hence, two lines are perpendicular, if and only if product of their slopes is -1 (i.e. $m_1 \cdot m_2 = -1$).

Note The slope of a perpendicular line is the negative reciprocal of the slope of given line.

Example 8. Check whether the pair of lines $x - 2y - 3 = 0$ and $2x + y - 1 = 0$ is perpendicular or not.

Sol. Given lines are $x - 2y - 3 = 0$ and $2x + y - 1 = 0$

$$\Rightarrow \quad 2y = x - 3 \text{ and } y = -2x + 1$$

$$\Rightarrow \quad y = \frac{1}{2}x - \frac{3}{2} \text{ and } y = -2x + 1$$

On comparing with $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively, we get

$$m_1 = \frac{1}{2} \text{ and } m_2 = -2$$

Now, $\quad m_1 \cdot m_2 = \frac{1}{2} \times (-2) = -1$

Hence, the given pair of lines is perpendicular.

Example 9. Find the value of p, for which the lines $2x + 3y - 7 = 0$ and $4y - px - 12 = 0$ are perpendicular to each other. ***[2009]***

Sol. Given lines are $2x + 3y - 7 = 0$ and $4y - px - 12 = 0$

$\Rightarrow \quad 3y = -2x + 7$ and $4y = px + 12$

$\Rightarrow \quad y = -\frac{2}{3}x + \frac{7}{3}$ and $y = \frac{p}{4}x + \frac{12}{4}$

On comparing with equation $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively, we get

$$m_1 = -\frac{2}{3} \text{ and } m_2 = \frac{p}{4}$$

Since, the given lines are perpendicular to each other.

$\therefore \quad m_1 \cdot m_2 = -1 \Rightarrow -\frac{2}{3} \times \frac{p}{4} = -1$

$\Rightarrow \quad p = \frac{3 \times 4}{2} = 3 \times 2 \Rightarrow p = 6$

Hence, the required value of p is 6.

Example 10. Find the equation of a line passing through the intersection of lines $4x + 3y = 1$ and $3x - y + 9 = 0$ and perpendicular to the line $3x - 2y + 5 = 0$.

Sol. Consider the equation, $3x - 2y + 5 = 0$ ...(i)

$\Rightarrow \quad 2y = 3x + 5 \Rightarrow y = \frac{3}{2}x + \frac{5}{2}$

On comparing it with $y = m_1x + c$, we get $m_1 = \frac{3}{2}$ ...(ii)

Now, the slope (m) of line perpendicular to the line (i) is given by $m = \frac{-1}{m_1} = \frac{-2}{3}$ [using Eq. (ii)]

i.e. the slope of required line $= m = \frac{-2}{3}$. ...(iii)

Now, let us find the intersection point of

$4x + 3y = 1$ ...(iv)

and $\quad 3x - y = -9$ ...(v)

On multiplying Eq. (v) by 3 and adding with Eq. (iv), we get

$9x - 3y + 4x + 3y = -27 + 1$

$\Rightarrow \quad 13x = -26 \Rightarrow x = -2$

On putting $x = -2$ in Eq. (iv), we get

$4(-2) + 3y = 1 \Rightarrow 3y = 1 + 8 = 9 \Rightarrow y = 3$

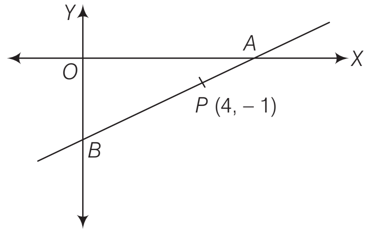
Thus, the intersection point is $(-2, 3)$.

Hence, the equation of line passing through $(-2, 3)$ and having slope $\frac{-2}{3}$ is $(y - 3) = \frac{-2}{3}(x + 2)$

$\Rightarrow \quad 3y - 9 = -2x - 4$

$\Rightarrow \quad 2x + 3y - 5 = 0$, which is the required equation.

Example 11. A line AB meets X-axis at A and Y-axis at B. P (4, – 1) divides AB in the ratio 1 : 2.

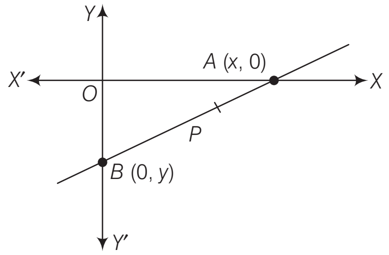


(i) Find the coordinates of A and B.

(ii) Find the equation of the line through P and perpendicular to AB. ***[2016]***

Sol.

(i) Let the coordinates of point A lying on X-axis be $(x, 0)$ and the coordinates of point B lying on Y-axis be $(0, y)$.



By using section formula,

$$4 = \frac{1 \times 0 + 2 \times x}{1 + 2} \quad \left[\because x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}\right]$$

$\Rightarrow \quad 4 = \frac{2x}{3} \Rightarrow x = \frac{12}{2} = 6$

and $\quad -1 = \frac{1 \times y + 2 \times 0}{1 + 2} \quad \left[\because y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right]$

$\Rightarrow \quad -1 = \frac{y}{3} \Rightarrow y = -3$

Hence, the coordinates of points A and B are (6, 0) and (0, – 3), respectively.

(ii) Slope of line $AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 0}{0 - 6} = \frac{1}{2}$

Let m be the slope of desired line, then

$$\frac{1}{2} \times m = -1 \Rightarrow m = -2$$

[$\because$ both lines are perpendicular to each other, $m_1m_2 = -1$]

Now, equation of a line passes through (4, – 1) having slope -2 is $(y + 1) = -2(x - 4)$ $\quad [\because y - y_1 = m(x - x_1)]$

$\Rightarrow \quad y + 1 = -2x + 8 \Rightarrow 2x + y - 7 = 0$

Topic Exercise 2

1. Check whether the pair of lines $7x + y + 3 = 0$ and $14x + 2y + 5 = 0$ is parallel or not.

2. If the lines $3(\alpha - 1)y - 6x = 2$ and $4y - 8x + 10 = 0$ are parallel, then find the value of α.

3. Find the equation of a line passing through the point $P(0, 5)$ and parallel to the line joining the points $A(18, 0)$ and $B(1, -1)$.

4. Find the equation of a line parallel to $x - 2y + 8 = 0$ and passing through the point (1, 2).

5. Find the equation of a line, which has the y-intercept 4; and is parallel to the line $2x - 3y - 7 = 0$. Hence, find the coordinates of the point, where it cuts the X-axis.

6. Find the equation of a straight line passing through the intersection of $2x + 5y - 4 = 0$ with X-axis and parallel to the line $3x - 7y + 8 = 0$.

7. $A(1, 4)$, $B(3, 2)$ and $C(7, 5)$ are the vertices of a ΔABC. Find
(i) the coordinates of the centroid G of ΔABC.
(ii) the equation of a line passing through G and parallel to AB. **[2002]**

8. Check whether the points $(-2, 9/7)$, $(-5, 0)$ and $(2, 3)$ are collinear or not.

9. If the three points $(k, 3)$, $(2, -4)$ and $(-k + 1, -2)$ are collinear, then find the value of k.

10. Check whether the pair of lines $x + 3y - 7 = 0$ and $3x - y - 21 = 0$ is perpendicular or not.

11. If the straight lines $3x - 5y = 7$ and $4x + ay + 9 = 0$ are perpendicular to one another, find the value of a. **[2018]**

12. The line passing through $A(-2, 3)$ and $B(4, b)$ is perpendicular to the line $2x - 4y = 5$. Find the value of b. **[2012]**

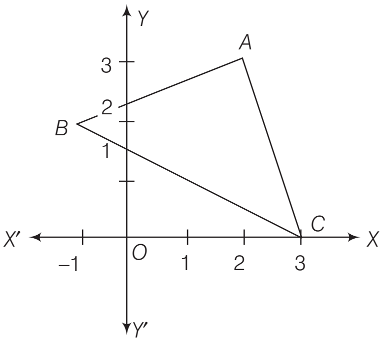
13. Find the equation of a line, that has y-intercept 4 and is perpendicular to the line joining $(2, -3)$ and $(4, 2)$.

14. Find the equation of the perpendicular bisector of the line joining the points (1, 3) and (3, 1).

15. Without using Pythagoras theorem, show that $A(4, 4)$, $B(3, 5)$ and $C(-1, -1)$ are the vertices of a right angled triangle.

16. The equation of a line is $3x + 4y - 7 = 0$. Find
(i) the slope of the line.
(ii) the equation of a line perpendicular to the given line and passing through the intersection of the lines $x - y + 2 = 0$ and $3x + y - 10 = 0$.

17. In the given figure,



write
(i) the coordinates of A, B and C.
(ii) the equation of a line passing through A and perpendicular to BC. **[2005]**

Hints and Answers

1. Do same as Example 1. **Ans.** Yes

2. Do same as Example 2. **Ans.** $\alpha = 2$

3. Do same as Example 3. **Ans.** $x - 7y + 35 = 0$

4. Hint Let the equation of line parallel to given line is $x - 2y + \lambda = 0$.
Put $x = 1$ and $y = 2$ in the equation, then determine λ.
Ans. $x - 2y + 3 = 0$

5. Hint Equation of line $y = mx + c$ in $c = 4$ and $m = \frac{2}{3}$.
Ans. $2x - 3y + 12 = 0$ and $(-6, 0)$

6. Hint The intersection point of given line on X-axis is (2, 0).
$\therefore$ Equation of line passing through (2, 0) and slope 3/7 is $y - 0 = \frac{3}{7}(x - 2)$.
Ans. $3x - 7y - 6 = 0$

7. Do same as Example 4.
Ans. (i) $\left(\frac{11}{3}, \frac{11}{3}\right)$
(ii) $3x + 3y = 22$

8. Do same as Example 5. **Ans.** Collinear

9. Do same as Example 6. **Ans.** $-\frac{1}{3}$

10. Do same as Example 8.

Ans. Yes

11. Do same as Example 9.

Ans. $a = \frac{12}{5}$

12. **Hint** Slope of AB, $m_1 = \frac{b-3}{6}$

and slope of line $2x - 4y = 5$, $m_2 = \frac{1}{2}$

We know that if the lines having slopes m_1 and m_2 are perpendicular, then $m_1 \cdot m_2 = -1$.

Ans. -9

13. **Hint** Slope of required line,

$$m = \frac{-1}{\left(\frac{2+3}{4-2}\right)} = -\frac{2}{5} \text{ and } c = 4$$

Ans. $2x + 5y - 20 = 0$

14. **Hint** The mid-point of line joining given point is $M(2, 2)$

Ans. $y - x = 0$

15. **Hint** Slope of AB is $m_1 = \frac{5-4}{3-4} = -1$

and slope of AC is $m_2 = \frac{(-1)-4}{(-1)-4} = 1$

Here, $m_1 \cdot m_2 = -1$

Ans. This shows that $AB \perp AC$, i.e. $\angle CAB = \frac{\pi}{2}$.

16. (i) **Hint** Compare the given equation with $y = mx + c$. **Ans.** $-\frac{3}{4}$

(ii) Do same as Example 10.

Ans. $4x - 3y + 4 = 0$

17. (i) **Hint** Measures the perpendicular distance from given points to the Y and X-axes.

Ans. $A(2, 3)$, $B(-1, 2)$ and $C(3, 0)$

(ii) **Hint** Slope of BC $(m_1) = \frac{y_2 - y_1}{x_2 - x_1}$

Let slope of line perpendicular to BC be m_2. Then, required equation of line, passing through A (2, 3) and having slope m_2, is $y - 3 = m_2 (x - 2)$.

Ans. $2x - y - 1 = 0$

CHAPTER EXERCISE

a 3 Marks Questions

1. If the side BC of an equilateral ΔABC is parallel to above X-axis, then find the slope of lines along sides BC, CA and AB. [Given, $\tan(90°+\theta) = -\cot\theta$]

2. Find the slope of a line, which passes through the origin and mid-point of the line segment joining the points $P(0, -4)$ and $B(8, 0)$.

3. Find the intercepts made by the line $2x - 3y + 12 = 0$ on the coordinates axes.

4. Find the equation of the straight lines parallel to

(i) X-axis and passing through the point $(-8, -7)$.

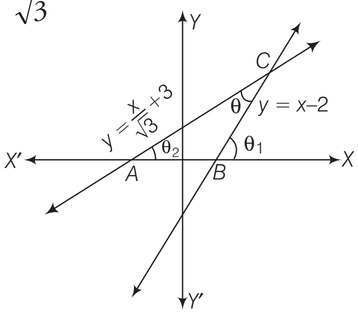
(ii) Y-axis and passing through the point $(1.8, 0.9)$.

5. Find the equation of the line passing through $(2, -5)$ and making an intercept of -3 on the Y-axis.

6. Find the slope and y-intercept made by the line

(i) $7x - 8y + 1 = 0$ (ii) $\dfrac{x}{1.5} + \dfrac{y}{0.5} = 1.$

7. The following figure represents the lines $y = x - 2$ and $y = \dfrac{x}{\sqrt{3}} + 3$.

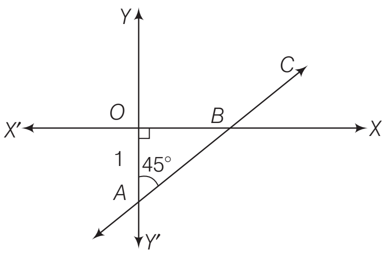


(i) Find the angles which the lines make with the positive direction of X-axis.

(ii) Determine the value of θ.

8. Find the equation of the line passing through the point $\left(9, \dfrac{1}{2}\right)$ and having slope $-\dfrac{3}{2}$.

9. Using the point-slope form, find the equation of the line AB.



10. Find the equation of the line intersecting the X-axis at a distance of three units to the left of origin with slope -2.

11. Find the equation of the line, which is passing through the points $(-1, 0)$ and $(0, -10)$.

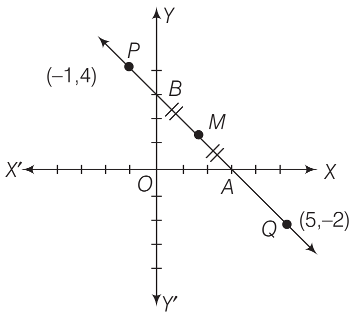
12. Find the equation of a line with x-intercept $= 5$ and passing through the point $(4, -7)$. ***[2009]***

13. (i) Write down the coordinates of the point P, that divides the line joining $A(-4, 1)$ and $B(17, 10)$ in the ratio $1 : 2$.

(ii) Calculate the distance OP, where O is origin.

(iii) Find the equation of a line AB.

14. In the given figure, a straight line passes through the points $P(-1, 4)$ and $Q(5, -2)$. It intersects the coordinate axes at points A and B. If M is the mid-point of the segment AB, then find



(i) the equation of the line.

(ii) the coordinates of A and B.

(iii) the coordinates of M. ***[2003]***

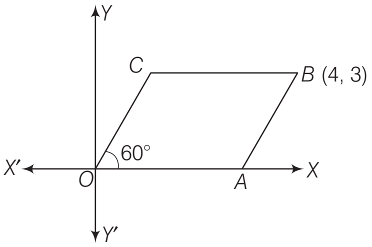
15. Show that the pair of lines $x + 9y - 7 = 0$ and $3x + 27y + 19 = 0$ is parallel.

16. Suppose two straight lines $3x - 2y = 5$ and $2x + ky + 7 = 0$. Find the value of k for which the given lines are parallel to each other.

17. Find the equation of the line through the point $A(0, -3)$ and parallel to the line joining the points $M(3, 2)$ and $N(-1, 0)$.

18. A quadrilateral has the vertices at the points $(-4, 2)$, $(2, 6)$, $(8, 5)$ and $(9, -7)$. Show that the mid-points of the sides of this quadrilateral are the vertices of a parallelogram by using slope formula.

19. In the given figure, $OABC$ is a parallelogram. Inclination of OC is $60°$ and the point B has coordinates (4, 3). Find the equations of all the sides of parallelogram $OABC$.



20. Show that the points $(-3, 2)$, $(2, -1)$ and $\left(-\dfrac{19}{4}, 4\right)$ are collinear.

21. If the points $(2, -5)$, $(p-1, p+4)$ and $(4, 3)$ are collinear, then the value of p.

22. Show that the pair of lines $3x - y = 0$ and $x + 3y - 3 = 0$ is perpendicular.

23. Given two straight lines $kx - 2y = 5$ and $2x + 3y + 7 = 0$. If the given pair of lines is perpendicular, then find the value of k.

24. Find the value of p, if the lines $y = 3x + 7$ and $y + px = 3$ are

(i) parallel to each other.

(ii) perpendicular to each other.

25. Find the value of a, if the lines $\dfrac{y}{2} = x - p$ and $ax + 5 = 3y$ are

(i) parallel to each other.

(ii) perpendicular to each other.

26. Lines $mx + 3y + 7 = 0$ and $5x - ny - 3 = 0$ are perpendicular to each other. Find the relation connecting m and n.

27. Find the equation of the line passing through the point of intersection of $7x + 6y = 71$ and $5x - 8y = -23$ and perpendicular to the line $4x - 2y = 1$.

28. Find the equation of a line drawn perpendicular to the line $\dfrac{x}{4} + \dfrac{y}{6} = 1$ through the point, where it meets the Y-axis.

29. Show that the perpendicular drawn from the point (4, 1) on the line segment joining (6, 5) and (2,– 1) divides it internally in the ratio 8 : 5.

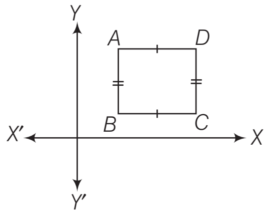
30. Points A (9, 2), B (1, 10) and C $(-7, -6)$ are the vertices of a ΔABC. Find the equation of

(i) the median of the triangle through A.

(ii) the altitude of the triangle through B.

b 4 Marks Questions

31. The side AB of a square $ABCD$ is parallel to the Y-axis as shown in the given figure.

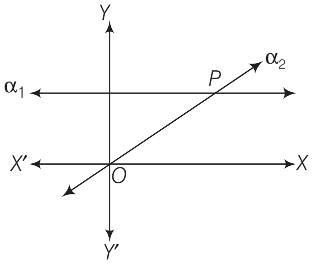


Calculate

(i) the slope of AD. (ii) the slope of BD.

(iii) the slope of AC. [Given, $\tan(90°+\theta) = -\cot\theta$]

32. The equation of a line α_1 is $y = -8$.



(i) Write the slope of the line α_2 if α_2, is the bisector of $\angle O$.

(ii) Write the coordinates of point P.

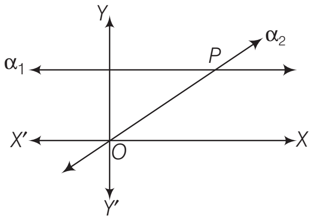
(iii) Find the equation of α_2.

33. If a line passes through the point (2,3) and cuts-off equal positive intercepts on the two axes of reference, then find the slope and equation of the line.

[Given, $\tan(90°+\theta) = -\cot\theta$]

34. Find the equation of the line passing through the point (1, 4) and intersecting the line $x - 2y - 11 = 0$ on the Y-axis.

35. A and B are two points on the X and Y-axes, respectively. $P(2, -3)$ is the mid-point of AB. Find the



(i) coordinate of A and B. (ii) slope of line AB.

(iii) equation of line AB.

36. The points P (5, 0), Q (– 2, 2) and R (3, 4) are the vertices of a ΔPQR. Find the equation of the median through P, Q and R, respectively.

37. Find the equation of the line, which passes through the point (3, 4) and the sum of its intercepts on the axes is 14.

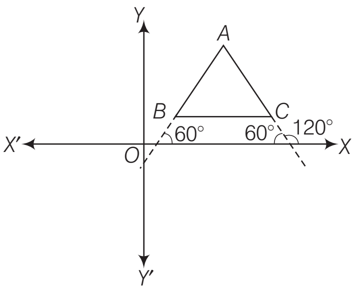
38. Find the equation of the line passing through the points P (5, 1) and Q (1, – 1). Hence, show that the points P, Q and R (11, 4) are collinear.

39. The equation of a line is $y = 3x - 5$. Write down the slope of this line and the intercept made by its on the Y-axis. Hence or otherwise, write down the equation of a line, which is parallel to the line and which passes through the point (0, 5).

40. $A(2,5)$, $B(-1, 2)$ and $C(5, 8)$ are the vertices of a triangle ABC, 'M' is a point on AB such that $AM : MB = 1:2$. Find the coordinates of 'M'. Hence find the equation of the line passing through the points C and M. ***[2018]***

Hints and Answers

1. Hint Clearly, $\angle ABC = \angle ACB = \angle BAC = 60°$



Since, BC is parallel to X-axis. Therefore, inclinations of lines along the sides AB, BC and CA are 60°, 0° and 120°, respectively.

$\therefore$ Slope of line along BC = tan 0°.

Slope of line along AB = tan 60°.

Slope of line along CA = tan 120°

$= \tan(90° + 30°) = -\cot 30°$

Ans. $0°, -\sqrt{3}, \sqrt{3}$

2. Hint Mid-point of P and B

$$= \left(\frac{0+8}{2}, \frac{-4+0}{2}\right) = (4, -2)$$

$\therefore$ Slope of a line $= \dfrac{-2-0}{4-0}$ **Ans.** $-\dfrac{1}{2}$

3. Hint x-intercept is the x-coordinate, of the point, where it cut the X-axis. For this put $y = 0$.

and y-intercept is the y-coordinate, of the point, where it cut the Y-axis. For this put $x = 0$.

Ans. x-intercept $= -6$, y-intercept $= 4$

4. (i) Do same as Example 6 of Topic 1.

Ans. $y + 7 = 0$

(ii) Do same as Example 7 of Topic 1.

Ans. $x - 1.8 = 0$

5. Do same as Example 9 of Topic 1.

Ans. $x + y + 3 = 0$

6. Do same as Example 10 of Topic 1.

Ans. (i) $\dfrac{7}{8}$ and $\dfrac{1}{8}$ (ii) $-\dfrac{1}{3}$ and $\dfrac{1}{2}$

7. (i) **Hint** On comparing with $y = mx + c$, we get

$$m_1 = 1 \text{ and } m_2 = \frac{1}{\sqrt{3}}$$ **Ans.** 45°, 30°

(ii) **Hint** In ΔABC, $\theta_1 = \theta_2 + \theta$

[$\because$ exterior angle of a triangle is equal to the sum of opposite interior angles]

Ans. 15°

8. Do same as Example 11 of Topic 1.

Ans. $3x + 2y - 28 = 0$

9. Hint $\because \angle OAB = 45°$

and $\angle AOB + \angle OBA + \angle OAB = 180°$

$\therefore \quad 90° + \angle OBA + 45° = 180°$

$\Rightarrow \quad \angle OBA = 45° = XBC$

[$\because$ vertically opposite angles]

$\therefore$ Slope of a line AB is $m = \tan 45° = 1$

It is clear from the figure that the coordinates of point A is $(0, -1)$.

Now, the equation of line AB passing through A (0, – 1) and having slope 1 is $y - y_1 = m(x - x_1)$.

Ans. $x - y - 1 = 0$

10. Hint Clearly, line passes through the point (– 3, 0) and having slope, $m = -2$.

Equation of line in point-slope form is

$y - y_1 = m(x - x_1)$. **Ans.** $2x + y + 6 = 0$

11. Do same as Example 13 of Topic 1.

Ans. $10x + y + 10 = 0$

12. Hint Given points are (5, 0) and (4, – 7).

Now, using the formula of straight line,

$y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ **Ans.** $7x - y - 35 = 0$

13. (i) **Hint** Coordinates of P

$$=\left(\frac{1\times 17+2\times(-4)}{1+2}, \frac{1\times 10+2\times 1}{1+2}\right).$$

Ans. (3, 4)

(ii) **Hint** The distance between O (0, 0) and P (3, 4) is

$$OP=\sqrt{(3-0)^2+(4-0)^2} \qquad \textbf{Ans. } 5$$

(iii) **Hint** The equation of a line passing through $A(-4, 1)$ and $B(17, 10)$ is

$$y-1=\frac{10-1}{17+4}[x-(-4)]. \quad \textbf{Ans. } 3x-7y+19=0$$

14. (i) **Hint** Equation of line passing through two points is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1). \quad \textbf{Ans. } x+y-3=0$$

(ii) **Hint** Here, the line $x+y-3=0$ meets the X and Y-axes, respectively.

On putting $y=0$ in $x+y-3=0$, we get $x=3$.

Now, putting $x=0$ in $x+y-3=0$, we get $y=3$.

Ans. (3, 0) and (0, 3)

(iii) **Hint** Coordinates of mid-point of $A(3, 0)$ and $B(0, 3)$

$$=\left(\frac{3+0}{2}, \frac{0+3}{2}\right) \quad \textbf{Ans. } \left(\frac{3}{2}, \frac{3}{2}\right)$$

15. Do same as Example 1 of Topic 2.

16. Do same as Example 2 of Topic 2. **Ans.** $k=-4/3$

17. Do same as Example 3 of Topic 2.

Ans. $x-2y-6=0$

18. **Hint** Let $A(-4, 2)$, $B(2, 6)$, $C(8, 5)$ and $D(9, -7)$ be the vertices of the given quadrilateral. Let P, Q, R and S be the mid-points of AB, BC, CD and DA, respectively. Then, the coordinates of P, Q, R and S are

$$P(-1, 4), Q\left(5, \frac{11}{2}\right), R\left(\frac{17}{2}, -1\right) \text{ and } S\left(\frac{5}{2}, -\frac{5}{2}\right),$$

respectively.

In order to prove that $PQRS$ is a parallelogram, it is sufficient to show that PQ is parallel to RS and $PQ=RS$.

Here, slope of PQ, $m_1=\dfrac{11/2-4}{5-(-1)}=\dfrac{1}{4}$

slope of RS, $m_2=\dfrac{-5/2+1}{5/2-17/2}=\dfrac{1}{4}$,

$$PQ=\sqrt{(5+1)^2+\left(\frac{11}{2}-4\right)^2}=\frac{\sqrt{153}}{2}$$

and $$RS=\sqrt{\left(\frac{5}{2}-\frac{17}{2}\right)^2+\left(-\frac{5}{2}+1\right)^2}=\frac{\sqrt{153}}{2}$$

19. **Hint** The equation of OA is $y=0$.

Side BC is parallel to X-axis and passes through $B(4, 3)$. So, the equation of BC is $y-3=0\,(x-4)$.

Side OC passing through $O(0, 0)$ and having slope $\tan 60°=\sqrt{3}$. So, its equation is $y-0=\sqrt{3}(x-0)$.

Now, slope of AB = slope of $OC=\sqrt{3}$

So, equation of side AB having point $B(4, 3)$ and the slope $\sqrt{3}$ is $y-3=\sqrt{3}\,(x-4)$.

Ans. Equation of OA is $y=0$, equation of AB is $\sqrt{3}x-y-4\sqrt{3}+3=0$, equation of BC is $y=3$ and equation of OC is $y=\sqrt{3}x$.

20. Do same as Example 5 of Topic 2.

21. Do same as Example 6 of Topic 2. **Ans.** $p=7$

22. Do same as Example 8 of Topic 2.

23. Do same as Example 9 of Topic 2. **Ans.** $k=3$

24. (i) Do same as Example 2 of Topic 2. **Ans.** $p=-3$

(ii) Do same as Example 9 of Topic 2. **Ans.** $p=\dfrac{1}{3}$

25. (i) Do same as Example 2 of Topic 2. **Ans.** $a=6$

(ii) Do same as Example 9 of Topic 2. **Ans.** $a=-\dfrac{3}{2}$

26. Do same as Example 9 of Topic 2. **Ans.** $5m=3n$

27. Do same as Example 10 of Topic 2.

Ans. $x+2y-17=0$

28. **Hint** $\dfrac{x}{4}+\dfrac{y}{6}=1 \Rightarrow 3x+2y=12$ ...(i)

If line (i) meets the Y-axis, then putting $x=0$ in Eq. (i), we get $y=6$

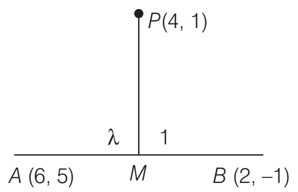
So, the point is (0, 6).

Slope of line (i) is $m_1=\dfrac{-3}{2}$.

Now, the equation of line having slope $\dfrac{2}{3}$ and passing through (0, 6) is given by $y-6=\dfrac{2}{3}(x-0)$.

Ans. $2x-3y+18=0$

29. **Hint** Let m be the slope of PM.



Since, as $PM \perp AB$

$\therefore$ Slope of $PM \times$ Slope of $AB=-1$

$$\Rightarrow \quad m\times\frac{-1-5}{2-6}=-1 \Rightarrow m=\frac{-2}{3}$$

Equation of line PM is $y-1=\frac{-2}{3}(x-4)$

$\Rightarrow \quad 2x+3y-11=0 \quad$ ...(i)

The coordinates of M are $\left(\frac{2\lambda+6}{\lambda+1}, \frac{-\lambda+5}{\lambda+1}\right)$

Since, M lies on line PM, so it will satisfies Eq. (i).

$\therefore \quad 2\left(\frac{2\lambda+6}{\lambda+1}\right)+3\left(\frac{-\lambda+5}{\lambda+1}\right)-11=0.$ **Ans.** $\lambda=\frac{8}{5}$

30. (i) **Hint** First, determine the mid-point of BC, then find the equation of line passing through two points using the formula,

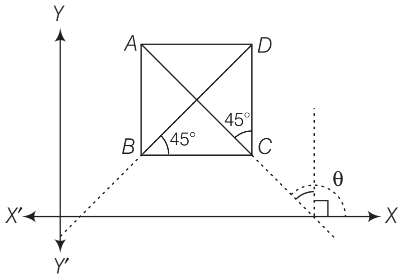
$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1).$$ **Ans.** $y=2$

(ii) **Hint** Now, determine the slope m of line AC, then find the equation of altitude through B.

i.e. $y-y_1=-\frac{1}{m}(x-x_1)$ **Ans.** $2x+y-12=0$

31. (i) **Hint** Line parallel to X-axis. **Ans.** 0

(ii) **Hint** Since, the diagonals of a square bisect the angles, therefore $\angle CBD$, i.e. $\theta=45°$.



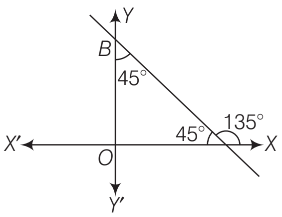
Ans. 1

(iii) **Hint** $\theta=135°$ **Ans.** -1

32. Do same as Example 12 of Topic 1.

Ans. (i) $m=1$ (ii) $(-8,-8)$ (iii) $y=x$

33. **Hint** Slope of a line, $m=\tan 135°$



Now, equation of line AB is $y-3=-1(x-2)$.

Ans. Slope $=-1$; $x+y=5$

34. **Hint** Since, line $x-2y-11=0$ intersect the Y-axis, putting $x=0$, we get $y=-\frac{11}{2}$.

$\therefore$ Coordinates on Y-axis is $\left(0,-\frac{11}{2}\right)$.

Use two-points form of equation of line formula.

Ans. $19x-2y-11=0$

35. (i) Do same as Example 11 of Topic 2.

Ans. $A(4,0)$ and $B(0,-6)$

(ii) **Hint** Use the formula, slope $=\frac{y_2-y_1}{x_2-x_1}$. **Ans.** $1\frac{1}{2}$

(iii) **Hint** Find equation of AB, using point-slope form.

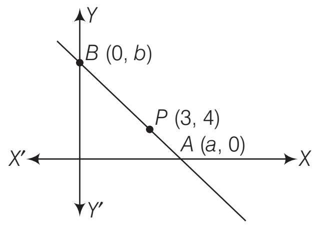
Ans. $2y=2x-12$

36. Do same as Example 14 of Topic 1.

Ans. $2x+3y=10$, $y=2$, $2x-y=2$

37. **Hint** Now, equation of line AB is

$$y-b=\frac{0-b}{a-0}(x-a) \Rightarrow y-b=\frac{-b}{a}(x-a)$$



Since, it passes through the point (3, 4).

$\therefore \quad 4-b=-\frac{b}{a}(3-a) \quad$ ...(i)

Also, given $a+b=14$. ...(ii)

Now, simplify Eqs. (i) and (ii).

Ans. $x+y=7$ and $4x+3y=24$

38. **Hint** First, use the formula,

$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$.

To prove collinear, show that R satisfy the equation of line.

Ans. $x-2y-3=0$

39. **Hint** Compare the given equation with $y=mx+c$. Further, equation of the line parallel to given line and passing through (0, 5) is $y-5=3(x-0)$.

Ans. $3,-5$ and $y=3x+5$

40. **Hint**

Coordinate of $M=\left(\frac{1\times -1+2\times 2}{1+2}, \frac{1\times 2+2\times 5}{1+2}\right)$

$\therefore$ Equation of line joining C and M is

$$y-4=\frac{8-4}{5-1}(x-1)$$

$\Rightarrow \quad y-4=\frac{4}{4}(x-1)$

Ans. $M=(1, 4)$; $y=x+3$

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1 If the straight lines $3x - 5y = 7$ and $4x + ay + 9 = 0$ are perpendicular to one another, find the value of a.

2 $A(2,5)$, $B(-1, 2)$ and $C(5, 8)$ are the vertices of a triangle ABC, 'M' is a point on AB such that $AM : MB = 1:2$. Find the coordinates of 'M'. Hence find the equation of the line passing through the points C and M.

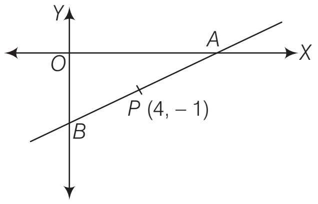
2017

3 $A(-1, 3)$, $B(4, 2)$ and $C(3, -2)$ are the vertices of a triangle.
 (i) Find the coordinates of the centroid G of the triangle.
 (ii) Find the equation of the line through G and parallel to AC.

2016

4 The slope of a line joining $P(6, k)$ and $Q(1-3k, 3)$ is $1/2$. Find
 (i) k. (ii) mid-point of PQ, using the value of 'k' found in part (i).

5 A line AB meets X-axis at A and Y-axis at B. $P(4, -1)$ divides AB in the ratio $1 : 2$.

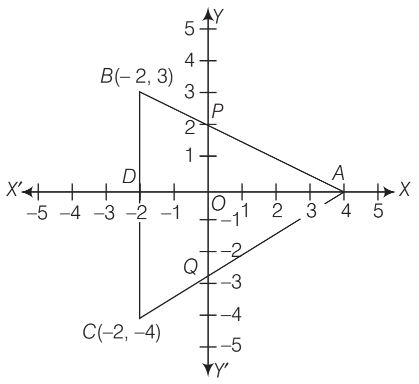


 (i) Find the coordinates of A and B.
 (ii) Find the equation of the line passing through P and perpendicular to AB.

2015

6 Three vertices of a parallelogram $ABCD$ taken in order are $A(3, 6)$, $B(5, 10)$ and $C(3, 2)$. Find
 (i) the coordinates of the fourth vertex D. (ii) the length of diagonal BD.
 (iii) the equation of side AB of the parallelogram $ABCD$.

7 In the given figure, ABC is a triangle and BC is parallel to Y-axis. AB and AC intersect the Y-axis at P and Q, respectively.



(i) Write the coordinates of A.
(ii) Find the length of AB and AC.
(iii) Find the ratio, in which Q divides AC.
(iv) Find the equation of the line AC.

2014

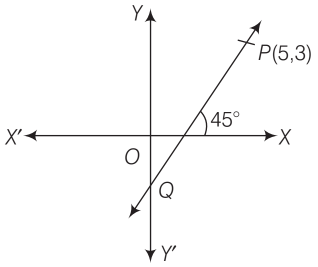
8 Find the value of a, for which the points $A(a, 3)$, $B(2, 1)$ and $C(5, a)$ are collinear. Also, find the equation of the line.

2013

9 In ΔABC, if $A(3, 5)$, $B(7, 8)$ and $C(1, -10)$, then find the equation of the median through A.

2012

10 The line through $P(5, 3)$ intersects Y-axis at Q.



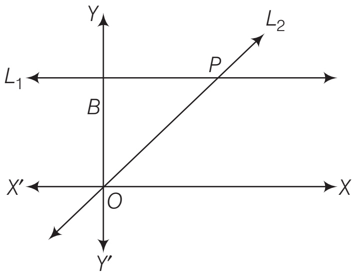
(i) Write the slope of the line.
(ii) Write the equation of the line.
(iii) Find the coordinates of Q.

11 The line passing through $A(-2, 3)$ and $B(4, b)$ is perpendicular to the line $2x - 4y = 5$.

Find the value of b.

2011

12 Given equation of line L_1 is $y = 4$.



(i) Write the slope of line L_2, if L_2 is bisector of $\angle O$.
(ii) Write the coordinates of point P.
(iii) Find the equation of L_2.

13 $ABCD$ is a parallelogram, where $A(x, y)$, $B(5, 8)$, $C(4, 7)$ and $D(2, -4)$. Find

(i) the coordinates of A.
(ii) the equation of diagonal BD.

* All these questions are completely covered in chapter either as solved examples or in topic/chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. The coordinates of the orthocentre of the triangle, whose vertices are $(-1, 3)$, $(2, -1)$ and $(0, 0)$, is

(a) $(4, 3)$ (b) $(-4, -3)$ (c) $(3, 4)$ (d) $(-3, -4)$

2. The point $P(-2, 4)$ is reflected in the line parallel to Y-axis at a distance 3 units to the left of Y-axis onto the point P'. The point Q is reflected in the origin onto the point $Q'(4, -1)$. The equation the line $P'Q$ is

(a) $x+4=0$ (b) $x-4=0$ (c) $y+4=0$ (d) $y-4=0$

3. If the slope of a line passing through the point $A(3, 2)$ is $3/4$, then the point on the line, which are 5 units away from the point A is

(a) $(-1, -1)$ (b) $(7, 5)$ (c) $(7, -5)$ (d) Both (a) and (b)

4. A ray of light coming from the point (1, 2) is reflected at a point A on the X-axis and then passes through the point (5, 3), then the coordinates of the point A is

(a) (0, 1) (b) (0, 13/5) (c) (13/5, 0) (d) (1, 0)

5. A line passes through the point $(3, -2)$. The locus of the middle point of the portion of the line intercepted between axes is

(a) $2x-3y=2xy$ (b) $3y+2x=-2xy$ (c) $2x+3y=2xy$ (d) $3y-2x=2xy$

6. If the image of the point (2, 1) with respect to the line mirror be (5, 2), then the equation of the mirror is

(a) $3x+y-12=0$ (b) $3x-y+12=0$ (c) $3x+y+12=0$ (d) None of these

7. Suppose the equation of the line through the point (0, 2) making an angle $\frac{2\pi}{3}$ with the positive X-axis. Find the equation of line parallel to the given line and below the origin, which is equidistant from origin.

[Given $\tan(\pi-\theta) = -\tan\theta$]

(a) $\sqrt{3}x-y-2=0$ (b) $\sqrt{3}x+y+2=0$ (c) $\sqrt{3}x-y+2=0$ (d) $\sqrt{3}x+y-2=0$

8. Suppose ABC be a triangle, whose centroid is G. If the coordinates of B, C and G are $(-2, 0)$, (3, 1) and (1, 5), then the equations of the sides AB and AC are respectively.

(a) $7x-2y+14=0, 13x+y-40=0$ (b) $7x+2y+14=0$, $x+13y-40=0$

(c) $7x-2y-14=0, 13x+13y-40=0$ (d) None of these

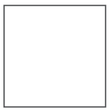
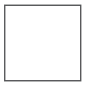
* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 403.

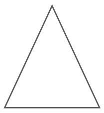
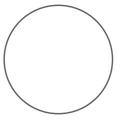
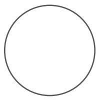
CHAPTER 13

Similarity

Two geometrical figures are said to be similar figures, if they have same shape but not necessarily the **same size**. A shape is said to be similar to other, if the ratios of their corresponding sides are equal and the corresponding angles are equal.

Some examples of similar figures are given below

(i)  and  (ii)  and 

(iii)  and  (iv)  and 

Two polygons of same number of sides are similar, if

(i) all the corresponding angles are equal.

(ii) all the corresponding sides are in the same ratio or proportion.

Comparison between Congruency and Similarity

If two figures are congruent, then it means their shape and size (area) both are equal, but for two similar figures, shape will necessarily be same but their sizes will not be same, rather their sizes will be proportional, i.e. one figure similar to other can be smaller or greater than to other. A figure similar to a given figure can be obtained by enlarging or reducing the given figure in a uniform manner.

e.g. All circles are similar (because the shape of all the circles are same) and can be obtained by reducing or enlarging the given circle.

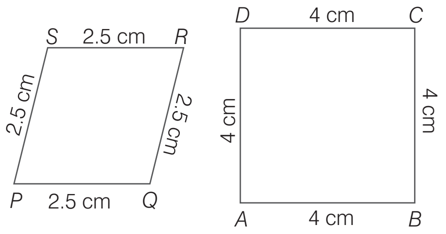
Scale Factor

The ratio that compares the measurements of two similar shapes, is called the scale factor or representative fraction. It is equal to the ratio of corresponding sides of two figures. We can use the ratio of corresponding sides to find unknown sides of similar shapes.

Chapter Objectives

- Similar Triangles and Criteria for Similarity of Two Triangles
- Areas of Similar Triangles, Maps and Models

Example 1. State whether the given quadrilaterals are similar or not.



Sol. Here, $\dfrac{PQ}{AB}=\dfrac{2.5}{4}=\dfrac{25}{10\times4}=\dfrac{5}{8}$

and $\dfrac{RQ}{BC}=\dfrac{2.5}{4}=\dfrac{25}{10\times4}=\dfrac{5}{8}$

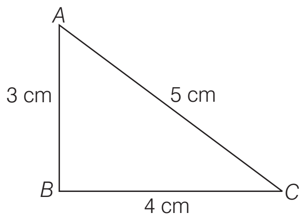
Clearly, the corresponding sides of quadrilaterals $ABCD$ and $PQRS$ are proportional but their corresponding angles are not equal.

Hence, quadrilaterals $ABCD$ and $PQRS$ are not similar.

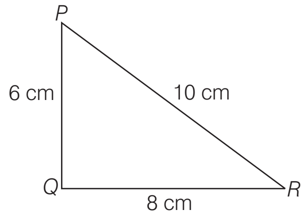
Similarity as a Size Transformation

Size transformation is the process, in which a geometrical figure is enlarged or reduced by a scale factor k, such that the resulting figure (the image) is similar to the given figure.

Consider a ΔABC having sides of length 4 cm, 3 cm and 5 cm.



From the figure it is clear that ΔABC is a right angle triangle, right angled at B. Now, we make one or more right angle triangle PQR by taking the sides double to the given triangle i.e. 6 cm, 8 cm and 10 cm.



We see that shapes of ΔABC and ΔPQR are similar but they have different sizes.

Thus, we can say that, two figures are similar if one can be obtained from the other by size transformation.

Example 2. A $\Delta\, ABC$ has been enlarged by a scale factor $k=2$ to the $\Delta\, PQR$. Calculate

(i) the length of AB, if $PQ=6$ cm.

(ii) the length of PR, if $AC=8$ cm.

Sol. As we know that if k is a scale factor, then each side of the resulting figure = k times the corresponding side of given figure. Therefore, we have

$$PQ=kAB \text{ and } PR=kAC \qquad \ldots\text{(i)}$$

(i) Now, as $PQ=6$ cm, therefore

$$AB=\frac{1}{k}PQ=\frac{1}{2}\times6=3\text{ cm} \qquad \text{[using Eq. (i)]}$$

(ii) As $AC=8$ cm, therefore

$$PR=kAC=2\times8=16\text{ cm} \qquad \text{[using Eq. (i)]}$$

Topic 1
Similar Triangles and Criteria for Similarity of Two Triangles

Similar Triangles

A triangle is a plane figure bounded by three lines. It has three sides and three angles.

Two triangles are said to be similar, if

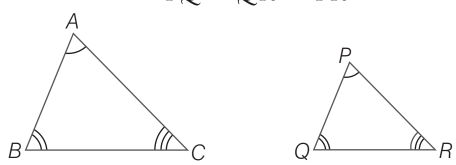
(i) their corresponding angles are equal.

(ii) their corresponding sides are proportional (i.e. the ratios between the lengths of corresponding sides are equal).

e.g. In ΔABC and ΔPQR, if

$$\angle A=\angle P,\quad \angle B=\angle Q,\quad \angle C=\angle R$$

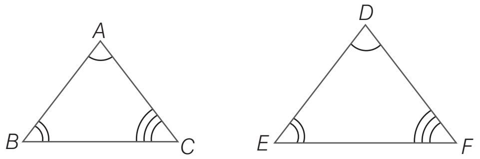
and $$\frac{AB}{PQ}=\frac{BC}{QR}=\frac{AC}{PR}.$$



Then, $\Delta ABC \sim \Delta PQR$, where symbol '$\sim$' is read as 'is similar to'.

Symbolic Expression of Two Similar Triangles

Similarity of two triangles should be expressed symbolically using correct correspondence of their vertices.



e.g. In ΔABC and ΔDEF, $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$, then we should write $\Delta ABC \sim \Delta DEF$, we cannot write $\Delta ABC \sim \Delta EDF$ or $\Delta BAC \sim \Delta DEF$.

Conversely (Converse of Similar Triangles) If ΔABC is similar to ΔPQR, then $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$ and $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$.

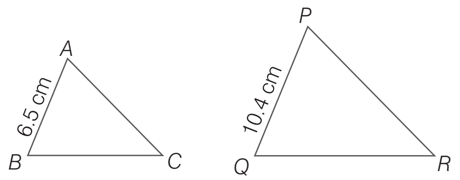
(i) If corresponding angles of two triangles are equal, then they are known as **equiangular triangles.**

(ii) The ratio of any two corresponding sides of two equiangular triangles is always the same.

(iii) All congruent figures are similar but the similar figures need not to be congruent.

Example 3. If $\Delta ABC \sim \Delta PQR$, $AB = 6.5$ cm, $PQ = 10.4$ cm and perimeter of $\Delta ABC = 60$ cm, find the perimeter of ΔPQR.

Sol. Given, $AB = 6.5$ cm and $PQ = 10.4$ cm



Since, $\Delta ABC \sim \Delta PQR$.

$$\therefore \quad \frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = \frac{6.5}{10.4} = \frac{65}{104}$$

[$\because$ corresponding sides of similar triangles are proportional]

i.e. $AB = \frac{65}{104} PQ$, $BC = \frac{65}{104} QR$, $AC = \frac{65}{104} PR$

Also given, perimeter of $\Delta ABC = 60$ cm

$$\therefore \quad AB + BC + AC = 60 \Rightarrow \frac{65}{104}(PQ + QR + PR) = 60$$

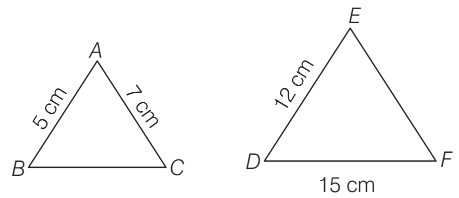
$$\Rightarrow \quad PQ + QR + PR = \frac{60 \times 104}{65} = 96 \text{ cm}$$

Hence, the perimeter of ΔPQR is 96 cm.

Example 4. It is given that $\Delta ABC \sim \Delta EDF$ such that $AB = 5$ cm, $AC = 7$ cm, $DF = 15$ cm and $DE = 12$ cm. Find the lengths of the remaining sides of the triangles.

Sol. Given, $\Delta ABC \sim \Delta EDF$

Also, $AB = 5$ cm, $AC = 7$ cm, $DF = 15$ cm and $DE = 12$ cm ...(i)



Since, $\Delta ABC \sim \Delta EDF$

$$\therefore \quad \frac{AB}{ED} = \frac{AC}{EF} = \frac{BC}{DF}$$

[$\because$ corresponding sides of similar triangles are proportions]

$$\Rightarrow \quad \frac{5}{12} = \frac{7}{EF} = \frac{BC}{15}$$

On taking first and second terms, we get

$$\frac{5}{12} = \frac{7}{EF} \Rightarrow EF = \frac{7 \times 12}{5} = 16.8 \text{ cm}$$

On taking first and third terms, we get

$$\frac{5}{12} = \frac{BC}{15} \Rightarrow BC = \frac{5 \times 15}{12} = 6.25 \text{ cm}$$

Hence, the lengths of remaining sides of the triangles are $EF = 16.8$ cm and $BC = 6.25$ cm.

Criteria for Similarity of Triangles

We have some criteria for similarity of two triangles involve relationship between less number of pairs of corresponding parts of two triangles instead of all the six pairs of corresponding parts. These criteria are discussed below

1. AA Similarity Criterion

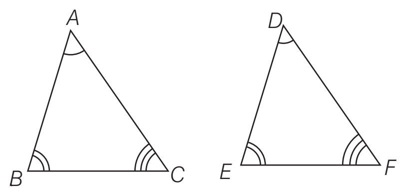
In two triangles, if corresponding two angles are equal, then their corresponding sides are in the same ratio, i.e. they are proportional and hence the two triangles are similar.

This AA similarity criterion also known as AAA (Angle-Angle-Angle) similarity criterion.

AA (Angle-Angle)

In AA (Angle-Angle) criterion for two ΔABC and ΔDEF,

$$\angle A = \angle D, \angle B = \angle E \text{ and } \angle C = \angle F$$



Hence, $\Delta ABC \sim \Delta DEF$.

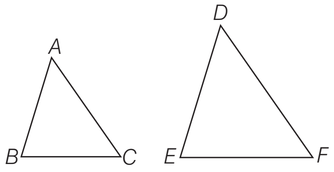
If two angles of a triangle are respectively equal to two angles of another triangle, then by angle sum property of a triangle, their third angle will also be equal and the two triangles will be similar, it is called **AA similarity criterion**.

2. SSS Similarity Criterion

If in two triangles, sides of one triangle are proportional (i.e. in the same ratio) to the sides of the other triangle, then their corresponding angles are equal and hence the two triangles are similar.

In SSS (Side-Side-Side) criterion,

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$



Hence, $\Delta ABC \sim \Delta DEF$.

In case of similarity of two triangles, it is not necessary to check both the conditions (AAA and SSS), because one condition implies the other.

3. SAS Similarity Criterion

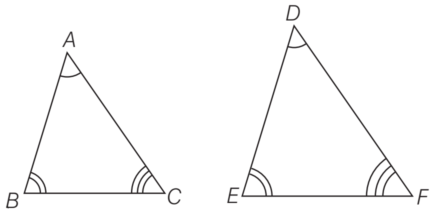
If one angle of a triangle is equal to one angle of the other triangle and the sides including these angles are proportional, then the two triangles are similar.

In SAS (Side-Angle-Side) criterion,

$$\frac{AB}{DE} = \frac{AC}{DF} \text{ and } \angle A = \angle D$$

or $$\frac{AB}{DE} = \frac{BC}{EF} \text{ and } \angle B = \angle E$$

or $$\frac{AC}{DF} = \frac{BC}{EF} \text{ and } \angle C = \angle F$$



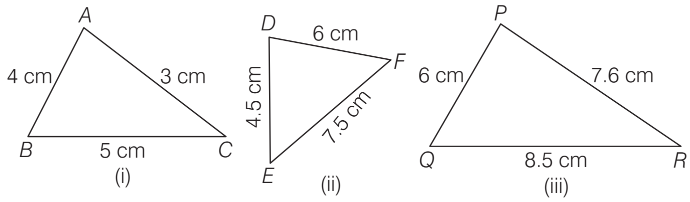
Hence, $\Delta ABC \sim \Delta DEF$.

For proving (or finding) the similarity, we use the criteria, AA, SAS and SSS. While proving (or finding) the congruency, we use the rule SAS, ASA, SSS and RHS.

Problems Based on Similarity of Triangles

There are many problems which can be solved or proved by using these three criterion of similarity. Some of such kind of problems are given below

Example 5. State which pairs of triangles in the given figure are similar? Also, state the similarity criterion used.



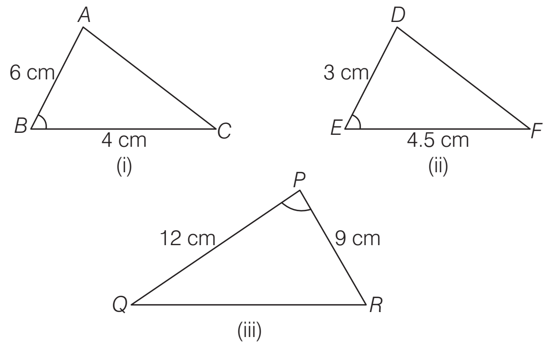
Sol. Here, $\frac{AB}{DF} = \frac{4}{6} = \frac{2}{3}$, $\frac{BC}{EF} = \frac{5}{7.5} = \frac{2}{3}$ and $\frac{AC}{DE} = \frac{3}{4.5} = \frac{2}{3}$

As, $\frac{AB}{DF} = \frac{BC}{EF} = \frac{AC}{DE}$

So, $\Delta ABC \sim \Delta DFE$ [by SSS similarity criterion]

Hence, figures (i) and (ii) are similar triangles, but no other pairs of triangles in the given figure are similar.

Example 6. State which of the two triangles given in the figure are similar. Also, state the similarity criterion used.



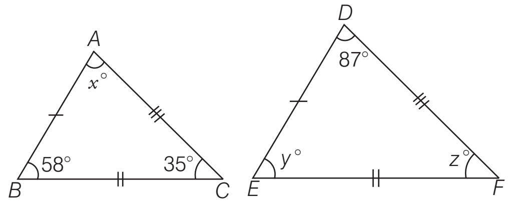
Sol. Here, $\frac{AB}{EF} = \frac{6}{4.5} = \frac{60}{45} = \frac{4}{3}$, $\frac{BC}{DE} = \frac{4}{3}$

So, $\frac{AB}{EF} = \frac{BC}{DE}$ and $\angle ABC = \angle FED$ [given]

$\Delta ABC \sim \Delta FED$ [by SAS similarity criterion]

Hence, figures (i) and (ii) are similar triangles, but no other pairs of triangles in the given figure are similar.

Example 7. Find the value of each of the pronumerals in the given pair of triangles. Give reason for your answer.



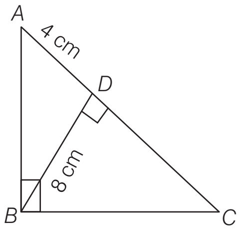
Sol. Given, $AB = DE$, $BC = EF$ and $AC = DF$

$\therefore$ $\Delta ABC \sim \Delta DEF$ [by SSS similarity criterion]

So, $x = 87°$, $y = 58°$ and $z = 35°$

[$\because$ corresponding angles of similar triangles are equal]

Example 8. In the given figure, $\angle ABC = 90°$ and $BD \perp AC$. If $BD = 8$ cm and $AD = 4$ cm, then find the value of CD.



Sol. Given, $BD = 8$ cm and $AD = 4$ cm

In ΔADB and ΔBDC,

$\angle BDA = \angle CDB$ [each 90°]

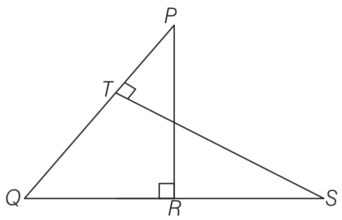
$\angle DBA = \angle DCB$ [each $90° - \angle A$]

$\therefore \quad \Delta ADB \sim \Delta BDC$ [by AA similarity criterion]

$\Rightarrow \quad \dfrac{BD}{CD} = \dfrac{AD}{BD}$ [$\because$ corresponding sides of similar triangles are proportional]

$\Rightarrow \quad CD = \dfrac{BD^2}{AD} = \dfrac{8^2}{4} = \dfrac{64}{4} = 16$ cm

Example 9. In the given figure, PQR and QST are two right angled triangles, right angled at R and T, respectively. Prove that $QR \times QS = QP \times QT$.



Sol. In ΔPRQ and ΔSTQ,

$\angle PRQ = \angle STQ$ [each 90°]

$\angle PQR = \angle SQT$ [common angle]

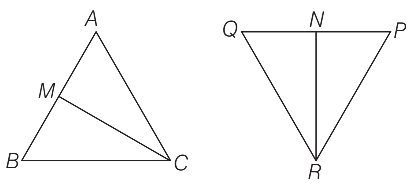
So, $\quad \Delta PRQ \sim \Delta STQ$ [by AA similarity criterion]

Then, $\quad \dfrac{QR}{QT} = \dfrac{QP}{QS}$

[$\because$ corresponding sides of similar triangles are proportional]

$\Rightarrow \quad QR \times QS = QP \times QT$ **Hence proved.**

Example 10. In the following figure, CM and RN are respectively the medians of ΔABC and ΔPQR.
If $\Delta ABC \sim \Delta PQR$, prove that



(i) $\Delta AMC \sim \Delta PNR$ (ii) $\dfrac{CM}{RN} = \dfrac{AB}{PQ}$

Sol.

(i) Given, $\Delta ABC \sim \Delta PQR$

$\therefore \quad \dfrac{AB}{PQ} = \dfrac{BC}{QR} = \dfrac{AC}{PR}$ ...(i)

and $\quad \angle A = \angle P, \angle B = \angle Q$ and $\angle C = \angle R$ ...(ii)

We know that the median bisects the opposite side.

$\therefore \quad AM = MB \Rightarrow AB = 2\,AM$ and $PN = NQ \Rightarrow PQ = 2PN$

From Eq. (i), we have

$\dfrac{AB}{PQ} = \dfrac{AC}{PR} \Rightarrow \dfrac{2AM}{2PN} = \dfrac{AC}{PR}$

$\Rightarrow \quad \dfrac{AM}{PN} = \dfrac{AC}{PR}$ ...(iii)

In ΔAMC and ΔPNR, $\angle A = \angle P$ [from Eq. (ii)]

and $\quad \dfrac{AM}{PN} = \dfrac{AC}{PR}$ [from Eq. (iii)]

So, $\quad \Delta AMC \sim \Delta PNR$ [by SAS similarity criterion]

(ii) We have, $\Delta AMC \sim \Delta PNR \Rightarrow \dfrac{AM}{PN} = \dfrac{AC}{PR} = \dfrac{CM}{RN}$

[$\because$ triangles are similar, hence corresponding sides will be proportional]

$\therefore \quad \dfrac{CM}{RN} = \dfrac{AC}{PR} \Rightarrow \dfrac{CM}{RN} = \dfrac{AM}{PN}$ [from Eq. (iii)]

$\Rightarrow \quad \dfrac{CM}{RN} = \dfrac{2AM}{2PN}$

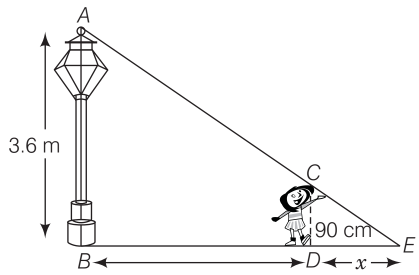
[multiplying numerator and denominator by 2 in RHS]

$\Rightarrow \quad \dfrac{CM}{RN} = \dfrac{AB}{PQ}$ [$\because AB = 2AM$ and $PQ = 2PN$]... (iv)

Example 11. A girl of height 90 cm is walking away from the base of a lamp-post at a speed of 1.2 m/s. If the lamp is 3.6 m above the ground, then find length of her shadow after 4s.

Sol. Let AB be the lamp-post, CD be the girl and D be the position of girl after 4 s.

Again, let $DE = x$ m be the length of shadow of the girl.



Given, $CD = 90$ cm $= 0.9$ m, $AB = 3.6$ m
and speed of the girl $= 1.2$ m/s

$\therefore$ Distance of the girl from lamp-post after 4 s,

$BD = 1.2 \times 4 = 4.8$ m [$\because$ distance = speed × time]

In ΔABE and ΔCDE,

$\angle B = \angle D$ [each 90°]

$\angle E = \angle E$ [common angle]

$\therefore \quad \Delta ABE \sim \Delta CDE$ [by AA similarity criterion]

$\Rightarrow \quad \dfrac{BE}{DE} = \dfrac{AB}{CD}$ ...(i)

[$\because$ corresponding sides of similar triangles are proportional]

On putting all the values in Eq. (i), we get

$$\frac{4.8+x}{x} = \frac{3.6}{0.9} \quad [\because BE = BD + DE = 4.8 + x]$$

$$\Rightarrow \quad \frac{4.8+x}{x} = 4 \Rightarrow 4.8 + x = 4x$$

$$\Rightarrow \quad 3x = 4.8 \Rightarrow x = \frac{4.8}{3} = 1.6\text{ m}$$

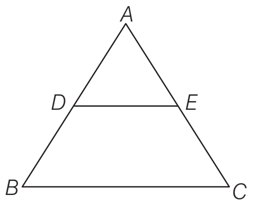
Hence, the length of her shadow after 4s is 1.6 m.

Basic Proportionality Theorem (BPT)

Here, we shall learn Basic Proportionality Theorem (BPT), its converse and their applications.

Theorem 1 **(Basic Proportionality Theorem)** *If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, then the other two sides are divided in the same ratio. This theorem is also known as Thales Theorem.*

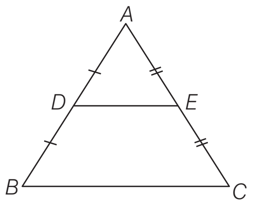
In the following figure $DE \parallel BC$, so $\frac{AD}{DB} = \frac{AE}{EC}$



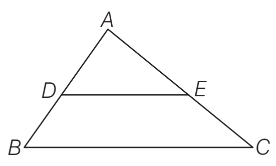
Theorem 2 **(Converse of Basic Proportionality Theorem)** *If a line divides any two sides of a triangle in the same ratio, then the line must be parallel to the third side.*

In the following figure, $\frac{AD}{DB} = \frac{AE}{EC}$

So, $DE \parallel BC$



Example 12. In the given figure, $DE \parallel BC$. If $AD = 3$ cm, $DB = 4$ cm and $AE = 6$ cm, find EC.



Sol. Given, in ΔABC, $DE \parallel BC$

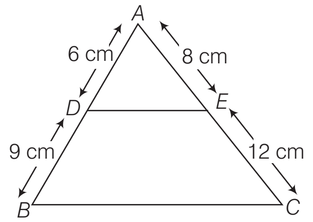
$$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC} \quad \text{[by basic proportionality theorem]}$$

$$\Rightarrow \quad \frac{3}{4} = \frac{6}{EC} \quad [\text{given, } AD = 3\text{cm}, DB = 4\text{cm and } AE = 6\text{cm}]$$

$$\therefore \quad EC = 8\text{cm}$$

Example 13. If D and E are points on the respective sides AB and AC of ΔABC such that $AD = 6$ cm, $BD = 9$ cm, $AE = 8$ cm, $EC = 12$ cm. Prove that $DE \parallel BC$.

Sol. In ΔABC, $\frac{AD}{DB} = \frac{6}{9} = \frac{2}{3}$ and $\frac{AE}{EC} = \frac{8}{12} = \frac{2}{3}$



So, $\frac{AD}{DB} = \frac{AE}{EC}$

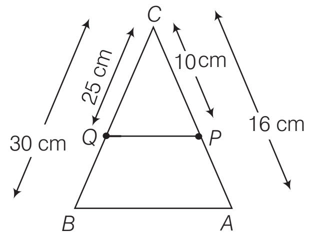
Hence, $DE \parallel BC$.

[by converse of basic proportionality theorem]

Hence proved.

Example 14. In ΔABC, points P and Q are on CA and CB, respectively such that $CA = 16$ cm, $CP = 10$ cm, $CB = 30$ cm and $CQ = 25$ cm. Is $PQ \parallel AB$?

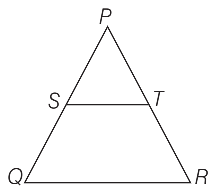
Sol. Given, $CQ = 25$cm, $CB = 30$cm, $CP = 10$cm and $CA = 16$ cm.



Now, $\frac{CQ}{CB} = \frac{25}{30} = \frac{5}{6}$ and $\frac{CP}{CA} = \frac{10}{16} = \frac{5}{8} \Rightarrow \frac{CQ}{CB} \neq \frac{CP}{CA}$

Then, by converse of basic proportionality theorem, PQ is not parallel to AB.

Example 15. In ΔPQR, if $ST \parallel QR$, $\frac{PS}{SQ} = \frac{3}{5}$ and $PR = 28$ cm then find PT.



Sol. Given, $ST \parallel QR$, $PS/SQ = 3/5$ and $PR = 28$cm

By using basic proportionality theorem, we get

$$\frac{PS}{SQ} = \frac{PT}{TR} \Rightarrow \frac{PS}{SQ} = \frac{PT}{PR - PT}$$

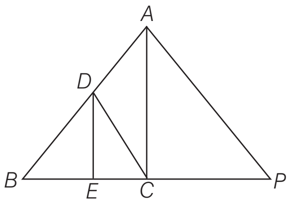
$$\Rightarrow \quad \frac{3}{5} = \frac{PT}{28 - PT} \Rightarrow 3(28 - PT) = 5PT$$

$$\Rightarrow \quad 84 = 5PT + 3PT$$

$$\Rightarrow \quad PT = \frac{84}{8} \Rightarrow PT = 10.5\text{ cm}$$

Hence, the length of PT is 10.5 cm.

Example 16. In the given figure, $DE \parallel AC$. If $DC \parallel AP$, where point P lies on BC produced, then prove that $\frac{BE}{EC} = \frac{BC}{CP}$.



Sol. Given, in ΔABC, $DE \parallel AC$.

So, $\frac{BE}{EC} = \frac{BD}{DA}$ [by basic proportionality theorem] ...(i)

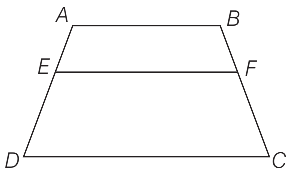
Also, in ΔAPB, $DC \parallel AP$ [given]

So, $\frac{BC}{CP} = \frac{BD}{DA}$ [by basic proportionality theorem] ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{BE}{EC} = \frac{BC}{CP}$$ **Hence proved.**

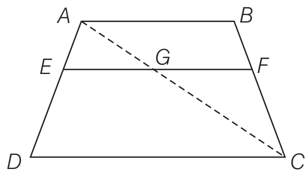
Example 17. In the given figure, $ABCD$ is a trapezium with $AB \parallel DC$. E and F are two points on non-parallel sides AD and BC respectively, such that EF is parallel to AB. Show that $\frac{AE}{ED} = \frac{BF}{FC}$.



Sol. **Given** In trapezium $ABCD$, $AB \parallel DC$ and $EF \parallel AB$

To prove $\frac{AE}{ED} = \frac{BF}{FC}$

Construction Join AC to intersect EF at G.



Proof Since, $AB \parallel DC$ and $EF \parallel AB$

$\therefore$ $EF \parallel DC$ [$\because$ lines parallel to the same line are also parallel to each other]

In ΔADC, $EG \parallel DC$ [$\because EF \parallel DC$]

By using basic proportionality theorem, $\frac{AE}{ED} = \frac{AG}{GC}$ ...(i)

In ΔABC, $GF \parallel AB$ [$\because EF \parallel AB$]

By using basic proportionality theorem,

$\frac{CG}{AG} = \frac{CF}{BF}$ or $\frac{AG}{GC} = \frac{BF}{CF}$ [reciprocal the terms] ...(ii)

From Eqs. (i) and (ii), we get $\frac{AE}{ED} = \frac{BF}{FC}$ **Hence proved.**

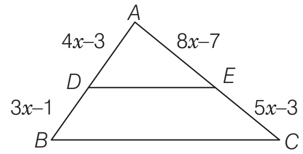
Example 18. In ΔABC, D and E are points on the sides AB and AC respectively, such that $DE \parallel BC$. If $AD = 4x - 3$, $AE = 8x - 7$, $BD = 3x - 1$ and $CE = 5x - 3$, then find the value of x.

Sol. Given, in ΔABC, $DE \parallel BC$

By Thales theorem, we get

$$\frac{AD}{DB} = \frac{AE}{EC} \Rightarrow \frac{4x-3}{3x-1} = \frac{8x-7}{5x-3}$$

[$\because AD = 4x - 3$, $DB = 3x - 1$, $AE = 8x - 7$, $EC = 5x - 3$]



$\Rightarrow$ $(4x-3)(5x-3) = (8x-7)(3x-1)$

$\Rightarrow$ $20x^2 - 12x + 9 - 15x = 24x^2 - 21x - 8x + 7$

$\Rightarrow$ $4x^2 - 2x - 2 = 0$

$\Rightarrow$ $2x^2 - x - 1 = 0$ [dividing both sides by 2]

$\Rightarrow$ $2x^2 - 2x + x - 1 = 0$ [by splitting the middle term]

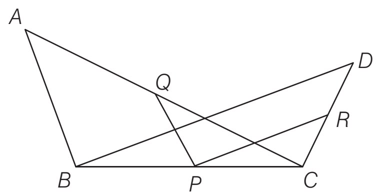
$\Rightarrow$ $2x(x-1) + 1(x-1) = 0 \Rightarrow (2x+1)(x-1) = 0$

$\therefore$ $x = -\frac{1}{2}$ or $x = 1$

If $x = -\frac{1}{2}$, then $AD = 4 \times -\frac{1}{2} - 3 = -5 < 0$ [not possible]

Hence, $x = 1$ is the required value.

Example 19. In the given figure, ΔABC and ΔDBC have same base BC and lie on the same side of BC. If $PQ \parallel BA$ and $PR \parallel BD$, then prove that $QR \parallel AD$.



Sol. **Given** ΔABC and ΔDBC have same base BC and lie on the same side of BC.

To prove $QR \parallel AD$

Construction Join QR and AD.

Proof In ΔABC, $PQ \parallel AB$

So, $\frac{CP}{PB} = \frac{CQ}{QA}$ [by basic proportionality theorem]...(i)

In ΔBCD, $PR \parallel BD$

So, $\frac{CP}{PB} = \frac{CR}{RD}$ [by basic proportionality theorem]...(ii)

From Eqs. (i) and (ii), $\frac{CQ}{QA} = \frac{CR}{RD}$

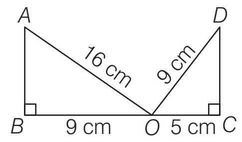
Hence, $QR \parallel AD$.

[by converse of basic proportionality theorem]

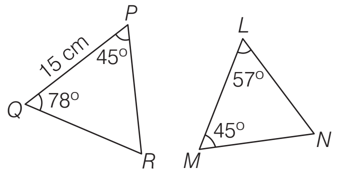
Hence proved.

Topic Exercise 1

1. State whether the given pairs of triangles are similar or not. In case of similarity, mention the criterion.

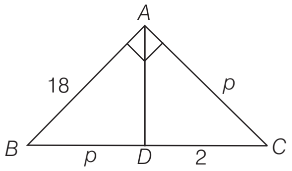


(i)

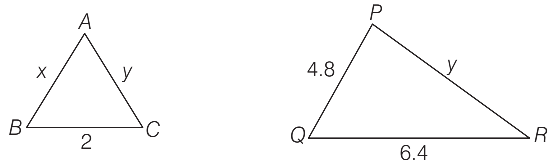


(ii)

2. In the given figure, if $\Delta ADB \sim \Delta ADC$, then find the value of p.

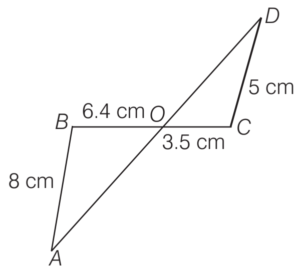


3. Find the value of unknown variables, if ΔABC and $\Delta\, PQR$ are similar.

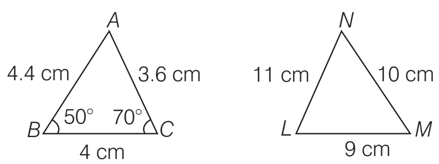


4. If $\Delta ABC \sim \Delta DFE$, $\angle A = 30°$, $\angle C = 50°$, $AB = 5$ cm, $AC = 8$ cm and $DF = 7.5$ cm, then find DE and $\angle F$.

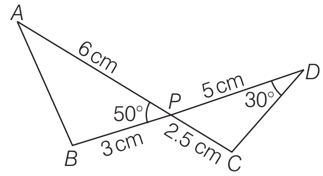
5. In the given figure, $\Delta\, OAB \sim \Delta OCD$. If $AB = 8$ cm, $BO = 6.4$ cm, $OC = 3.5$ cm and $CD = 5$ cm, then find the values of OA and DO.



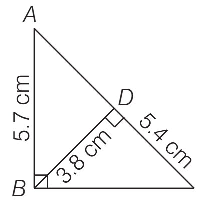
6. In the given figures, find $\angle MLN$.



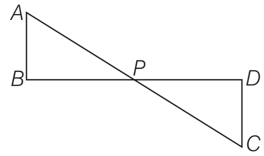
7. In the given figure, two line segments AC and BD intersect each other at the point P such that $PA = 6$ cm, $PB = 3$ cm, $PC = 2.5$ cm, $PD = 5$ cm, $\angle APB = 50°$ and $\angle CDP = 30°$. Then, find $\angle PBA$.



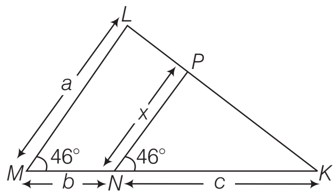
8. In the given figure, $\angle ABC = 90°$ and $BD \perp AC$. If $AB = 5.7$ cm, $BD = 3.8$ cm and $CD = 5.4$ cm, find the value of BC.



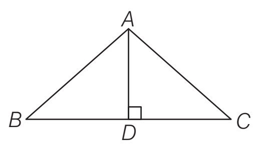
9. In the given figure, if $\angle A = \angle C$, $AB = 6$ cm, $BP = 15$ cm, $AP = 12$ cm and $CP = 4$ cm, find the lengths of PD and CD.



10. In the given figure, $\angle M = \angle N = 46°$. Express x in terms of a, b and c, where a, b and c are the lengths of LM, MN and NK, respectively.



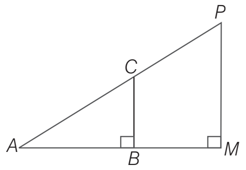
11. In the given figure, if $\angle BAC = 90°$ and $AD \perp BC$, prove that $AD^2 = BD \cdot CD$.



12. A 15 m high tower casts a shadow 24 m long at a certain time and at the same time, telephone pole casts a shadow 16 m long. Find the height of the telephone pole.

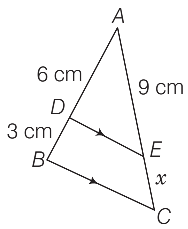
13. P and Q are the points on sides AB and AC, respectively of ΔABC. If $AP = 3$ cm, $PB = 6$ cm, $AQ = 5$ cm and $QC = 10$ cm, show that $BC = 3PQ$.

14. ΔABC and ΔAMP are two right angled triangles, right angled at B and M, respectively. Prove that $CA \times MP = PA \times BC$.



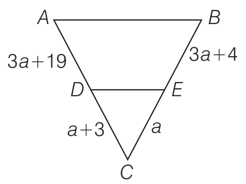
15. In a ΔABC, P and Q are points in AB and AC, respectively and $PQ \parallel BC$. Prove that the median bisects PQ.

16. In the given figure, find the value of x, if $DE \parallel BC$.

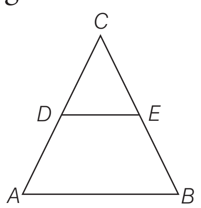


17. In ΔABC, P and Q are points on AB and AC such that $PQ \parallel BC$. If $AP = 4$ cm, $AQ = 3$ cm and $PB = 2$ cm, then find the value of AC.

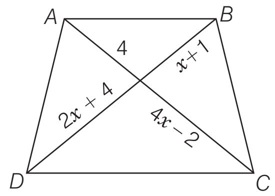
18. What value of a will make $DE \parallel AB$ in the figure given below?



19. In the given figure, if $\dfrac{AD}{DC} = \dfrac{BE}{EC}$ and $\angle CDE = \angle CED$, then prove that ΔCAB is an isosceles triangle.



20. In the given figure, if $AB \parallel CD$, then find the value of x.



Hints and Answers

1. (i) **Hint** Corresponding sides are not in proportion.
Ans. Not similar

(ii) **Hint** From ΔLMN,

$$\angle LMN + \angle MNL + \angle MLN = 180°$$
$$\Rightarrow \quad 45° + \angle MNL + 57° = 180°$$
$$\Rightarrow \quad \angle MNL = 78°$$
$$\therefore \quad \angle QPR = \angle LMN = 45°$$
$$\text{and} \quad \angle PQR = \angle MNL = 78°$$

So, $\Delta PQR \sim \Delta MNL$ [by AA similarity criterion]

Ans. Yes similar

2. **Hint** As, $\Delta ADB \sim \Delta ADC$

$$\Rightarrow \quad \frac{AB}{AC} = \frac{DB}{DC} \Rightarrow \frac{18}{p} = \frac{p}{2}$$

Ans. $p = 6$ cm

3. **Hint** As, ΔABC and ΔPQR are similar,

$$\text{i.e.} \quad \frac{AB}{PQ} = \frac{BC}{QR} \Rightarrow \frac{x}{4.8} = \frac{2}{6.4} \Rightarrow x = 1.5$$

$$\text{and} \quad \frac{AC}{PR} = \frac{BC}{QR} \Rightarrow \frac{4}{y} = \frac{2}{6.4} \Rightarrow y = 12.8$$

Ans. $x = 1.5$ cm and $y = 12.8$ cm

4. **Hint** As $\Delta ABC \sim \Delta DFE$

$$\angle D = \angle A = 30°, \ \angle C = \angle E = 50°$$
$$\therefore \quad \angle B = \angle F = 180° - (50° + 30°) = 100°$$
$$\text{Now,} \quad \frac{AB}{DF} = \frac{AC}{DE} \Rightarrow \frac{5}{7.5} = \frac{8}{DE} \Rightarrow DE = 12 \text{ cm}$$

Ans. $DE = 12$ cm, $\angle F = 100°$

5. **Hint** As $\Delta OAB \sim \Delta OCD$,

$$\frac{OA}{OC} = \frac{AB}{CD} = \frac{BO}{OD} \Rightarrow \frac{OA}{3.5} = \frac{8}{5} = \frac{6.4}{DO}$$

Ans. $OA = 5.6$ cm, $DO = 4$ cm

6. **Hint** First prove $\Delta ABC \sim \Delta LNM$ by SSS similarity criterion, then find $\angle MLN$. **Ans.** 60°.

7. **Hint** Prove $\Delta APB \sim \Delta DPC$ by SAS similarity criterion. **Ans.** 100°.

8. Do same as Example 8. **Ans.** $BC = 8.1$ cm

9. **Hint** First prove $\Delta APB \sim \Delta CPD$ by SAS similarity criterion, then find PD and CD.
Ans. $PD = 5$ cm, $CD = 2$ cm

10. **Hint** ΔPNK and ΔLMK, are similar by AA similarity criterion.

$$\frac{NK}{MK} = \frac{PN}{LM} \Rightarrow \frac{c}{b+c} = \frac{x}{a}$$

$$\Rightarrow \quad x = \frac{ac}{b+c}$$

11. **Hint** $\Delta ADB \sim \Delta ADC$, then $\dfrac{BD}{AD} = \dfrac{AD}{CD}$

$$\Rightarrow \quad AD^2 = BD \cdot CD.$$

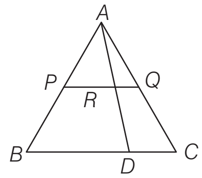
12. Do same as Example 11.
Ans. 10 m

13. **Hint** Here, $\dfrac{AP}{AB} = \dfrac{AQ}{AC} = \dfrac{1}{3}$ and $\angle A = \angle A$ common, therefore ΔAPQ and ΔABC are similar by SAS similarity criterion,

i.e. $\dfrac{AP}{AB} = \dfrac{AQ}{AC} = \dfrac{PQ}{BC} = \dfrac{1}{3} \Rightarrow BC = 3PQ$

14. **Hint** Prove ΔABC and ΔAMP are similar by AA similarity criterion, take ratio $\dfrac{AC}{AP} = \dfrac{BC}{MP}$.

15. **Hint** In ΔAPR and ΔABD, $\angle APR = \angle ABD$



$$\angle ARP = \angle ADB$$

$\therefore \quad \Delta APR \sim \Delta ABD$ [by AA similarity criterion]

$$\frac{PR}{BD} = \frac{AR}{AD} \quad \ldots(i)$$

Similarly, we can prove $\dfrac{AR}{AD} = \dfrac{RQ}{DC}$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{PR}{BD} = \frac{RQ}{DC} \Rightarrow PR = RQ$$

16. Do same as Example 12. **Ans.** $x = 4.5$ cm

17. Do same as Example 12. **Ans.** $AC = 4.5$ cm

18. Do same as Example 18. **Ans.** $a = 2$

19. **Hint** In ΔABC, $\dfrac{AD}{DC} = \dfrac{BE}{EC}$

$\Rightarrow \quad DE \parallel AB$ [by converse of BPT]

$\therefore \quad \angle CDE = \angle A$ and $\angle CED = \angle B$

Since, $\angle CDE = \angle CED$

$\angle A = \angle B \Rightarrow CB = CA$

$\therefore$ ΔCAB is an isosceles triangle.

20. **Hint** Use the result 'the diagonals of a trapezium divide each other proportionally.' **Ans.** $x = 3$

Topic 2
Areas of Similar Triangles, Maps and Models

We have learnt that the corresponding sides of two similar triangles are in the same ratio and corresponding altitudes of two similar triangles are in the same ratio as the ratio of their corresponding sides.

So, the ratio of the areas of two similar triangles is the same as the ratio of the squares of their corresponding sides.

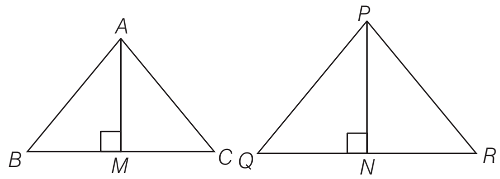
Now, we dicuss about these theorem given below

Theorem 1 *The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.*

Given $\Delta ABC \sim \Delta PQR$

To prove $\dfrac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta PQR)} = \left(\dfrac{AB}{PQ}\right)^2 = \left(\dfrac{AC}{PR}\right)^2 = \left(\dfrac{BC}{QR}\right)^2$

Construction Draw altitudes AM and PN of ΔABC and ΔPQR.



Proof Since, $\Delta ABC \sim \Delta PQR$

$\therefore \quad \angle A = \angle P, \quad \angle B = \angle Q$ and $\angle C = \angle R$ ...(i)

Also, $\dfrac{AB}{PQ} = \dfrac{AC}{PR} = \dfrac{BC}{QR}$ ...(ii)

[$\because$ in similar triangles, corresponding angles are equal and corresponding sides are in the same ratio]

In ΔABM and ΔPQN,

$$\angle B = \angle Q \quad \text{[from Eq. (i)]}$$

$$\angle AMB = \angle PNQ \quad \text{[each 90°]}$$

$\therefore$ $\Delta ABM \sim \Delta PQN$ [by AA similarity criterion]

So, $$\frac{AM}{PN} = \frac{AB}{PQ} = \frac{BM}{QN} \quad \text{...(iii)}$$

[$\because$ in similar triangles, corresponding sides are in the same ratio]

Since, AM and PN are altitudes.

$\therefore$ $\text{ar}(\Delta ABC) = 1/2 \times BC \times AM$ ...(iv)

and $\text{ar}(\Delta PQR) = 1/2 \times QR \times PN$ ...(v)

On dividing Eq. (iv) by Eq. (v), we get

$$\frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta PQR)} = \frac{BC \times AM}{QR \times PN}$$

$$= \frac{BC}{QR} \times \frac{AM}{PN} = \frac{AB}{PQ} \times \frac{AB}{PQ} \quad \text{[from Eqs. (ii) and (iii)]}$$

$$= (AB/PQ)^2 \quad \text{...(vi)}$$

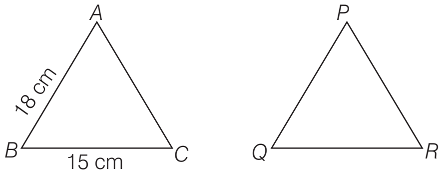
From Eqs. (ii) and (vi), we get

$$\frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{AC}{PR}\right)^2 = \left(\frac{BC}{QR}\right)^2$$

Hence proved.

Example 1. If $\Delta ABC \sim \Delta QRP$, $\dfrac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta QRP)} = \dfrac{9}{4}$, $AB = 18$ cm and $BC = 15$ cm, then find PR.

Sol. Given, $\Delta ABC \sim \Delta QRP$, $AB = 18$ cm, $BC = 15$ cm



We know that the ratio of areas of two similar triangles is equal to the ratio of squares of their corresponding sides.

$\therefore$ $$\frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta QRP)} = \frac{(BC)^2}{(RP)^2}$$

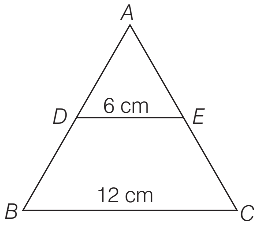
But $$\frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta QRP)} = \frac{9}{4} \quad \text{[given]}$$

$\therefore$ $$\frac{(15)^2}{(RP)^2} = \frac{9}{4} \quad [\because BC = 15 \text{ cm}]$$

$\Rightarrow$ $$(RP)^2 = \frac{225 \times 4}{9} = 100$$

$\therefore$ $RP = 10$ cm [taking positive square root]

Example 2. In the given figure, if $DE \parallel BC$, find the ratio of ar (ΔADE) and ar ($DECB$).



Sol. Given, $DE \parallel BC$, $DE = 6$ cm and $BC = 12$ cm.

In ΔABC and ΔADE,

$$\angle ABC = \angle ADE \quad \text{[corresponding angles]}$$

$$\angle ACB = \angle AED \quad \text{[corresponding angles]}$$

and $$\angle A = \angle A \quad \text{[common angle]}$$

$\therefore$ $\Delta ABC \sim \Delta ADE$ [by AAA similarity criterion]

We know that the ratio of areas of two similar triangles is equal to the ratio of squares of their corresponding sides.

$\therefore$ $$\frac{\text{ar}(\Delta ADE)}{\text{ar}(\Delta ABC)} = \frac{(DE)^2}{(BC)^2} = \frac{(6)^2}{(12)^2} = \left(\frac{1}{2}\right)^2$$

$\Rightarrow$ $$\frac{\text{ar}(\Delta ADE)}{\text{ar}(\Delta ABC)} = \frac{1}{4}$$

Let ar (ΔADE) $= k$, then ar (ΔABC) $= 4k$

Now, ar ($DECB$) = ar (ΔABC) − ar(ΔADE)

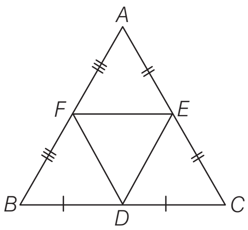
$= 4k - k = 3k$

$\therefore$ Required ratio = ar (ΔADE) : ar ($DECB$)

$= k : 3k = 1 : 3$

Example 3. D, E and F are the mid-points of the sides BC, CA and AB, respectively of ΔABC. Determine the ratio of the area of ΔDEF and ΔABC.

Sol. Since, D and E are the mid-points of the sides BC and CA, respectively of ΔABC.



Therefore, $DE \parallel BA \Rightarrow DE \parallel BF$

[by mid-point theorem] ...(i)

Since, F and E are the mid-points of AB and AC, respectively of ΔABC.

$\therefore$ $FE \parallel BC \Rightarrow FE \parallel BD$ ...(ii)

From Eqs. (i) and (ii), we conclude that $BDEF$ is a parallelogram.

$\therefore$ $\angle B = \angle DEF$ [opposite angles of a parallelogram] ...(iii)

Similarly, $AFDE$ is a parallelogram.

$\therefore \angle A = \angle FDE$ [opposite angles of a parallelogram]...(iv)

In ΔABC and ΔDEF, we have

$\angle B = \angle DEF$ [from Eq. (iii)]

$\angle A = \angle FDE$ [from Eq. (iv)]

$\therefore \quad \Delta ABC \sim \Delta\, DEF$ [by AA similarity criterion]

Since, the ratio of the area of two similar triangles is equal to the ratio of the square of any two corresponding sides.

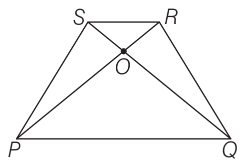
$$\therefore \quad \frac{\text{ar}(\Delta DEF)}{\text{ar}(\Delta ABC)} = \frac{DE^2}{AB^2} = \frac{\left(\frac{AB}{2}\right)^2}{AB^2} = \frac{1}{4}$$

[$\because$ mid-point theorem, $DE = \frac{1}{2}AB$]

Hence, ar (ΔDEF): ar $(\Delta ABC) = 1 : 4$.

Example 4. Diagonals of a trapezium $PQRS$ intersect each other at the point O, $PQ \parallel RS$ and $PQ = 3RS$. Find the ratio of the areas of ΔPOQ and ΔROS.

Sol. Given, $PQRS$ is a trapezium in which $PQ \parallel RS$ and $PQ = 3RS$.



$$\Rightarrow \quad \frac{PQ}{RS} = \frac{3}{1} \quad \text{...(i)}$$

In ΔPOQ and ΔROS,

$\angle SOR = \angle QOP$ [vertically opposite angles]

$\angle SRP = \angle RPQ$ [alternate angles]

$\therefore \quad \Delta POQ \sim \Delta ROS$ [by AA similarity criterion]

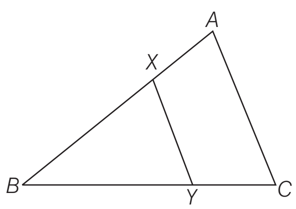
By property of area of similar triangle,

$$\frac{\text{ar}(\Delta POQ)}{\text{ar}(\Delta ROS)} = \frac{(PQ)^2}{(RS)^2} = \left(\frac{3}{1}\right)^2 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \frac{\text{ar}(\Delta POQ)}{\text{ar}(\Delta SOR)} = \frac{9}{1}$$

Hence, the required ratio is 9 : 1.

Example 5. In the given figure, the line segment XY is parallel to side AC of ΔABC and it divides the triangle into two parts of equal areas. Find the ratio $\frac{AX}{AB}$.



Sol. Given, $XY \parallel AC$

Then, $\angle BXY = \angle BAC$ [corresponding angles]

and $\angle BYX = \angle BCA$ [corresponding angles]

$\therefore \quad \Delta ABC \sim \Delta XBY$ [by AA similarity criterion]

Then, $$\frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta XBY)} = \left(\frac{AB}{XB}\right)^2 \quad \text{...(i)}$$

[$\because$ ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides]

Also, given that XY divides the triangle into two parts of equal areas.

$$\therefore \quad \frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta XBY)} = \frac{2}{1} \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$\left(\frac{AB}{XB}\right)^2 = \frac{2}{1} \Rightarrow \frac{AB}{XB} = \sqrt{\left(\frac{2}{1}\right)} = \frac{\sqrt{2}}{1}$$

[taking positive square root]

$$\Rightarrow \quad \frac{XB}{AB} = \frac{1}{\sqrt{2}} \quad \text{[reciprocal the terms]}$$

On multiplying both sides by -1 and then adding 1 on both sides, we get

$$1 - \frac{XB}{AB} = 1 - \frac{1}{\sqrt{2}} \Rightarrow \frac{AB - XB}{AB} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{AX}{AB} = \frac{\sqrt{2}-1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

[multiplying numerator and denominator by $\sqrt{2}$ in RHS]

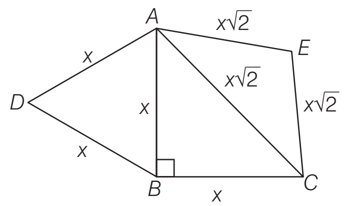
$$\Rightarrow \quad \frac{AX}{AB} = \frac{2-\sqrt{2}}{2}$$

which is the required ratio.

Example 6. Prove that the area of the equilateral triangle described on the side of an isosceles right angled triangle is half the area of the equilateral triangle described on its hypotenuse.

Sol. **Given** A ΔABC in which $\angle ABC = 90°$ and $AB = BC$.

ΔABD and ΔCEA are equilateral triangles.



To prove ar $(\Delta ABD) = \frac{1}{2}$ ar (ΔCEA)

Proof Let $AB = BC = x$ units

Now, $CA = \sqrt{AB^2 + BC^2} = \sqrt{x^2 + x^2} = x\sqrt{2}$ units

[by Pythagoras theorem]

In ΔABD and ΔCEA, each angle is 60° as they are equilateral triangle.

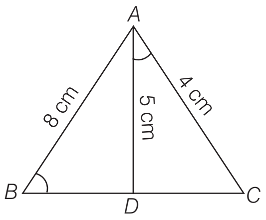
$\therefore \quad \Delta ABD \sim \Delta CEA$

Since, the ratio of the area of two similar triangles is equal to the ratio of the squares of their corresponding sides.

$$\therefore \quad \frac{\text{ar}(\Delta ABD)}{\text{ar}(\Delta CEA)} = \frac{AB^2}{CE^2} = \frac{x^2}{(x\sqrt{2})^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

Hence, ar $(\Delta ABD) = \frac{1}{2}$ ar (ΔCEA).

Example 7. In ΔABC, $\angle ABC = \angle DAC$, $AB = 8$ cm, $AC = 4$ cm and $AD = 5$ cm.



(i) Prove that ΔACD is similar to ΔBCA.
(ii) Find BC and CD.
(iii) Find area of ΔACD : area of ΔABC. ***[2014]***

Sol. **Given** $\angle ABC = \angle DAC$, $AB = 8$ cm, $AC = 4$ cm, $AD = 5$ cm

(i) **To prove** $\Delta ACD \sim \Delta BCA$

Proof In ΔACD and ΔBCA,

$\angle ACD = \angle BCA$ [common angle]

and $\angle ABC = \angle DAC$ [given]

$\therefore$ $\Delta ACD \sim \Delta BCA$ [by AA similarity criterion]

Hence proved.

(ii) Since, $\Delta ACD \sim \Delta BCA$

$$\therefore \quad \frac{AC}{BC} = \frac{CD}{CA} = \frac{AD}{BA}$$

[$\because$ if two triangles are similar, then their corresponding sides are proportional]

$$\Rightarrow \quad \underset{\text{I}}{\frac{4}{BC}} = \underset{\text{II}}{\frac{CD}{4}} = \underset{\text{III}}{\frac{5}{8}}$$

On taking I and III terms, we get

$$\frac{4}{BC} = \frac{5}{8} \Rightarrow BC = \frac{4 \times 8}{5} = \frac{32}{5} = 6.4 \text{ cm}$$

On taking II and III terms, we get

$$\frac{CD}{4} = \frac{5}{8} \Rightarrow CD = \frac{4 \times 5}{8} = \frac{5}{2} = 2.5 \text{ cm}$$

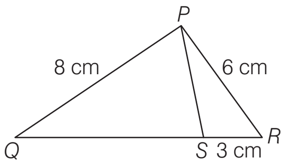
(iii) Since, $\Delta ACD \sim \Delta BCA$

$$\therefore \quad \frac{\text{ar}(\Delta ACD)}{\text{ar}(\Delta ABC)} = \frac{(AC)^2}{(BC)^2} = \frac{(4)^2}{(6.4)^2} = \frac{16}{40.96} = \frac{1600}{4096} = \frac{25}{64}$$

[$\because$ ratio of the areas of two triangles is equal to the squares of the ratio of their corresponding sides]

Hence, area of ΔACD : area of $\Delta ABC = 25 : 64$.

Example 8. PQR is a triangle. S is a point on the side QR of ΔPQR such that $\angle PSR = \angle QPR$. Given $QP = 8$ cm, $PR = 6$ cm and $SR = 3$ cm.



(i) Prove that $\Delta PQR \sim \Delta SPR$.
(ii) Find the length of QR and PS.
(iii) Find $\dfrac{\text{area of } \Delta PQR}{\text{area of } \Delta SPR}$. ***[2017]***

Sol.

(i) In ΔPQR and ΔSPR, we have

$\angle PSR = \angle QPR$ [given]

$\angle SRP = \angle PRQ$ [common angle]

So, by AA similarity criterion,

$\Delta PQR \sim \Delta SPR$

(ii) $\because$ $\Delta PQR \sim \Delta SPR$

$$\therefore \quad \frac{QR}{PR} = \frac{RP}{RS} \Rightarrow \frac{QR}{6} = \frac{6}{3} \Rightarrow QR = \frac{6 \times 6}{3} = 12 \text{ cm}$$

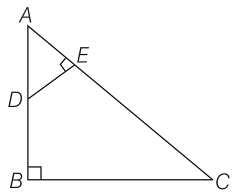
Also, $\dfrac{PQ}{PS} = \dfrac{RP}{RS} \Rightarrow \dfrac{8}{PS} = \dfrac{6}{3}$

$$\Rightarrow \quad PS = \frac{8 \times 3}{6} \Rightarrow PS = 4 \text{ cm}$$

(iii) We know that the ratio of areas of two similar triangle is equal to the ratio of the squares of their corresponding sides.

$$\therefore \quad \frac{\text{area of } \Delta PQR}{\text{area of } \Delta SPR} = \frac{RP^2}{RS^2} = \frac{(6)^2}{(3)^2} = \frac{36}{9} = 4$$

Example 9. ABC is a right angled triangle with $\angle ABC = 90°$. D is any point on AB and DE is perpendicular to AC.



(i) Prove that $\Delta ADE \sim \Delta ACB$.
(ii) If $AC = 13$ cm, $BC = 5$ cm and $AE = 4$ cm, find DE and AD.
(iii) Find area of ΔADE : area of quadrilateral $BCED$. ***[2015]***

Sol.

(i) In ΔADE and ΔACB,

$\angle AED = \angle B$ [$90°$ each]

$\angle DAE = \angle BAC$ [common angle]

$\therefore$ $\Delta ADE \sim \Delta ACB$ [by AA similarity criterion]

(ii) In ΔABC,

$AC = 13$ cm, $BC = 5$ cm

Using Pythagoras theorem in ΔABC,

$$AC^2 = AB^2 + BC^2$$

$$\Rightarrow \quad AB = \sqrt{AC^2 - BC^2} = \sqrt{(13)^2 - (5)^2}$$

[taking positive square root both sides]

$$= \sqrt{169 - 25} = 12 \text{ cm}$$

Since, $\Delta ADE \sim \Delta ACB$

$$\therefore \quad \frac{AD}{AC} = \frac{DE}{CB} = \frac{AE}{AB}$$

$$\Rightarrow \quad \frac{AD}{13} = \frac{DE}{5} = \frac{4}{12}$$ [$\because$ $AE = 4$ cm, given]

On taking, $\dfrac{AD}{13} = \dfrac{4}{12} \Rightarrow AD = \dfrac{13}{3} = 4.33\,\text{cm}$

On taking $\dfrac{DE}{5} = \dfrac{4}{12} \Rightarrow DE = \dfrac{5}{3} = 1.66\,\text{cm}$

(iii) Area of $\Delta ADE = \dfrac{1}{2} \times \text{Base} \times \text{Height}$

$$= \frac{1}{2} \times AE \times DE = \frac{1}{2} \times 4 \times \frac{5}{3} = \frac{10}{3} = 3.33\text{ cm}^2$$

Area of $\Delta ABC = \dfrac{1}{2} \times 12 \times 5 = 30\text{ cm}^2$

Area of quadrilateral $BCED$

$= \text{Area of } \Delta ABC - \text{Area of } \Delta AED$

$= (30 - 3.33)\text{ cm}^2$

$= 26.67\text{ cm}^2$

Hence, area of ΔADE : area of quadrilateral $BCED$

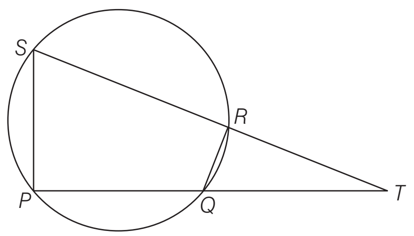
$= 3.33 : 26.67\text{ cm}^2$

Example 10. In the given figure, $PQRS$ is a cyclic quadrilateral, PQ and SR produced meet at T.

(i) Prove $\Delta TPS \sim \Delta TRQ$.

(ii) Find SP, if $TP = 18$ cm, $RQ = 4$ cm and $TR = 6$ cm.

(iii) Find area of quadrilateral $PQRS$, if area of $\Delta PTS = 27\text{ cm}^2$.



[2016]

Sol. **Given** $PQRS$ is a cyclic quadrilateral, PQ and RS are produced such that it intersect at T.

To prove $\Delta TPS \sim \Delta TRQ$

Proof In cyclic quadrilateral $PQRS$,

$$\angle SPQ + \angle SRQ = 180° \quad \text{... (i)}$$

[$\because$ sum of opposite angles of a cyclic quadrilateral is 180°]

$$\angle SRQ + \angle QRT = 180° \quad \text{[linear pair] ...(ii)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$(\angle SPQ + \angle SRQ) - (\angle SRQ + \angle QRT) = 180° - 180°$$

$\Rightarrow \quad \angle SPQ - \angle QRT = 0$

$\Rightarrow \quad \angle SPQ = \angle QRT$

Now, in ΔTPS and ΔTRQ,

$\angle SPQ = \angle QRT$ [proved above]

and $\angle PTS = \angle RTQ$ [common]

$\Rightarrow \Delta TPS \sim \Delta TRQ$ [by AA similarity criterion]

Hence proved.

(ii) Since, ΔTPS and ΔTRQ are similar.

$\therefore \quad \dfrac{TP}{TR} = \dfrac{PS}{RQ}$

[$\because$ in similar triangles, corresponding sides are proportional]

$\Rightarrow \quad \dfrac{18}{6} = \dfrac{PS}{4}$

$\therefore \quad PS = \dfrac{18 \times 4}{6} = 12\text{ cm}$

(iii) Since, ΔTPS and ΔTRQ are similar.

$\therefore \quad \dfrac{\text{ar}(\Delta TPS)}{\text{ar}(\Delta TRQ)} = \left(\dfrac{TP}{TR}\right)^2$

[$\because$ ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides]

$\Rightarrow \quad \dfrac{27}{\text{ar}(\Delta TRQ)} = \left(\dfrac{18}{6}\right)^2 \quad \left[\begin{array}{l}\because TP = 18\text{ cm}, TR = 6\text{ cm and} \\ \text{ar}(\Delta TPS) = 27\text{ cm}^2\text{, given}\end{array}\right]$

$\Rightarrow \quad \text{ar}(\Delta TRQ) = \dfrac{27 \times (6)^2}{(18)^2}$

$\Rightarrow \quad \text{ar}(\Delta TRQ) = \dfrac{27 \times 6 \times 6}{18 \times 18} = 3\text{ cm}^2$

$\therefore \quad \text{ar (quadrilateral } PQRS) = \text{ar}(\Delta TPS) - \text{ar}(\Delta TRQ)$

$= 27 - 3 = 24\text{ cm}^2$

Maps

The maps of a plane figure and the actual figure are similar to one another. If the map of a plane figure is drawn to the scale $1 : x$, then scale factor $k = \dfrac{1}{x}$.

(i) Length of the map $= k \times$ (Length of the actual figure)

(ii) Breadth of the map

$= k \times$ (Breadth of the actual figure)

(iii) Area of the map $= k^2 \times$ (Area of the actual figure)

The ratio that compares the measurements of two similar shapes, is called the scale factor or representative fraction. It is equal to the ratio of corresponding sides of two figures.

Example 11. The scale of map is 1 : 50000. In the map, a triangular plot ABC of land has the following dimensions $AB = 2$ cm, $BC = 3.5$ cm and $\angle ABC = 90°$. Calculate

(i) the actual length of the side BC (in km) of the land.

(ii) the area of the plot (in km^2).

Sol. Given, scale factor, $k = \dfrac{1}{50000}$, $AB = 2\,\text{cm}$

and $BC = 3.5\,\text{cm}$ ...(i)

(i) Length BC of triangular plot on the map

$= k$ (Length BC of actual triangular plot)

$\Rightarrow$ Length BC of triangular plot on the map $= \left(\dfrac{1}{50000}\right)$

(Length BC of actual triangular plot)

$= 50000$ (Length BC of triangular plot on the map)

$= 50000 \times 3.5 = 175000\,\text{cm}$

$= \dfrac{175000}{10^5}\,\text{km} \qquad \left[\because 1\text{ cm} = \dfrac{1}{10^5}\text{ km}\right]$

$= \dfrac{1.75 \times 10^5}{10^5} = 1.75\,\text{km}$

(ii) Area of ΔABC in map $= \frac{1}{2} \times AB \times BC = \frac{1}{2} \times 2 \times 3.5 = 3.5 \text{cm}^2$

$\because$ Area of ΔABC in the map $= k^2$ (Area of actual ΔABC)

$\therefore$ Area of ΔABC in the map

$$= \left(\frac{1}{50000}\right)^2 (\text{Area of actual } \Delta ABC)$$

Area of the actual ΔABC

$$= 50000 \text{ (Area of } \Delta ABC \text{ on the map)}$$
$$= (50000)^2 \times 3.5 = 2500000000 \times 3.5$$
$$= 25 \times 35 \times 10^7 = 875 \times 10^7 \text{ km}^2$$
$$= \frac{875 \times 10^7}{10^5 \times 10^5} \text{ km}^2 \quad \left[\because 1 \text{ cm} = \frac{1}{10^5} \text{ km}\right]$$
$$= \frac{875}{10^3} \text{ km}^2 = 0.875 \text{ km}^2$$

Models

The models of a solid and the actual solid are similar to one another.

If the model of a solid is drawn to the scale $1 : x$, then scalar factor $k = \frac{1}{x}$

(i) Each side of the model
$$= k \times (\text{Corresponding side of the actual solid})$$

(ii) Surface area of the model
$$= k^2 \times (\text{Surface area of the actual solid})$$

(iii) Volume of the model
$$= k^3 \times (\text{Volume of the actual solid})$$

Example 12. The dimensions of the model of a multistoried building are 1 m × 60 cm × 1.25 m. If the model is drawn to a scale 1 : 60, then find the actual dimension of the building in metres.

Sol. Given, scale factor, $k = \frac{1}{60}$

$\therefore$ Length of building on the model
$$= k \text{ (Length of actual building)}$$

$\Rightarrow$ Length of building on the model
$$= \frac{1}{60} \text{ (Length of actual building)}$$

Length of actual building
$$= 60 \text{ (Length of building on the model)} = 60 \times 1 \text{ m}$$

$\therefore$ Length of actual building = 60 m

Similarly, breadth of actual building
$$= \frac{1}{k} \text{ (Breadth of building on the model)}$$
$$= 60 \times 60 = 3600 \text{ cm}$$
$$= \frac{3600}{100} \text{ m} = 36 \text{ m} \quad \left[\because 1 \text{ cm} = \frac{1}{100} \text{ m}\right]$$

$\therefore$ Breadth of actual building = 36 m

and height of actual building
$$= \frac{1}{k} \text{ (Height of building on the model)}$$
$$= 60 \times 1.25 \text{ m} = 75 \text{ m}$$

Hence, the actual dimensions of the building are 60 m × 36 m × 75 m.

Example 13. A model of a ship is made to a scale of 1 : 250. Find

(i) the length of the ship, if the length of its model is 1.2 m.

(ii) the area of the deck of the ship, if the area of the deck of its model is 1.6 m^2.

(iii) the volume of its model, when the volume of the ship is 1 km^3.

Sol. Given, scale factor, $k = \frac{1}{250}$

$$\therefore \quad \frac{1}{k} = 250$$

(i) Actual length of the ship
$$= \frac{1}{k} \text{ (Length of the model of the ship)}$$
$$= 250 \times 1.2 = 300 \text{ m}$$

(ii) Actual area of deck of the ship
$$= \frac{1}{k^2} \text{ (area of the deck of model of ship)}$$
$$= (250)^2 \times 1.6 \text{ m}^2 = 62500 \times 1.6 \text{ m}^2 = 100000 \text{ m}^2$$

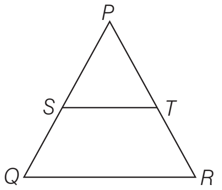
(iii) Volume of the model of ship
$$= k^3 \text{ (Volume of the actual ship)}$$
$$= \frac{(1000)^3}{(250)^3} = \left(\frac{1000}{250}\right)^3 = (4)^3 = 64 \text{ m}^3$$

Topic Exercise 2

1. Suppose ΔABC and ΔDEF are similar. Area of ΔABC is 9 cm^2 and area of ΔDEF is 64 cm^2. If $DE = 5.1$ cm, then find the value of AB.

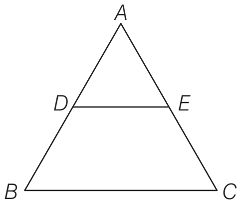
2. Suppose ΔABC and ΔDEF are similar and $AB = \frac{1}{3} DE$, then find ar (ΔABC) : ar (ΔDEF).

3. In the given figure, S and T are points on the sides PQ and PR respectively of ΔPQR, such that $PT = 2$ cm, $TR = 4$ cm and $ST \parallel QR$. Find the ratio of the areas of ΔPST and ΔPQR.



4. The areas of two similar triangles are 25 cm^2 and 81 cm^2, respectively. Find the ratio of their corresponding sides.

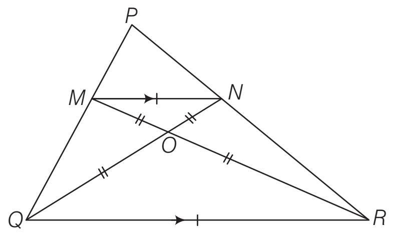
5. In the given figure, $DE \parallel BC$. $DE = 4$ cm, $BC = 8$ cm, area of $\Delta ADE = 25$ sq cm. Find the area of ΔABC.



6. If the areas of two similar triangles are 81 cm^2 and 49 cm^2 respectively, then find the ratio of their corresponding medians.

7. Two isosceles right angled triangles have equal vertical angles and their areas are in the ratio $36:25$. Find the ratio of their corresponding heights.

8. In ΔPQR, MN is parallel to QR and $\dfrac{PM}{MQ} = \dfrac{2}{3}$.

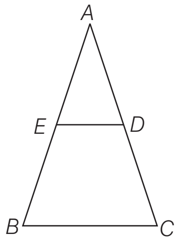


(i) Find $\dfrac{MN}{QR}$.

(ii) Prove that ΔOMN and ΔORQ are similar.

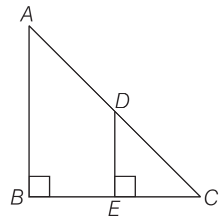
(iii) Find area of ΔOMN : area of ΔORQ. **[2018]**

9. In the given figure, ΔAED and trapezium $EBCD$ are such that the area of trapezium is three times the area of the triangle. Find the ratio $\dfrac{AE}{AB}$.



10. In an isosceles ΔABC, if $AB = AC = 25$ cm and altitude from A on BC is 24 cm, then find BC.

11. In the given figure, AB and DE are perpendicular to BC.

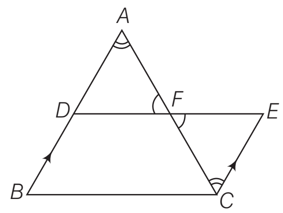


(i) Prove that $\Delta ABC \sim \Delta DEC$.

(ii) If $AB = 6$ cm, $DE = 4$ cm and $AC = 15$ cm, then calculate CD.

(iii) Find the ratio of area of ΔABC : area of ΔDEC. **[2013]**

12. In the given figure, ABC and CEF are two triangles, where BA is parallel to CE and $AF : AC = 5:8$.



(i) Prove that $\Delta ADF \sim \Delta CEF$.

(ii) Find AD, if $CE = 6$ cm.

(iii) If DE is parallel to BC, then find area of ΔADF : area of ΔABC. **[2009]**

13. On a map drawn to a scale of 1 : 50000, a rectangular plot of land $ABCD$ has dimensions $AB = 6$ cm, $BC = 8$ cm and all angles are right angles. Find

(i) the actual length of the diagonal distance AC of the plot in km.

(ii) the actual area of the plot in sq km. **[2018]**

14. A model of a square $PQRS$ is made to a scale of 0.006. If the area of the square on the model is 150 cm^2, then find the actual area of the square.

15. A model of a ship is made to a scale of 1 : 200.

(i) If the length of the model is 4 m. Calculate the length of the ship.

(ii) If the area of the deck of the ship is 160000 m^2, then find the area of the deck of the model.

(iii) The volume of the model is 100 L. Calculate the volume of the ship (in m^3).

Hints and Answers

1. Do same as Example 1. **Ans.** $AB = 1.91$ cm

2. **Hint** $\dfrac{\text{ar } \Delta ABC}{\text{ar } \Delta DEF} = \dfrac{(AB)^2}{(DE)^2}$ **Ans.** $\dfrac{1}{9}$

3. **Hint** $\dfrac{\text{ar } \Delta PST}{\text{ar } \Delta PQR} = \dfrac{(PT)^2}{(PR)^2}$ **Ans.** $\dfrac{1}{9}$

4. **Hint** Use the fact that the ratio of area of two similar triangles is equal to the ratio of the square of its corresponding sides. **Ans.** $\dfrac{5}{9}$

5. **Hint** $\dfrac{\text{ar } \Delta ADE}{\text{ar } \Delta ABC} = \dfrac{(DE)^2}{(BC)^2}$ **Ans.** 100 cm^2

6. **Hint** The ratio of areas of two similar triangles is equal to the ratio of the square of their median. **Ans.** 9 : 7

7. **Hint** Use the fact that ratio of area of two similar triangles is equal to the ratio of the square of its corresponding sides. **Ans.** $\dfrac{6}{5}$

8. **Hint**

(i) Here, $\Delta PMN \sim \Delta PQR$ [by AA similarity criteria]

$$\therefore \quad \frac{MN}{QR} = \frac{PM}{PQ} = \frac{2}{5} \quad \ldots\text{(i)}$$

We have, $\dfrac{PM}{MQ} = \dfrac{2}{3} \Rightarrow \dfrac{MQ}{PM} = \dfrac{3}{2}$

$$\Rightarrow \quad \frac{MQ}{PM} + 1 = \frac{3}{2} + 1$$

Ans. $\dfrac{MN}{QR} = \dfrac{2}{5}$

(ii) In ΔOMN and ΔORQ,

$\angle OMN = \angle ORQ$ [alternate interior angles]

$\angle ONM = \angle OQR$ [alternate interior angles]

$\therefore$ By AA similarity criteria, $\Delta OMN \sim \Delta ORQ$

(iii) Clearly, $\dfrac{(\Delta OMN)}{(\Delta ORQ)} = \left(\dfrac{MN}{QR}\right)^2$

Ans. 4 : 25

9. **Hint** $\Delta AED \sim \Delta ABC$

So $\dfrac{\text{ar }(\Delta AED)}{\text{ar }(ABC)} = \dfrac{\text{ar }(\Delta AED)}{\text{ar }(\Delta AED) + \text{ar (trapezium } EBCD)}$

$$= \frac{(AE)^2}{(AB)^2}$$

Ans. 1 : 2

10. Do same as Example 7. **Ans.** 16 cm

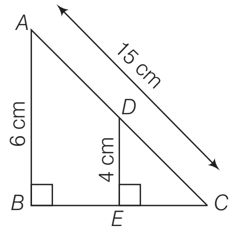
11. (i) **Hint** In ΔABC and ΔDEC,

$\angle ABC = \angle DEC = 90°$

and $\angle ACB = \angle DCE$ [common angle]

$\therefore$ $\Delta ABC \sim \Delta DEC$ [by AA at similarity criterion]

(ii) **Hint** Since, $\Delta ABC \sim \Delta DEC$ [proved above]



$$\therefore \quad \frac{AB}{DE} = \frac{AC}{DC} \Rightarrow \frac{6}{4} = \frac{15}{DC}$$

Ans. 10 cm

(iii) **Hint** Since, $\Delta ABC \sim \Delta DEC$

$$\therefore \quad \frac{\text{ar }(\Delta ABC)}{\text{ar }(\Delta DEC)} = \frac{(AB)^2}{(DE)^2}$$

Ans. 9 : 4

12. (i) **Hint** In ΔADF and ΔCEF, we have

$\angle AFD = \angle CFE$, $\angle DAF = \angle FCE$

$\therefore$ $\Delta ADF \sim \Delta CEF$ [by AA at similarity criterion]

(ii) **Hint** Since, $\Delta ADF \sim \Delta CEF$

$$\therefore \quad \frac{AD}{CE} = \frac{AF}{CF}$$

$$\left[\because \frac{AC}{AF} = \frac{8}{5} \Rightarrow \frac{AC}{AF} - 1 = \frac{8}{5} - 1 \Rightarrow \frac{CF}{AF} = \frac{3}{5}\right]$$

$$\Rightarrow \quad \frac{AD}{6} = \frac{5}{3} \Rightarrow AD = \frac{5 \times 6}{3}$$

$$\therefore \quad AD = 10 \text{ cm}$$

(iii) **Hint** Here, $\dfrac{AF}{AC} = \dfrac{AD}{AB}$ $\ldots$(i)

and $\angle DAF = \angle BAC$ [common angle]

$\therefore$ $\Delta ABC \sim \Delta ADF$

Then, $\dfrac{\text{ar }(\Delta ADF)}{\text{ar }(\Delta ABC)} = \dfrac{(AF)^2}{(AC)^2}$

Ans. 25 : 64

13. Do same as Example 11.
Ans. (i) 5 km (ii) 12 km^2

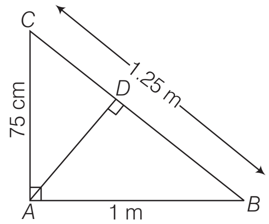
14. Do same as Example 13 (ii).
Ans. 25000 cm^2

15. Do same as Example 13.
Ans. (i) 800 m (ii) 4 m^2 (iii) 800000 m^3

CHAPTER EXERCISE

a 3 Marks Questions

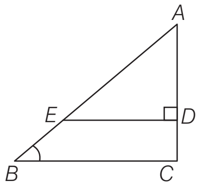
1. In the given figure, $\angle CAB = 90°$, $AD \perp BC$ and $\Delta BDA \sim \Delta BAC$. If $AC = 75$ cm, $AB = 1$ m and $BC = 1.25$ m, then find the value of AD.



2. A vertical stick 1 m long casts a shadow 80 cm long. At the same time a tower casts a shadow 30 m long. Determine the height of the tower.

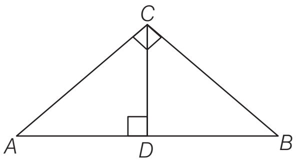
3. The perimeters of two similar ΔABC and ΔPQR are 18 cm and 12 cm, respectively. If $PQ = 5$ cm, then find AB.

4. In ΔABC, if $\angle ADE = \angle B$, then prove that $\Delta ADE \sim \Delta ABC$. Also, if $AD = 7.6$ cm, $AE = 7.2$ cm, $BE = 4.2$ cm and $BC = 8.4$ cm, find DE.

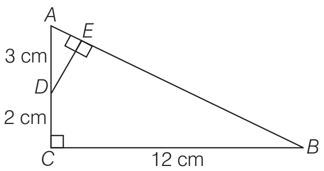


5. In the given figure, $\Delta ACB = 90°$ and $CD \perp AB$.

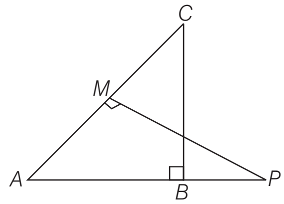
Prove that $\dfrac{BC^2}{AC^2} = \dfrac{BD}{AD}$.



6. In the given figure, ABC is a right angled triangle, right angled at C and $DE \perp AB$. Prove that $\Delta ABC \sim \Delta ADE$ and hence find the lengths of AE and DE.

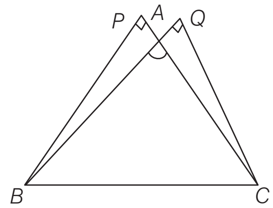


7. In the given figure, ΔABC and ΔAMP are right angled at B and M respectively. Given, $AC = 10$ cm, $AP = 15$ cm and $PM = 12$ cm.

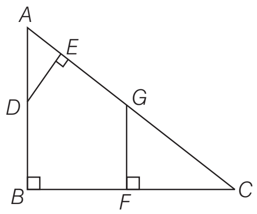


(i) Prove that $\Delta ABC \sim \Delta AMP$.
(ii) Find AB and BC. **[2012]**

8. In ΔABC, $\angle A$ is obtuse, $PB \perp PC$ and $QC \perp QB$. Prove that $AB \times AQ = AC \times AP$.

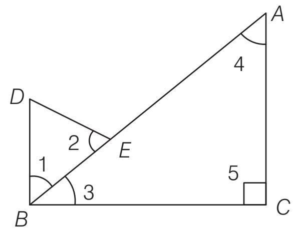


9. In the given figure, if $AB \perp BC$, $DE \perp AC$ and $GF \perp BC$, then prove that $\Delta ADE \sim \Delta GCF$.

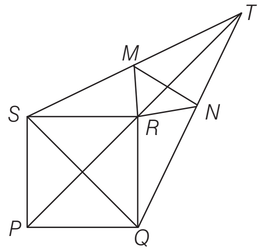


10. In the given figure, $DB \perp BC$ and $DE \perp AB$.

Prove that $\dfrac{BE}{DE} = \dfrac{AC}{BC}$.



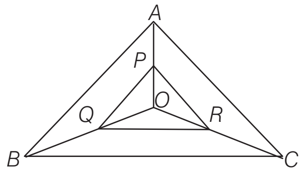
11. In the given figure, T is the exterior point on the diagonal PR of a parallelogram $PQRS$. SR produced meets OT at N and QR produced meets ST at M. Prove that $MN \parallel SQ$.



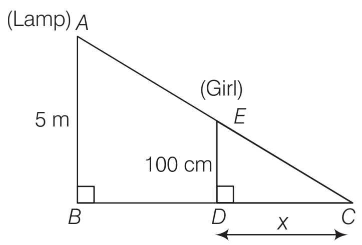
12. In the given figure,

$\dfrac{OP}{OA} = \dfrac{OR}{OC}$ and $QR \parallel BC$.

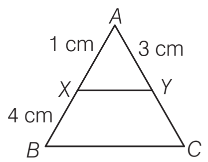
Prove that $PQ \parallel AB$.



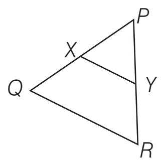
13. A girl of height 100 cm is walking away from the base of a lamp-post at a speed of 1.9 m/s. If the lamp is 5 m above the ground, find the length of her shadow after 4 s.



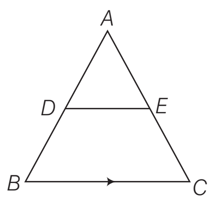
14. In the given figure, $XY \parallel BC$. Find the length of YC.



15. In the given figure, $XY \parallel QR$, $PQ/XQ = 7/3$ and $PR = 6.3$ cm. Find the value of YR.

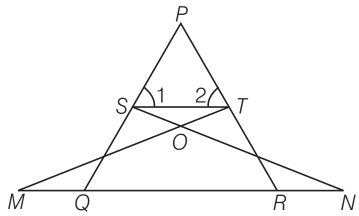


16. In ΔABC, $DE \parallel BC$, so that $AD = (7x - 4)$ cm, $AE = (5x - 2)$ cm, $DB = (3x + 4)$ cm and $EC = 3x$ cm. Then, find the value of x.



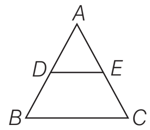
17. $ABCD$ is a trapezium in which $AB \parallel DC$. P and Q are points on sides AD and BC respectively such that $PQ \parallel AB$. If $PD = 18$ cm, $BQ = 35$ cm and $QC = 15$ cm, find the value of AD.

18. In the given figure, if $\angle 1 = \angle 2$ and $\Delta NSQ \cong \Delta MTR$, prove that $\Delta PTS \sim \Delta PRQ$.



19. In ΔABC and ΔDEF, if $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF} = \dfrac{5}{9}$, find the ratio of ar (ΔABC) : ar (ΔDEF).

20. In ΔABC shown below, $DE \parallel BC$. If $BC = 8$ cm, $DE = 6$ cm and area of $\Delta ADE = 45\ \text{cm}^2$, what is the area of ΔABC?

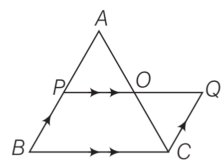


21. ABC is a triangle in which $\angle A = 90°$, $AN \perp BC$, $BC = 12$ cm and $AC = 5$ cm. Find the ratio of the areas of ΔANC and ΔABC.

22. If $\Delta ABC \sim \Delta DEF$, such that $AB = 1.2$ cm and $DE = 1.4$ cm, then find the ratio of areas of ΔABC and ΔDEF.

23. Prove that the area of the ΔBCE described on one side BC of a square $ABCD$ as base, is one-half the area of the similar ΔACF described on the diagonal AC as base.

24. In ΔABC, $AP : PB = 2 : 3$, PO is parallel to BC and is extended to Q, so that CQ is parallel to BA.

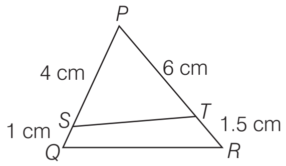


Find

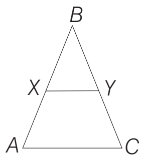
(i) area of ΔAOP : area of ΔABC.

(ii) area of ΔAPO : area of ΔCQO. **[2008]**

25. In the given figure, PS, SQ, PT and TR are 4 cm, 1 cm, 6 cm and 1.5 cm, respectively. Prove that $ST \parallel QR$. Also, find $\dfrac{\text{ar}(\Delta PST)}{\text{ar (trapezium } QRTS)}$.



26. In the given figure, in ΔABC, $XY \parallel AC$ and XY divides the ΔABC into two regions such that ar $(\Delta BXY) = 2\,\text{ar}\,(ACYX)$. Determine AX/AB.



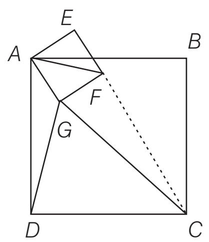
27. The model of a building is constructed with scale factor 1 : 30.

(i) If the height of the model is 80 cm, then find the actual height of the building in metres.

(ii) If the actual volume of a tank at the top of the building is 27 m^3, then find the volume of the tank on the top of the model. ***[2009]***

b 4 Marks Questions

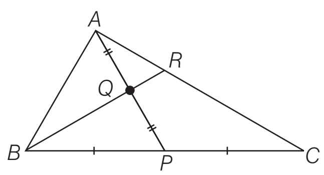
28. In the given figure, $ABCD$ and $AEFG$ are two squares.



Prove that

(i) $\dfrac{AF}{AG} = \dfrac{AC}{AD}$ (ii) $\Delta ACF \sim \Delta ADG$

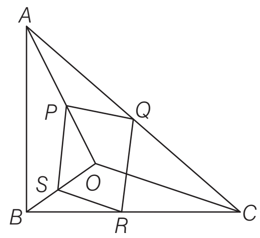
29. In the following figure of ΔABC, P is the mid-point of BC and 'Q' is the middle point of AP. If extended BQ meets AC in R, prove that $RA = \dfrac{1}{3} CA$.



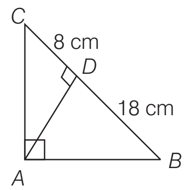
30. In a trapezium $ABCD$, $AB \parallel DC$ and $DC = 2AB$. $EF \parallel AB$, where E and F lie on BC and AD respectively, such that $\dfrac{BE}{EC} = \dfrac{4}{3}$. Diagonal DB intersects EF at G. Prove that $7EF = 11AB$.

31. D is the mid-point of side BC of a ΔABC. AD is bisected at the point E and BE produced cuts AC at the point X. Prove that $BE : EX = 3 : 1$.

32. In the given figure, if $PQRS$ is a parallelogram and $AB \parallel PS$, prove that $OC \parallel SR$.



33. In the given figure, ABC is a right angled triangle with $\angle BAC = 90°$.

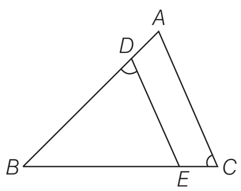


(i) Prove that $\Delta ADB \sim \Delta CDA$.

(ii) If $BD = 18$ cm and $CD = 8$ cm, then find AD.

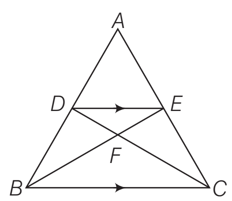
(iii) Find the ratio of the area of ΔADB to the area of ΔCDA. ***[2011]***

34. In the given figure, ABC is a triangle with $\angle EDB = \angle ACB$. Prove that $\Delta ABC \sim \Delta EBD$. If $BE = 6$ cm, $EC = 4$ cm, $BD = 5$ cm and area of $\Delta BED = 9\text{ cm}^2$, then calculate



(i) the length of AB. (ii) the area of ΔABC. ***[2010]***

35. In the given figure, ABC is a triangle, DE is parallel to BC and $\dfrac{AD}{BD} = \dfrac{3}{2}$.

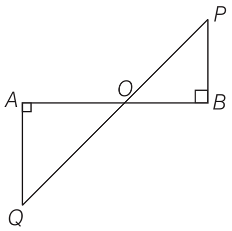


(i) Determine the ratio $\dfrac{AD}{AB}$ and $\dfrac{DE}{BC}$.

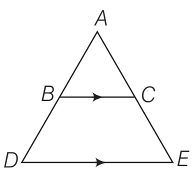
(ii) Prove that ΔDEF is similar to ΔCBF. Also, find $\frac{EF}{FB}$.

(iii) What is the ratio of the areas of ΔDEF and ΔCBF? **[2007]**

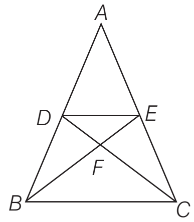
36. In the given figure, PB and QA are perpendiculars to the line segment AB. If $PO = 6$ cm, $QO = 9$ cm and the area of $\Delta POB = 120\ \text{cm}^2$, then find the area of ΔAOQ. **[2006]**



37. In the given figure, BC is parallel to DE, area of $\Delta ABC = 25\ \text{cm}^2$, area of trapezium $BCED = 24\ \text{cm}^2$ and $DE = 14$ cm. Calculate the length of BC. **[2002]**



38. In the given figure, if $DE \parallel BC$ and $AD : DB = 5 : 4$, then find $\frac{\text{ar}(\Delta DFE)}{\text{ar}(\Delta CFB)}$.



39. The scale of a map is 1 : 200000. A plot of land of area 20 km^2 is to be represented on the map. Find

(i) the number of kilometres on the ground, which is represented by 1 cm on the map.

(ii) the area (in km^2), that can be represented by $1\,\text{cm}^2$.

(iii) the area on the map, that represents the plot of land.

40. In a size transformation, the area of the image of an object of area 6 m^2 is 37.5 cm^2. Calculate

(i) the scale factor.

(ii) the volume of an object, if the volume of its image is 360 cm^3.

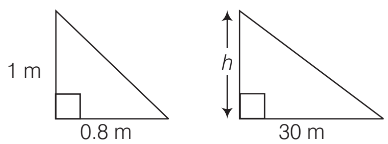
Hints and Answers

1. Hint Since, $\Delta BDA \sim \Delta BAC$

$$\therefore \quad \frac{AD}{AC} = \frac{BA}{BC}$$

Ans. 60 cm

2. Hint Let h be the height of the tower.



[$\because$ 80 cm = 0.8 m]

Since, it is clear that both the triangles will be similar.

$$\therefore \quad \frac{1}{h} = \frac{0.8}{30}$$

Ans. 37.5 m

3. Hint Since, $\Delta ABC \sim \Delta PQR$

$$\therefore \quad \frac{AB}{PQ} = \frac{18}{12}$$

$$\Rightarrow \quad AB = \frac{18 \times 5}{12} = 7.5\ \text{cm}$$

4. Hint In ΔADE and ΔABC,

$$\angle ADE = \angle ABC \quad \text{[given]}$$

$$\Rightarrow \quad \angle DAE = \angle BAC \quad \text{[common angle]}$$

So, $\Delta ADE \sim \Delta ABC$ [by AA similarity criterion]

Then, $$\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}$$

$$\Rightarrow \quad \frac{7.6}{7.2 + 4.2} = \frac{DE}{8.4}$$

Ans. $DE = 5.6$ cm

5. Hint Here, $\Delta ADC \sim \Delta ACB$ and $\Delta BDC \sim \Delta BCA$

$$\therefore \quad \frac{AD}{AC} = \frac{AC}{BA} \quad \text{and} \quad \frac{BD}{BC} = \frac{BC}{AB}$$

$$\Rightarrow \quad AC^2 = AB \times AD \quad \text{and} \quad BC^2 = AB \times BD$$

6. Hint Use AA similarity criterion to prove $\Delta ABC \sim \Delta ADE$.

Ans. $AE = \frac{15}{13}$ cm and $DE = \frac{36}{13}$ cm

7. (i) **Hint** In ΔABC and ΔAMP,

$$\angle ABC = \angle AMP = 90^\circ \quad \text{[given]}$$

$$\angle BAC = \angle PAM \quad \text{[common angles]}$$

$$\therefore \quad \Delta ABC \sim \Delta AMP \quad \text{[by AA similarity criterion]}$$

(ii) **Hint** Since, $\Delta ABC \sim \Delta AMP$ [prove above]

Now, taking $\dfrac{AC}{AP} = \dfrac{BC}{MP}$

$$\therefore \quad \frac{10}{15} = \frac{BC}{12} \Rightarrow BC = 8\text{ cm}$$

In ΔABC, right angled at B, using Pythagoras theorem,

$$AB = \sqrt{AC^2 - BC^2} = \sqrt{(10)^2 - (8)^2}$$

Ans. 8 cm, 6 cm

8. **Hint** Prove that $\Delta APB \sim \Delta AQC$.

[by AA similarily criterion]

9. **Hint** Since, $AB \perp BC$ and $GF \perp BC$, therefore $AB || GF$ and hence $\angle BAC = \angle FGC$ (corresponding angles). Now, use AA similarity criteria in ΔADE and ΔGCF.

10. **Hint** Since, $DB \perp BC$ and $AC \perp BC$, therefore $DB || AC$ and hence $\angle 1 = \angle 4$ (alternate angles). Now, use AA similarity criteria in ΔBDE and ΔABC

11. **Hint** Use BPT in ΔSTP and ΔQTP.

12. **Hint** In ΔAOB, $PQ || AB$

$$\frac{OP}{PA} = \frac{OQ}{QB} \quad \text{[by basic proportionality theorem]}$$

13. Do same as Example 11 of Topic 1. **Ans.** 190 cm

14. Do same as Example 12 of Topic 1. **Ans.** 12 cm

15. Do same as Example 15 of Topic 1. **Ans.** 2.7 cm

16. Do same as Example 18 of Topic 1. **Ans.** 4 cm

17. Do same as Example 17 of Topic 1. **Ans.** 60 cm

18. **Hint** Since, $\Delta NSQ \cong \Delta MTR$

$$\therefore \quad SQ = TR \quad \text{...(i)}$$

Also, $\quad \angle 1 = \angle 2$

$$\Rightarrow \quad PT = PS \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $\dfrac{PS}{SQ} = \dfrac{PT}{TR}$

$$\Rightarrow \quad ST || QR$$

Hence, $\Delta PTS \sim \Delta PRQ$ by AAA similarity criterion.

19. **Hint** $\Delta ABC \sim \Delta DEF$, $\dfrac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta DEF)} = \left(\dfrac{AB}{DE}\right)^2$

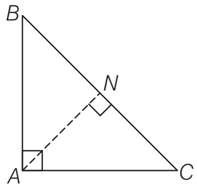
Ans. 25 : 81

20. **Hint** $\Delta ADE \sim \Delta ABC$ [by AA similarity criterion]

$$\Rightarrow \quad \frac{\text{ar}(\Delta ADE)}{\text{ar}(\Delta ABC)} = \left(\frac{DE}{BC}\right)^2$$

Ans. 80 cm^2

21. **Hint** Here, $\Delta ANC \sim \Delta BAC$



$$\Rightarrow \quad \frac{\text{ar}(\Delta ANC)}{\text{ar}(\Delta ABC)} = \left(\frac{AC}{BC}\right)^2$$

Ans. 25 : 144

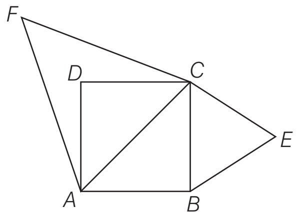
22. **Hint** We know that the ratio of areas of two similar triangles is equal to square of the ratio of their corresponding sides.

$$\therefore \quad \frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta DEF)} = \left(\frac{AB}{DE}\right)^2 = \left(\frac{1.2}{1.4}\right)^2$$

Ans. 36 : 49

23. **Hint** Here, $AB = BC = CD = DA$

and $\quad AC = \sqrt{2}\, BC$



Since, $\Delta BCE \sim \Delta ACF$ [given]

$$\therefore \quad \frac{\text{ar}(\Delta BCE)}{\text{ar}(\Delta ACF)} = \left(\frac{BC}{AC}\right)^2$$

$$\Rightarrow \quad \frac{\text{ar}(\Delta BCE)}{\text{ar}(\Delta ACF)} = \left(\frac{BC}{\sqrt{2}BC}\right)^2$$

24. (i) **Hint** Here, $\dfrac{AP}{PB} = \dfrac{2}{3}$ ...(i)

$$\Rightarrow \quad \frac{AP}{AP + PB} = \frac{2}{2+3}$$

Also, given $\quad PO || BC$

i.e. $\quad PQ || BC$ and $PB || QC$

So, $PBCQ$ is a parallelogram.

$$\therefore \quad BP = CQ$$

From Eq. (i), we get

$$\frac{AP}{PB} = \frac{2}{3} \Rightarrow \frac{AP}{CQ} = \frac{2}{3} \quad \text{...(iii)}$$

Also, $\Delta APO \sim \Delta ABC$ by AA similarity criterion

Then, $\dfrac{\text{ar}(\Delta APO)}{\text{ar}(\Delta ABC)} = \dfrac{(AP)^2}{(AB)^2}$

Ans. 4 : 25

(ii) **Hint** Here, $\Delta AOP \sim \Delta CQO$ by AA similarity criterion.

Then, $\dfrac{\text{ar}(\Delta APO)}{\text{ar}(\Delta CQO)} = \dfrac{AP^2}{CQ^2}$

Ans. 4 : 9

25. Hint Here, $\dfrac{PS}{SQ} = \dfrac{PT}{TR} = \dfrac{4}{1}$

By converse of BPT, $ST \parallel QR$.

Ans. $\dfrac{16}{9}$

26. Do same as Example 5 of Topic 2.

Ans. $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}}$

27. Hint Here, $\quad k = \dfrac{1}{30}$

(i) Actual height of the building

$$= \frac{1}{k} \times (\text{Height of the model})$$

$$= 30 \times 80 = 2400 \text{ cm}$$

Ans. 24 m $\quad [\because 1 \text{ m} = 100 \text{ cm}]$

(ii) Volume of the model tank $= k$ (Volume of the tank)

$$= \left(\frac{1}{30}\right)^3 \times 27 \text{ m}^3$$

Ans. 1 L $\quad [\because 1 \text{ L} = 0.001 \text{ m}^3]$

28. Hint

(i) $\angle AGF = \angle ADC$ $\quad$ [each 90°]

and $\angle GAF = \angle DAC$ $\quad$ [each 45°]

$\therefore \quad \Delta AGF \sim \Delta ADC$

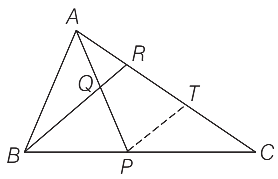
(ii) $\angle DAC = \angle GAF$

$\Rightarrow \angle DAC - \angle GAC = \angle GAF - \angle GAC$

$\Rightarrow \angle DAG = \angle CAF$

and $\dfrac{AG}{AD} = \dfrac{AF}{AC}$ $\quad$ [from part (i)]

29. Hint



In ΔCBR, $\quad PT \parallel BR$

$\therefore \quad \dfrac{CT}{TR} = \dfrac{CP}{BP}$

$\Rightarrow \quad \dfrac{CT}{TR} = 1$ $\quad [\because P$ is mid-point of $BC]$

$\Rightarrow \quad CT = TR$ $\quad \ldots$(i)

In ΔAPT, $QR \parallel PT$

$\therefore \quad \dfrac{AQ}{QP} = \dfrac{AR}{RT}$

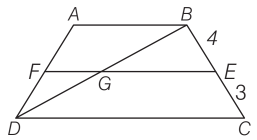
$\Rightarrow \quad 1 = \dfrac{AR}{RT}$ $\quad [\because Q$ is mid-point of $AP]$

$\Rightarrow \quad AR = RT$ $\quad \ldots$(ii)

From Eqs. (i) and (ii), we get $AR = RT = CT$

$\therefore \quad AR = \dfrac{1}{3} AC$

30. Hint $\dfrac{AF}{FD} = \dfrac{BE}{EC} = \dfrac{4}{3} \Rightarrow \dfrac{AD}{FD} = \dfrac{7}{3}$



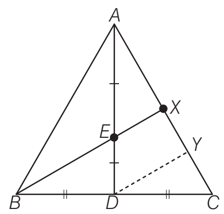
Also, $\Delta DFG \sim \Delta DAB$ and $\Delta BGE \sim \Delta BDC$

$\Rightarrow \quad \dfrac{DF}{DA} = \dfrac{FG}{AB}$ and $\dfrac{GE}{DC} = \dfrac{BE}{BC}$

$\Rightarrow \quad \dfrac{3}{7} = \dfrac{FG}{AB}$ and $\dfrac{GE}{2AB} = \dfrac{4}{7}$

Now, $\quad EF = FG + GE = \dfrac{3}{7} AB + \dfrac{8}{7} AB$

31. Hint Draw $DY \parallel BX$. Then, $\Delta BCX \sim \Delta DCY$



$\Rightarrow \quad \dfrac{BX}{DY} = \dfrac{BC}{DC}$

$\Rightarrow \quad \dfrac{BX}{DY} = 2$ $\quad \ldots$(i)

Similarly, $\Delta AEX \sim \Delta ADY$

$\Rightarrow \quad \dfrac{EX}{DY} = \dfrac{1}{2}$ $\quad \ldots$(ii)

From Eqs. (i) and (ii), we get

$$\frac{BX}{EX} = 4$$

$\Rightarrow \quad BE + EX = 4EX$

32. Hint In ΔOPS and ΔOAB,

$PS \parallel AB$

$\angle POS = \angle AOB$ $\quad$ [common angle]

$\angle OSP = \angle OBA$ $\quad$ [corresponding angles]

$\therefore \quad \Delta OPS \sim \Delta OAB$ $\quad$ [by AA similarity criterion]

Then, $\quad \dfrac{PS}{AB} = \dfrac{OS}{OB}$ $\quad \ldots$(i)

In ΔCQR and ΔCAB,

$QR \parallel PS \parallel AB$

$\angle QCR = \angle ACB$ $\quad$ [common angle]

$$\angle CRQ = \angle CBA \quad \text{[corresponding angle]}$$

$\therefore \quad \Delta CQR \sim \Delta CAB$ [by AA similarity criterion]

Then, $\dfrac{QR}{AB} = \dfrac{CR}{CB}$

$\Rightarrow \quad \dfrac{PS}{AB} = \dfrac{CR}{CB}$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{OS}{OB} = \frac{CR}{CB} \text{ or } \frac{OB}{OS} = \frac{CB}{CR}$$

On subtracting 1 from both sides, we get

$$\frac{OB}{OS} - 1 = \frac{CB}{CR} - 1$$

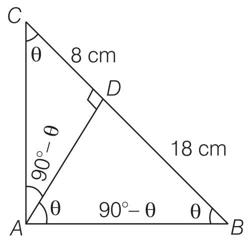
$$\Rightarrow \quad \frac{OB - OS}{OS} = \frac{CB - CR}{CR} \Rightarrow \frac{BS}{OS} = \frac{BR}{CR}$$

By converse of basic proportionality theorem, we get the result.

33. (i) **Hint** In ΔADB and ΔCDA, we have

$$\angle DAB = \angle DCA = \theta$$

and $\angle CAD = \angle ABD = 90° - \theta$



$\therefore \; \Delta ADB \sim \Delta CDA$ [by AA similarity criterion]

(ii) **Hint** Since, $\Delta ADB \sim \Delta CDA$

$\therefore \quad \dfrac{AD}{CD} = \dfrac{BD}{AD}$

$\Rightarrow \quad AD^2 = BD \times CD$

Ans. 12 cm

(iii) **Hint** $\dfrac{\text{ar}(\Delta ADB)}{\text{ar}(\Delta CDA)} = \dfrac{(AD)^2}{(CD)^2}$

Ans. 9 : 4

34. **Hint** In ΔABC and ΔEBD, we have

$\angle ACB = \angle EDB$ [given]

$\angle ABC = \angle EBD$ [common angle]

$\therefore \quad \Delta ABC \sim \Delta EBD$ [by AA similarity criterion]

(i) Since, $\Delta ABC \sim \Delta EBD$

$\therefore \quad \dfrac{AB}{EB} = \dfrac{BC}{BD} = \dfrac{AC}{ED}$

Now, taking first two terms, we get

$$\frac{AB}{EB} = \frac{BC}{BD}$$

$\Rightarrow \quad \dfrac{AB}{6} = \dfrac{BE + EC}{5}$

Ans. 12 cm

(ii) Since, $\Delta ABC \sim \Delta EBD$

$\therefore \quad \dfrac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta EBD)} = \dfrac{(AB)^2}{(EB)^2}$ **Ans.** 36 cm^2

35. (i) **Hint** $\dfrac{AD}{DB} = \dfrac{3}{2}$

$\Rightarrow \quad \dfrac{AD}{AD + DB} = \dfrac{3}{3+2}$

[using componendo property]

$\Rightarrow \quad \dfrac{AD}{AB} = \dfrac{3}{5}$

In ΔADE and ΔABC, we have

$$\angle DAE = \angle BAC$$

and $\quad \angle ADE = \angle ABC$

$\therefore \quad \Delta ADE \sim \Delta ABC$ [by AA similarity criterion]

Then, $\dfrac{DE}{BC} = \dfrac{AD}{AB}$

Ans. $\dfrac{AD}{AB} = \dfrac{DE}{BC} = \dfrac{3}{5}$

(ii) **Hint** In ΔDEF and ΔBCF, we have

$$\angle FED = \angle FBC$$

and $\quad \angle DFE = \angle BFC$

$\therefore \quad \Delta DEF \sim \Delta CBF$ [by AA criterion similarity]

Then, $\dfrac{DE}{BC} = \dfrac{EF}{FB}$

Ans. $\dfrac{EF}{FB} = \dfrac{3}{5}$

(iii) **Hint** Since, $\Delta DEF \sim \Delta CBF$

$\therefore \quad \dfrac{\text{ar}(\Delta DEF)}{\text{ar}(\Delta CBF)} = \dfrac{DE^2}{BC^2}$

Ans. 9 : 25

36. **Hint** In ΔAOQ and ΔBOP, we have

$$\angle QAO = \angle OBP = 90°$$

and $\quad \angle AOQ = \angle POB$

$\therefore \quad \Delta AOQ \sim \Delta BOP$ [by AA similarity criterion]

Then, $\quad \dfrac{\text{ar}(\Delta AOQ)}{\text{ar}(\Delta BOP)} = \dfrac{(OQ)^2}{(PO)^2}$

$\therefore \quad \text{ar}(\Delta AOQ) = \left(\dfrac{9}{6}\right)^2 \times 120$

Ans. 270 cm^2

37. Hint ar $(\Delta ADE) = $ ar $(\Delta ABC) + $ ar (trapezium $BCED$)

$$= 25 + 24 = 49 \text{cm}^2$$

Since, $BC \parallel DE$

In ΔABC and ΔADE, we have

$$\angle BAC = \angle DAE \text{ and } \angle ABC = \angle ADE$$

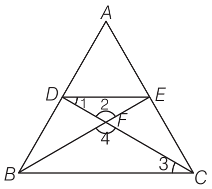
So, $\Delta ABC \sim \Delta ADE$

$$\therefore \quad \frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta ADE)} = \frac{BC^2}{DE^2}$$

Ans. $BC = 10$ cm

38. Hint $\Delta ADE \sim \Delta ABC$

$$\Rightarrow \quad \frac{AD}{AB} = \frac{DE}{BC} \quad \ldots\text{(i)}$$



and $\frac{AD}{DB} = \frac{5}{4} \Rightarrow \frac{AD}{AB} = \frac{5}{9}$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{DE}{BC} = \frac{5}{9}$$

Here, $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$

$\therefore \quad \Delta DFE \sim \Delta CFB$

$$\Rightarrow \quad \frac{\text{ar}(\Delta DFE)}{\text{ar}(\Delta CFB)} = \left(\frac{DE}{BC}\right)^2$$

Ans. 25 : 81

39. (i) **Hint** Given, scale factor,

$$k = \frac{1}{200000}$$

$\therefore$ 1 cm on map represents 200000 cm on the ground.

$\Rightarrow \quad$ 200000 cm = 2 km

Ans. 2 km

(ii) **Hint** Since, 1 cm on the map represents 2 km on the ground.

Ans. 4 km^2

(iii) **Hint** Area of plot on the map $= k^2$ (Area of plot on the ground)

$$\text{Area of plot on the map} = \frac{20}{(200000)^2} \text{ km}^2$$

$$= \frac{20 \times 10^5 \times 10^5}{(2 \times 10^5)^2} \text{cm}^2 \quad [\because 1 \text{ km} = 10^5 \text{ cm}]$$

Ans. 5 cm^2

40. (i) **Hint** Area of the model $= k^2$ (Area of the object)

$$\Rightarrow \quad 37.5 \text{ m}^2 = k^2 \times 6 \text{cm}^2$$

$$\Rightarrow \quad \frac{1}{k^2} = \frac{6 \text{ m}^2}{37.5 \text{ cm}^2}$$

$$= \frac{6 \times 100 \times 100 \text{ cm}^2}{37.5 \text{ cm}^2}$$

$$= \frac{6 \times 100 \times 100 \times 10}{375}$$

Ans. $\frac{1}{40}$

(ii) **Hint** Volume of an object

$$= \frac{1}{k^3}(\text{Volume of the model})$$

$$= (40)^3 \times 360$$

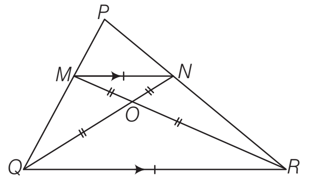
Ans. 23040000 cm^3

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1. In ΔPQR, MN is parallel to QR and $\dfrac{PM}{MQ}=\dfrac{2}{3}$.



(i) Find $\dfrac{MN}{QR}$.

(ii) Prove that ΔOMN and ΔORQ are similar.

(iii) Find area of ΔOMN : area of ΔORQ.

2. On a map drawn to a scale of 1 : 50000, a rectangular plot of land $ABCD$ has dimensions $AB = 6$ cm, $BC = 8$ cm and all angles are right angles.

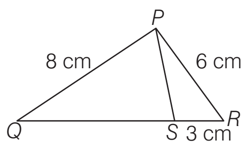
Find

(i) the actual length of the diagonal distance AC of the plot in km.

(ii) the actual area of the plot in sq km.

2017

3. PQR is a triangle. S is a point on the side QR of ΔPQR such that $\angle PSR = \angle QPR$. Given $QP = 8$ cm, $PR = 6$ cm and $SR = 3$ cm.



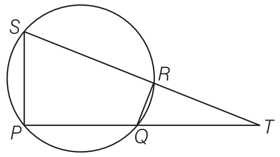
(i) Prove $\Delta PQR \sim \Delta SPR$.

(ii) Find the length of QR and PS.

(iii) Find $\dfrac{\text{area of } \Delta PQR}{\text{area of } \Delta SPR}$.

2016

4. In the adjoining figure, $PQRS$ is a cyclic quadrilateral, PQ and SR produced meet at T.



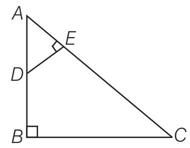
(i) Prove $\Delta TPS \sim \Delta TRQ$.

(ii) Find SP, if $TP = 18$ cm, $RQ = 4$ cm and $TR = 6$ cm.

(iii) Find area of quadrilateral $PQRS$, if area of $\Delta PTS = 27\text{ cm}^2$.

2015

5. ABC is a right angled triangle with $\angle ABC = 90°$. D is any point on AB and DE is perpendicular to AC.



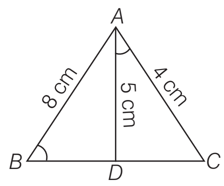
(i) Prove that $\Delta ADE \sim \Delta ACB$.

(ii) If $AC = 13$ cm, $BC = 5$ cm and $AE = 4$ cm, find DE and AD.

(iii) Find area of ΔADE : area of quadrilateral $BCED$.

2014

6. In ΔABC, $\angle ABC = \angle DAC$, $AB = 8$ cm, $AC = 4$ cm and $AD = 5$ cm.



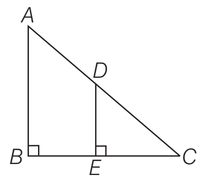
(i) Prove that ΔACD is similar to ΔBCA.

(ii) Find BC and CD.

(iii) Find, area of ΔACD : area of ΔABC.

2013

7. In the adjoining figure, AB and DE are perpendicular to BC.



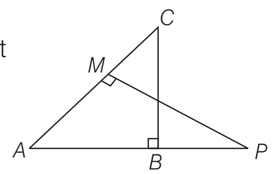
(i) Prove that $\Delta ABC \sim \Delta DEC$.

(ii) If $AB = 6$ cm, $DE = 4$ cm and $AC = 15$ cm, then calculate CD.

(iii) Find the ratio of the area of ΔABC : area of ΔDEC.

2012

8. In the adjoining figure, ΔABC and ΔAMP are right angled at B and M, respectively. Given, $AC = 10$ cm, $AP = 15$ cm and $PM = 12$ cm.

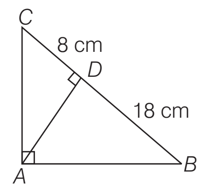


(i) Prove that $\Delta ABC \sim \Delta AMP$.

(ii) Find AB and BC.

2011

9. In the adjoining figure, ABC is a right angled triangle with $\angle BAC = 90°$.



(i) Prove that $\Delta ADB \sim \Delta CDA$.

(ii) If $BD = 18$ cm, $CD = 8$ cm, find AD.

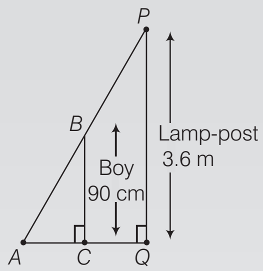
(iii) Find the ratio of the area of ΔADB is to area of ΔCDA.

* All these questions are completely covered in chapter either as solved examples or in topic/chapter exercise.

CHALLENGERS*

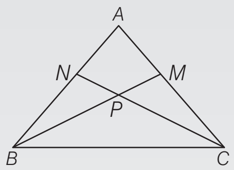
A Set of Brain Teasing Questions for Exercise of Your Mind

1. A boy of height 90 cm is walking away from the base of a lamp-post at a speed of 1.2 m/s. If the lamp is 3.6 m above the ground, then the length of his shadow after 6 s is



(a) 1.2 m (b) 2.4 m
(c) 3.6 m (d) None of these

2 In the figure given below, $AM : MC = 3 : 4$, $BP : PM = 3 : 2$ and $BN = 12$ cm. Then, AN equals to

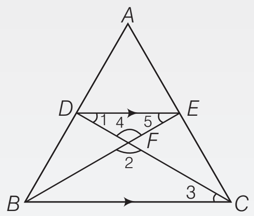


(a) 10 cm (b) 12 cm
(c) 14 cm (d) 16 cm

3 Diagonal AC of a rectangle $ABCD$ is produced to the point E such that $AC : CE = 2 : 1$, $AB = 8$ cm and $BC = 6$ m. The length of DE is

(a) $2\sqrt{19}$ cm (b) 15 cm (c) $3\sqrt{17}$ cm (d) 13 cm

4. In the given figure, $DE \parallel BC$ and $AD : DB = 5 : 4$. Find the ratio of areas of ΔDEF and ΔCFB.

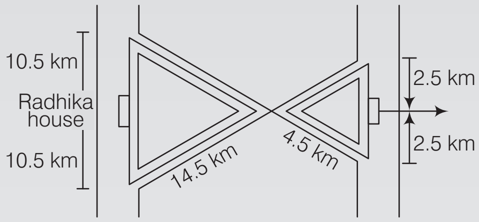


(a) 25 : 81 (b) 81 : 25
(c) 16 : 25 (d) 25 : 16

5. Through the mid-point M of the side CD of a parallelogram $ABCD$, the line BM is drawn intersecting AC at L and AD produced at E. The values of EL and ar(ΔAEL) are respectively

(a) BL and ar(ΔCBL) (b) ar(ΔCBL) and BL
(c) $2BL$ and 4 ar(ΔCBL) (d) 4 ar(ΔCBL) and $2BL$

6. Radhika wants to visit her friend who recently moved to a new house. The road map between Radhika's home and her friend's as well as the distance known to Radhika are as shown in the figure given below

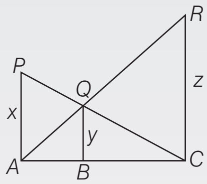


To reach the friend's house, the shortest distance which Radhika has to travel, is

(a) 30.95 km (b) 32.5 km (c) 28.5 km (d) 35.35 km

7. In the given figure, PA, QB and RC are each perpendicular to AC. The value of

$$\frac{[\text{ar}(\Delta PAC)\,\text{ar}(\Delta QBA) + \text{ar}(\Delta RCA)\,\text{ar}(\Delta QBC)]}{\text{ar}(\Delta QBC)\,\text{ar}(\Delta QBA)}$$ is



(a) $\dfrac{x^2+y^2}{z^2}$ (b) $\dfrac{x^2+z^2}{y^2}$ (c) $\dfrac{z^2+y^2}{x^2}$ (d) None of these

8. O is the point of intersection of the diagonals AC and BD of a trapezium $ABCD$ with $AB \parallel DC$. Through O, a line segment PQ is drawn parallel to AB meeting AD in P and BC in Q, then OP is equal to

(a) $OP = OQ$ (b) $OP = 2OQ$
(c) $OQ = 2\,OP$ (d) $OP = \dfrac{1}{3}OQ$

9. In a ΔPQR, L and M are two points on base QR, such that $\angle LPQ = \angle QRP$ and $\angle RPM = \angle RQP$. Then, which of the following is/are true

(i) $\Delta PQL \sim \Delta RPM$ (ii) $QL \times RM = PL \times PM$
(iii) $PQ^2 = QR \cdot QL$

(a) Both (i) and (ii) (b) Both (ii) and (iii)
(c) Both (i) and (iii) (d) All the three

*These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 406.

CHAPTER

14

Locus

Chapter Objectives

- Theorems Based on Locus
- Locus in Some Standard Cases

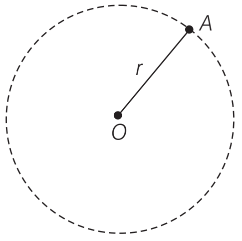
Locus is a Latin word which means 'Location' or 'Place'. The locus of a point is the path travelled by it moving under given geometrical conditions.

Or

Locus is the set of all those points, which satisfy one or more geometrical conditions.

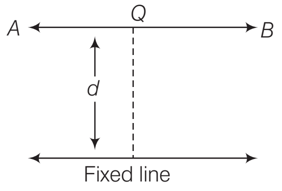
The concept (or meaning) of 'Locus' is best explained as

1. When a point A moves in the plane, such that it is always equidistant from the fixed point O in the same plane, then the path followed by A is a circle.



Thus, the locus of point A is a circle.

2. Let a point Q moves such that its distance from a fixed line is always equal to d. Then, point Q will trace out a straight line AB parallel to the fixed line. Thus, the locus of a point Q is the straight line AB.



Some important point related to locus are as follow

(i) Every point which satisfies one or more geometrical conditions, lies on the locus and *vice-versa*.

(ii) A point which does not satisfy the one or more geometrical conditions, cannot lie on the locus and *vice-versa*.

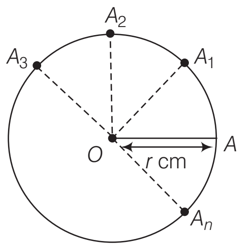
(iii) The locus of a point moving in a plane under a given geometrical condition (or conditions), is always a straight or curved line.

(iv) To find the locus of a moving point, take few points which satisfy the given geometrical conditions and then join them.

Theorems Based on Locus

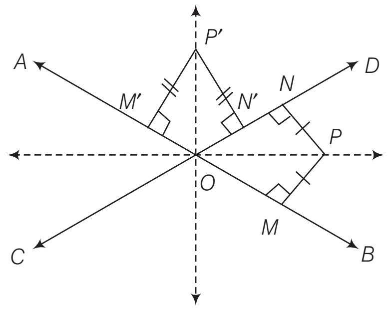
There are various theorems based on locus, which are as follow

Theorem 1 *The locus of a point equidistant from a fixed point, is a circle with the fixed point as centre and fixed distance as radius.*



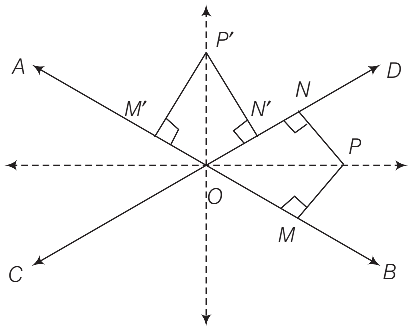
In the given figure, O is a fixed points and $OA_1, OA_2, \dots, OA_n$ all are equidistance. Then, path traced by points $A_1, A_2, \dots, A_n$ is the circle.

Theorem 2 *Locus of a point equidistant from two intersecting straight lines, is the bisector of the angles between the lines.*



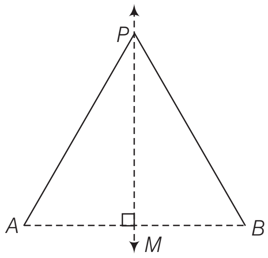
In the given figure, P is the locus of a point which is equidistant from lines OD and OB. Then, line OP will be the bisector of intersecting lines.

Theorem 3 **(Converse of Theorem 2)** *Any point on the bisector of an angle is equidistant from the arms of that angle.*



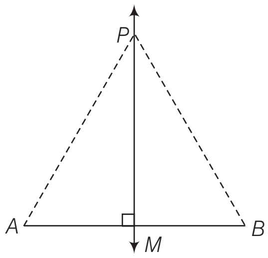
In the given figure, line OP is the angle bisector of lines OD and OB. Then, point P will be the equidistant from OD and OB.

Theorem 4 *The locus of a point equidistant from two given fixed points is the perpendicular bisector of the line joining the two fixed points.*



In the given figure, P is the locus of a point, which is equidistant from points A and B. Then, line PM will be the perpendicular bisector of AB.

Theorem 5 **(Converse of Theorem 4)** *Any point on the perpendicular bisector of a line segment joining two fixed points is equidistant from the fixed points.*

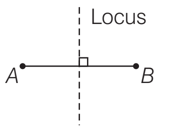


In the given figure, line PM is the perpendicular bisector of AB. Then, the distances from point P to the fixed points are equal, i.e. $AP = BP$.

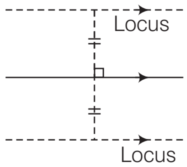
Locus in Some Standard Cases

There are various cases, which are given below

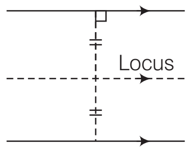
1. The locus of a point, which is equidistant from two fixed points, is the perpendicular bisector of the joining two fixed points.



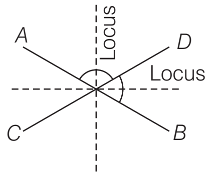
2. The locus of a point, which is equidistant from a given straight line, is the straight lines parallel to the given line (i.e. above or below it).



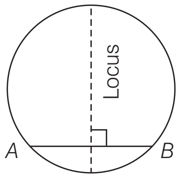
3. The locus of a point, which is equidistant from two given parallel line, is a mid-way line between the two given line.



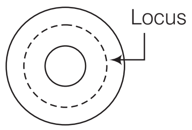
4. The locus of a point, which is equidistant from two intersecting straight lines, is angular bisector of the angles formed by the two intersecting straight lines.



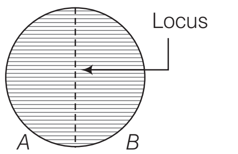
5. The locus of inside point of a circle, which is equidistant from two given points on the circle, is diameter of the circle perpendicular to the chord formed by the two fixed points.



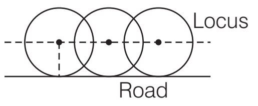
6. The locus of a point, which is equidistant from concentric circles, is the circumference of the circle concentric to the two given circles and lying mid-way between them.



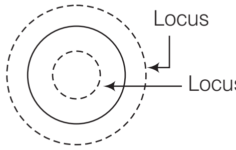
7. The locus of a mid-points of all parallel chords is a diameter of the circle, which is perpendicular to the given chords.



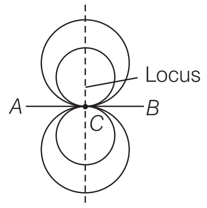
8. The locus of centre of a wheel, which is moving on a horizontal road is a straight line parallel to the road and at a distance equal to the radius of wheel.



9. The locus of a point, which is equidistant from a given circle, is a pair of circles concentric with the given circle.



10. The locus of a centres of a circles, which is touching a given line at a given point, is a straight line perpendicular to the given line at the given point.

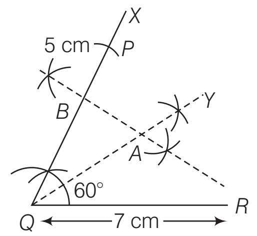


Example 1. Draw an $\angle PQR = 60°$, having $PQ = 5$ cm and $QR = 7$ cm. Find a point A equidistant from PQ and QR and also equidistant from P and Q.

Sol. ***Steps of construction***

1. Draw a line segment $QR = 7$ cm.
2. Take Q as centre and draw an $\angle XQR = 60°$.
3. From Q, mark an arc of 5 cm, which intersects XQ at P.
4. Draw the bisector QY of $\angle PQR$.

   Draw the perpendicular bisector AB of PQ, so that it intersects QY at A.

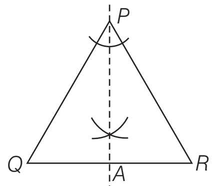


Thus, A is the required point, which is equidistant from PQ and QR, and also from P and Q.

Example 2. Find the locus of a point on the base of an isosceles triangle equidistant from its sides.

Sol. Let the given isosceles triangle be ΔPQR, whose base is QR.

Now, draw the bisector of $\angle QPR$, which meets base QR at A.

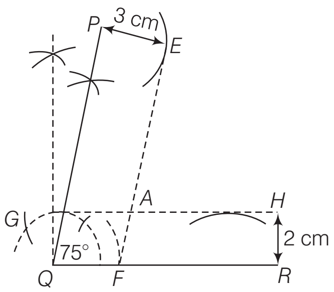


Thus, A is the required locus of a point, which is equidistant from the sides PQ, and PR of an isosceles triangle.

Example 3. Draw an $\angle PQR = 75°$. Find a point A, such that A is at a distance of 3 cm from PQ and 2 cm from QR.

Sol. ***Steps of construction***

1. Draw $\angle PQR = 75°$.
2. Draw $EF \parallel PQ$ at a distance of 3 cm from PQ.
3. Draw $GH \parallel QR$ at a distance of 2 cm from QR, which intersects EF at A.

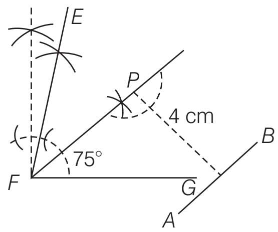


Thus, A is the required point.

Example 4. Construct $\angle EFG = 75°$. Mark a point P equidistant from EF and FG, such that distance from another line AB is 4 cm.

Sol. ***Steps of construction***

1. Draw $\angle EFG = 75°$.
2. Draw a bisector of $\angle EFG$ and take any point P on it.
3. Draw $AB \parallel FP$ at a distance of 4 cm.

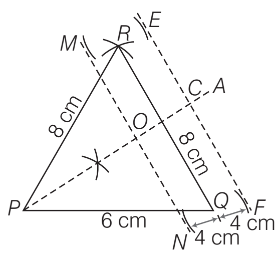


Thus, P is the required point, which is equidistant from EF and FG.

Example 5. Construct a ΔPQR with $PQ = 6$ cm and $PR = QR = 8$ cm. Find the points equidistant from PQ and PR, and also 4 cm from QR. Measure the distance between the two points obtained.

Sol. ***Steps of construction***

1. Draw a line segment $PQ = 6$ cm.
2. Draw arcs of radii 8 cm each from P and Q to intersect at R.
3. Join PR and QR. Thus, ΔPQR is formed.



4. Draw the bisector of $\angle RPQ$, draw a line PA passing through this bisector.
5. Draw $MN \parallel QR$ and $EF \parallel QR$ at a distance of 4 cm each, which intersects PA at O and C, respectively. Measure OC.
   Thus, O and C are the required points and distance OC is 8.2 cm (approx.).

Example 6. Construct a ΔABC with $AB = 7$ cm, $BC = 8$ cm and $\angle ABC = 60°$. Also, locate by construction, the point P, such that

(i) P is equidistant from B and C.

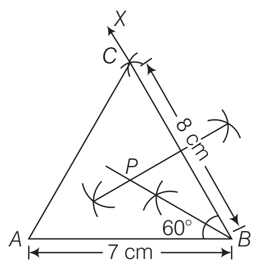
(ii) P is equidistant from AB and BC.

(iii) Measure and record the length of PB. ***[2000]***

Sol. ***Steps of construction***

(i) 1. Draw a line segment $AB = 7$ cm.
2. Draw an $\angle XBA = 60°$ at point B with the help of pair of compasses.
3. From point B, cut $BC = 8$ cm from BX.
4. Join A and C, which is the required ΔABC.
5. Draw perpendicular bisector of line BC. The point situated on this line will be equidistant from B and C.

(ii) 6. Draw angular bisector of $\angle ABC$. Any point situated on this angular bisector is equidistant from lines AB and BC.
7. The intersection point of perpendicular bisector and angular bisector is the required point P, which is equidistant from lines AB and BC and also from points B and C.



(iii) On measuring, the length of PB is 4.5 cm.

Example 7. A straight line AB is 8 cm long. Locate, by construction the locus of a point, which is

(i) equidistant from A and B.

(ii) always 4 cm long from the line AB.

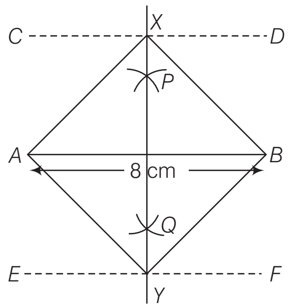
(iii) Mark two points X and Y, which are 4 cm from AB and equidistant from A and B. Name the figure $AXBY$. ***[2008]***

Sol. ***Steps of construction***

(i) 1. Draw a line segment $AB = 8$ cm.
2. Draw perpendicular bisector PQ of AB, which is the locus of a point, equidistant from A and B.

(ii) 3. Draw CD and EF both parallel to opposite sides of AB and each at a distance of 4 cm from AB. Thus, CD and EF are the required loci.

(iii) 4. Extends PQ on both sides, which meets the lines CD and EF at X and Y respectively, which are at a distance 4 cm from AB.



5. Join AX, XB, BY and YA.
Hence, the figure formed $AXBY$ is a square.

Example 8. Use ruler and compass only for this question.

(i) Construct ΔABC, where $AB = 3.5$ cm, $BC = 6$ cm and $\angle ABC = 60°$.

(ii) Construct the locus of points inside the triangle, which are equidistant from BA and BC.

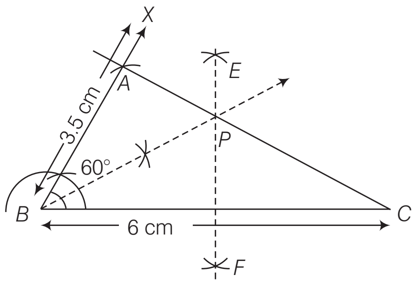
(iii) Construct the locus of points inside the triangle, which are equidistant from B and C.

(iv) Mark the point P, which is equidistant from AB and BC; and also equidistant from B and C. Measure and record the length of PB. ***[2010]***

Sol. Steps of construction

(i) 1. Draw a line segment $BC = 6$ cm.
2. From the point B, draw $\angle XBC = 60°$.
3. Taking B as centre, cut $BA = 3.5$ cm from BX.
4. Join AC. Thus, we get the required ΔABC.

(ii) 5. Draw angle bisector of $\angle ABC$, which is the locus of points inside the triangle, which are equidistant from BA and BC.



(iii) 6. Draw perpendicular bisector EF of BC, which intersects the angular bisector at point P. Then, EF is the locus of points inside the triangle, which are equidistant from B and C.

(iv) 7. The intersection point of angle bisector and EF is the required point P, which is equidistant from AB, BC, B and C. On measuring, the length of PB is 3.5 cm.

Example 9. Construct a ΔBCP, given $BC = 5$ cm, $BP = 4$ cm and $\angle CBP = 45°$.

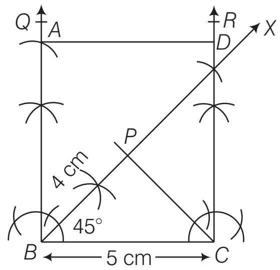
(i) Complete the rectangle $ABCD$, such that
(a) P is equidistant from AB and BC.
(b) P is equidistant from C and D.

(ii) Measure and record the length of AB. ***[2007]***

Sol. Steps of construction

(i) (a) 1. Draw a line segment $BC = 5$ cm.
2. Draw a ray making $\angle XBC = 45°$ at B.
3. Cut $BP = 4$ cm from BX and join PC. Then, ΔBCP is the required triangle.
4. Draw $BQ \perp BC$ and $CR \perp BC$. Here, PB is the bisector of $\angle QBC$, so it is equidistant from BC and QB.

(b) 5. Now, cut BQ at A, such that $PA = PC$ and cut CR at D, such that $PC = PD$.



6. Join AD. Thus, the rectangle $ABCD$ is the required rectangle.

(ii) On measuring, the length of AB is 5.7 cm (approx.).

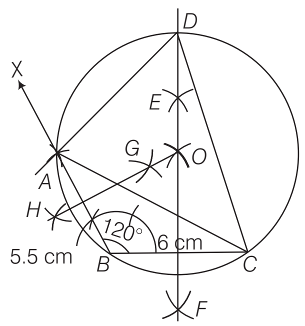
Example 10. Construct a ΔABC, in which base $BC = 6$ cm, $AB = 5.5$ cm and $\angle ABC = 120°$.

(i) Construct a circle circumscribing the ΔABC.

(ii) Draw a cyclic quadrilateral $ABCD$, so that D is equidistant from B and C. ***[2012]***

Sol. Steps of construction

(i) 1. Draw a line segment $BC = 6$ cm.
2. From the point B, draw $\angle XBC = 120°$.
3. Taking radius 5.5 cm with B as centre, cut $BA = 5.5$ cm from BX.
4. Join AC, which is the required ΔABC.
5. Draw perpendicular bisectors EF of BC and GH of AB. They both intersect at point O.
6. Draw a circle with O as centre and OA or OC or OB as radius. Then, we get the required circumcircle.



(ii) 7. Extend line FE in the direction of E, which meets the circle at point D and which is equidistant from B and C.

8. Join AD and CD.

Hence, $ABCD$ is the required cyclic quadrilateral.

Example 11. Use a ruler and compasses only for this question.

(i) Construct a ΔABC with $AB = 3.5$ cm, $BC = 6$ cm and $\angle ABC = 120°$.

(ii) In the same diagram, draw a circle with BC as diameter. Find a point P on the circumference of the circle, that is equidistant from AB and BC.

(iii) Measure $\angle BCP$. *[2013, 05]*

Sol. ***Steps of construction***

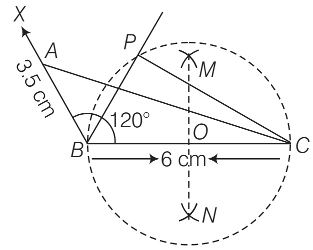
(i) 1. Draw a line segment $BC = 6$ cm.

2. From the point B, draw $\angle XBC = 120°$.

3. Taking length 3.5 cm with B as centre, cut $BA = 3.5$ cm from BX.

4. Join A to C. Thus, we get the required ΔABC.

(ii) 5. Draw perpendicular bisector MN of BC, which meets BC at point O.



6. Draw a circle with O as centre and OC or OB as radius.

7. Draw angle bisector of $\angle ABC$, which intersects the circumference of the circle at P. Then, point P is equidistant from AB and BC.

8. Join BP and CP.

(iii) On measuring, we get $\angle BCP = 30°$.

Example 12. Using a ruler and a compass construct a ΔABC in which $AB = 7$ cm, $\angle CAB = 60°$ and $AC = 5$ cm. Construct the locus of

(i) points equidistant from AB and AC.

(ii) points equidistant from BA and BC.

Hence, construct a circle touching the three sides of the triangle internally. *[2017]*

Sol. ***Steps of construction***

***Step* I** Draw a line segment $AB = 7$ cm.

***Step* II** Taking A as centre and some radius, draw an arc of a circle which intersect AB, say at P.

***Step* III** Taking P as centre and with the same radius as before drawn an arc intersecting previously draw arc, say at a point Q.

***Step* IV** Draw the ray AX passing through Q, then $\angle XAB = 60°$

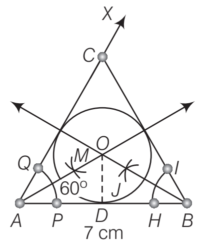
***Step* V** With A as centre and radius equal to 5 cm, draw an arc cutting AX at C.

***Step* VI** Join CB.

Thus, ΔABC is obtained.

(i) **Locus is the bisector of $\angle BAC$**

***Step* VII** Taking P and Q as centre and with the radius more than $\frac{1}{2}PQ$, draw arcs to intersect each other, say at M.



***Step* VIII** Join MA, which is required locus of point equidistant from AB and AC.

[∵ any point on the angle bisector of an angle is equidistant from its arms]

(ii) **Locus is the bisector of $\angle ABC$**

***Step* IX** Taking B as centre and draw an arc of a circle, which intersects BA at H and BC at I.

***Step* X** Taking H and I as centres and with the radius more than $\frac{1}{2}HI$, draw two arcs intersect each other say at J.

***Step* XI** Join JB, which is required locus of points equidistant from BA and BC.

***Step* XII** Suppose both bisectors AM and BJ intersect at a point say O.

***Step* XIII** Draw perpendicular from point O on any side of the triangle, say OD on AB.

***Step* XIV** Taking O as centre and OD as radius, draw a circle, which is required incircle.

Example 13. Use ruler and compasses only for the following question. All construction lines and arcs must be clearly shown. *[2016]*

(i) Construct a ΔABC in which $BC = 6.5$ cm, $\angle ABC = 60°$, $AB = 5$ cm.

(ii) Construct the locus of points at a distance of 3.5 cm from A.

(iii) Construct the locus of points equidistant from AC and BC.

(iv) Mark 2 points X and Y which are at a distance of 3.5 cm from A and also equidistant from AC and BC. Measure XY.

Sol. Given, $BC = 6.5$ cm, $\angle ABC = 60°$ and $AB = 5$ cm.

Steps of construction

(i) ***Step* I** Draw $BC = 6.5$ cm.

***Step* II** At point B, make $\angle ZBC = 60°$ with the help of compasses.

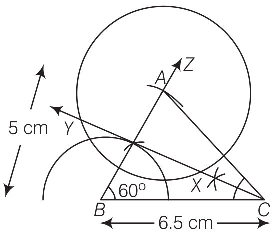
***Step* III** Taking radius 5 cm, with B as centre cut $BA = 5$ cm from BZ.

***Step* IV** Join A to C.

Thus, ΔABC is the required triangle.

(ii) Draw a circle with A as centre and radius equal to 3.5 cm, which cut the ΔABC at point X and Y. Also, it is the locus of a point equidistant from point A.

(iii) Draw angle bisector of $\angle ACB$, which is the locus of points equidistant from AC and BC.



(iv) Since, the points are equidistant from point A and also equidistant from AC and BC.

Therefore, the required point will be the point of intersection of angle bisector of $\angle ACB$ and circle.

On measuring, we get $XY = 5.2$ cm.

Example 14. Use graph paper for this question. (Take 1 cm = 1 unit on both axes.)

(i) Plot the points $A(1, 1)$, $B(5, 3)$ and $C(2, 7)$.

(ii) Construct the locus of points equidistant from A and B.

(iii) Construct the locus of points equidistant from AB and AC.

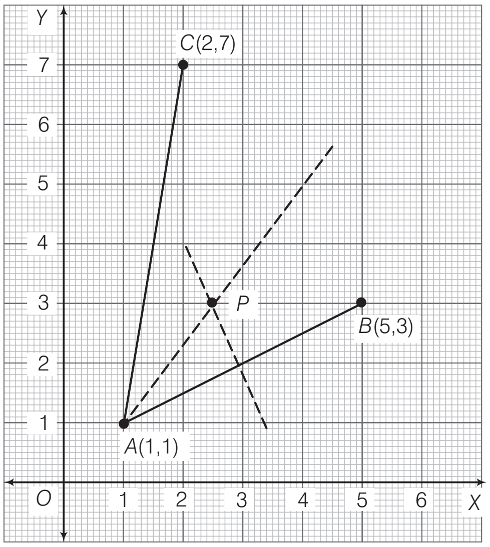
(iv) Locate the point P such that $PA = PB$ and P is equidistant from AB and AC.

(v) Measure and record the length PA (in cm).

Sol. Take 1 cm = 1unit on both axes.

(i) Plot the given points $A(1,1)$, $B(5,3)$ and $C(2,7)$.

(ii) As the locus of points equidistant from A and B is the right bisector of AB, so construct the right bisector of segment AB.



(iii) As the locus of the points equidistant from AB and AC is the bisector of $\angle BAC$, so construct the bisector of $\angle BAC$.

(iv) P is the point of intersection of the right bisector of the segment AB and the bisector of $\angle BAC$, where $PA = PB$.

(v) Length $PA = 2.5$ cm approx.

CHAPTER EXERCISE

a 3 Marks Questions

1. Find the locus of a point at a distance 3 cm from a fixed line AB.

2. A point P moves such that its distance from a fixed line AB is always the same. What is the relation between AB and the path travelled by P?

3. Construct $\angle ABC = 135°$ with $AB = 4$ cm and $BC = 6$ cm. Also, construct the locus of a point, that is equidistant from the arms of the angle.

4. Construct $\angle PQR = 150°$ with $PQ = 5$ cm and $QR = 6.5$ cm. Also, construct the locus of a point, that is equidistant from the arms of the angle.

5. Draw a line segment and construct the locus of a point, that is equidistant from the ends of the line segment $EF = 7$ cm.

6. Draw a line segment and construct the locus of a point, that is equidistant from the ends of the line segment $AB = 4$ cm.

7. Draw two straight lines PQ and RS, intersecting at C, such that $\angle PCR = 70°$. Construct the loci of a point, that is equidistant from the lines.

8. Draw two straight lines AB and CD, intersecting at P, such that $\angle APC = 45°$. Construct the loci of a point, that is equidistant from the lines.

9. Construct $\angle ABC = 60°$. Mark a point O equidistant from AB and BC, such that distance from another line PQ is 2 cm.

10. Construct a ΔEFG, in which $EF = 5$ cm, $FG = 6$ cm and $GE = 5.5$ cm. Find the point O equidistant from F and G and also equidistant from EF and FG.

11. Construct a ΔABC, in which $AB = 7$ cm, $BC = 5.5$ cm and $CA = 4$ cm. Find the point P equidistant from B and C; and also equidistant from AB and BC.

12. What is the locus of the vertices of isosceles triangles having a common base?

13. What is the locus of the vertices of an equilateral triangle having common base?

14. What is the locus of point A, which is equidistant from four given non-collinear points?

15. Draw an angle of 105°. Find a point B, such that B is at a distance of 5 cm from the base and 3 cm from the arm of the given angle.

b 4 Marks Questions

16. Attempt this question on the graph paper. [Take 2 boxes = 1 unit]

(i) Draw the line $x = 7$. Now, the same graph paper, draw the locus of the point, which moves in such a way that its distance from the given line is always equal to 2 units.

(ii) Draw the line $y = 3$ in the different graph paper. Now, draw the locus of the point on this graph, which moves in such a way that its distance from the given line is always equal to 3 units.

17. Use graph paper for this question.

Take 10 small divisions = 1 unit on both the axes.

(i) Plot the points P (2,2), Q (6,4) and R (3,8).

(ii) Construct the locus of points equidistant from P and Q.

(iii) Construct the locus of points equidistant from PQ and PR.

(iv) Locate the point A, such that $AP = AQ$ and A is equidistant from PQ and PR.

(v) Measure and record the length AP (in cm).

18. Plot the points E (2,9), F (−1, 3) and C (6, 3) on a graph paper. On same graph paper, draw the locus of point E, so that area of ΔEFG remains same as E moves.

19. Construct a ΔMNO with $MN = 4$ cm and $MO = NO = 7$ cm. Find the points equidistant from MN and MO and also 5 cm from NO. Measure the distance between the two points obtained.

20. Using the ruler and compasses only, construct a ΔFGO, such that $FG = 4.5$ cm, $FO = 3$ cm and $\angle GFO = 45°$. Complete the rectangle $EFGH$, such that

(i) O is equidistant from EF and FG.

(ii) O is equidistant from G and H.

21. Using only ruler and compasses, construct $\angle ABC = 120°$, where $AB = BC = 5$ cm.

(i) Mark two points D and E, which satisfy the condition that they are equidistant from both BA and BC.

(ii) In the figure obtained in part (i), join AE and EC. Describe the figures **[2002]**

(a) $ABCD$ (b) $\angle ABD$ (c) ABE

22. Without using set square or a protractor, construct a ΔABC, in which $AB = 4$ cm, $BC = 5$ cm and $\angle ABC = 120°$. Locate the point P, such that $\angle BAP = 90°$ and $BP = CP$. Also, measure the length of BP.

23. Use a ruler and a pair of compasses only for this question.

(i) Construct a ΔPQR with $PQ = 4.5$ cm, $QR = 7$ cm and $\angle PQR = 135°$.

(ii) In the same diagram, draw a circle with QR as diameter. Find a point P on the circumference of the circle, that is equidistant from PQ and QR.

(iii) Measure $\angle QRP$.

24. Using ruler and compasses, construct

(i) a ΔABC in which $AB = 5.5$ cm, $BC = 3.4$ cm and $CA = 4.9$ cm.

(ii) the locus of points equidistant from A and C.

(iii) a circle touching AB at A and passing through C. **[2009]**

25. (i) Construct a ΔLMN, in which $LM = 3.5$ cm, $MN = 4$ cm and $NL = 5$ cm, using ruler and compasses only.

(ii) Construct the locus of points equidistant from L and N.

(iii) Construct a circle touching LM at L and passing through N.

26. Construct a ΔABC with $AB = 5.5$ cm, $AC = 6$ cm and $\angle BAC = 105°$. Hence,

(i) construct the locus of points equidistant from BA and BC.

(ii) construct the locus of points equidistant from B and C.

(iii) mark the point which satisfies the above two loci as P. Measure and write the length of PC. **[2015]**

27. Construct a ΔABC, such that $AB = 7$ cm, $AC = 5$ cm and $BC = 6$ cm. Construct a point which is equidistant from all the three sides of the triangle. Name the point.

28. In ΔABC, bisectors of interior angles at A and C intersect each other at point O.
Prove that

(i) point O is equidistant from all the three sides of the triangle.

(ii) OA bisects $\angle ABC$.

29. Prove that the locus of a point equidistant from the arms of an angle is the bisector of the angle.

30. What is the locus of the following?

(i) The centre of a wheel of a cycle.

(ii) The door-handle as the door opens.

31. Show that the locus of the centres of all circles, passing through two given points A and B is the perpendicular bisector of line segment AB.

32. Find the locus of centres of circles, which touch two intersecting lines.

33. P and Q are two different points. Prove that the locus of a point R, such that $\angle PRQ = 90°$, is a circle with mid-point O of PQ as centre and OP as radius.

34. A is a fixed point on the circumference of a circle of radius 4 cm with centre O. M is the mid-point of a variable chord AB. State the locus of M and justify your answer.

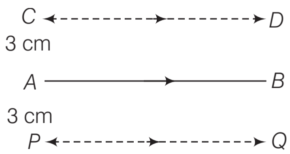
35. Use ruler and a pair of compasses only for this question. Draw a circle of radius 4 cm and mark two chords AB and AC of the circle of length 6 cm and 5 cm, respectively.

(i) Construct the locus of points inside the circle, that are equidistant from A and C. Prove your construction.

(ii) Construct the locus of points inside the circle, that are equidistant from AB and AC.

Hints and Answers

1. Hint

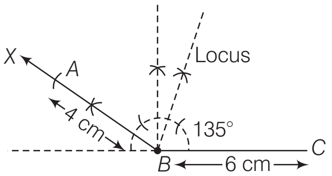


The locus of the points at a distance 3 cm from a fixed line is a pair of straight lines parallel to the given fixed line.

2. Do same as Q. 1.

Ans. A pair of straight lines parallel to AB.

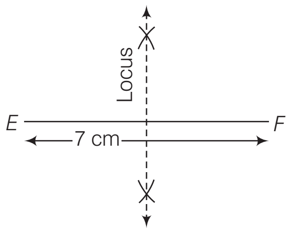
3. Hint



The locus of a point equidistant from AB and BC is the bisector of $\angle ABC$.

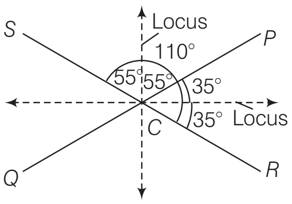
4. Do same as Q. 3.

5. Hint



6. Do same as Q. 5.

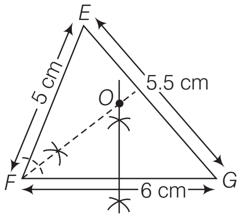
7. Hint



8. Do same as Q. 7.

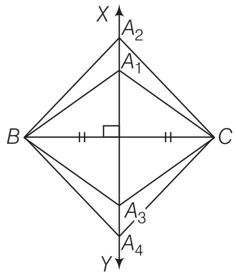
9. Do same as Example 4.

10. Hint



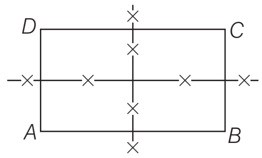
11. Do same as Q. 9.

12. Hint



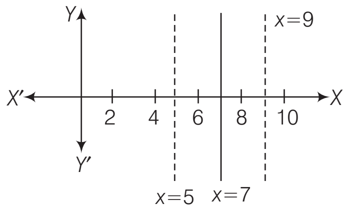
13. Do same as Q. 12.

14. Hint Let four given non-collinear points be A, B, C and D. Now, draw perpendicular bisector of each side.



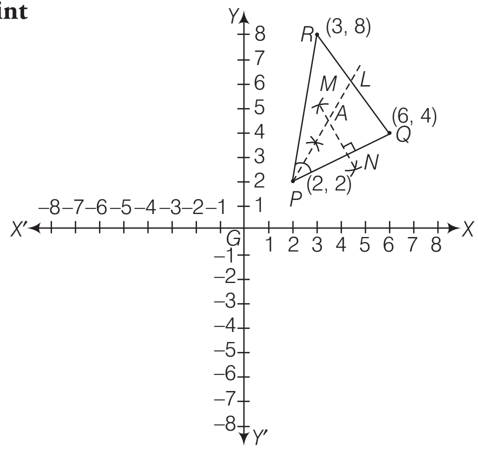
15. Do same as Example 3.

16. Hint (i)



(ii) Do same as part (i).

17. Hint



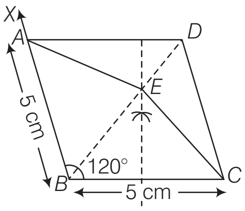
Ans. Measure of AP is 1.9 cm (approx.).

18. Do same as Q. 17.

19. Do same as Example 5.

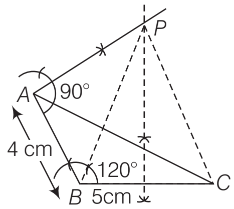
20. Do same as Example 9.

21. Hint



Ans. (a) Rhombus (b) Bisector of $\angle ABC$ (c) Triangle

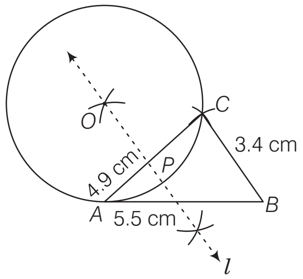
22. Hint



Ans. Measure of $BP = 6.8$ cm.

23. Do same as Example 11. **Ans.** (iii) 22°

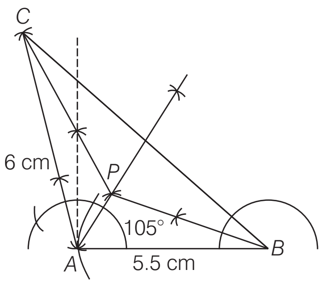
24. Hint



(ii) Point P.

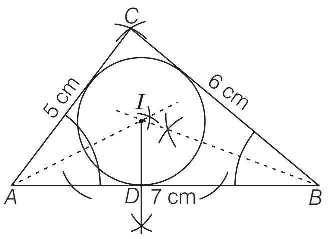
25. Do same as Q. 24.

26. Hint



Ans. (iii) The length of PC is 4.8 cm.

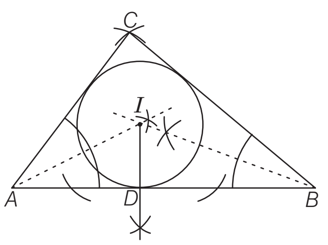
27. Hint



The point is I.

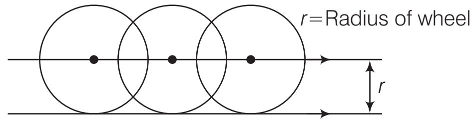
28. Do same as Q. 27.

29. Hint

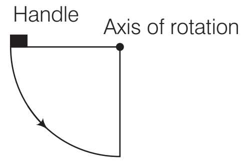


30. Hint

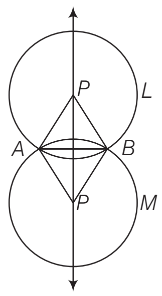
(i) Locus of the centre of a wheel of a cycle is a straight line parallel to the given level of road at a distance equal to the radius of the wheel.



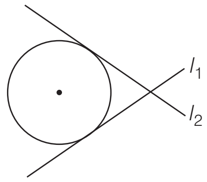
(ii) Locus of the door-handle as the door opens, is the quadrant of the circle with its centre at the axis of rotation of the door and radius equal to the distance between door-handle and axis of rotation.



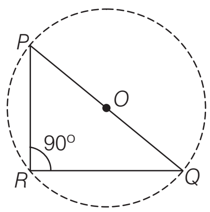
31. Hint



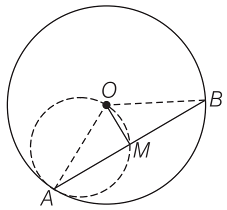
32. Hint



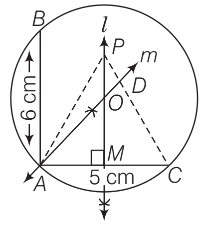
33. Hint



34. Hint



35. Hint



ARCHIVES* (Last 8 Years)

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2017

1 Using a ruler and a compass construct a ΔABC in which $AB = 7$ cm, $\angle CAB = 60°$ and $AC = 5$ cm. Construct the locus of

(i) points equidistant from AB and AC.

(ii) points equidistant from BA and BC.

Hence, construct a circle touching the three sides of the triangle internally.

2016

2 Use ruler and compasses only for the following question. All construction lines and arcs must be clearly shown.

(i) Construct a ΔABC in which $BC = 6.5$ cm, $\angle ABC = 60°$, $AB = 5$ cm.

(ii) Construct the locus of points at a distance of 3.5 cm from A.

(iii) Construct the locus of points equidistant from AC and BC.

(iv) Mark 2 points X and Y which are at a distance of 3.5 cm from A and also equidistant from AC and BC. Measure XY.

2015

3 Construct a ΔABC with $AB = 5.5$ cm, $AC = 6$ cm and $\angle BAC = 105°$. Hence,

(i) construct the locus of points equidistant from BA and BC.

(ii) construct the locus of points equidistant from B and C.

(iii) mark the point which satisfies the above two loci as P. Measure and write the length of PC.

2013

4 Use a ruler and compasses only for this question.

(i) Construct a ΔABC with $AB = 3.5$ cm, $BC = 6$ cm and $\angle ABC = 120°$.

(ii) In the same diagram, draw a circle with BC as diameter. Find a point P on the circumference of the circle, that is equidistant from AB and BC.

(iii) Measure $\angle BCP$.

2012

5 Construct a ΔABC, in which base $BC = 6$ cm, $AB = 5.5$ cm and $\angle ABC = 120°$.

(i) Construct a circle circumscribing the ΔABC.

(ii) Draw a cyclic quadrilateral $ABCD$, so that D is equidistant from B and C.

* All these questions are completely covered in chapter either as solved examples or in chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. $ABCD$ is a quadrilateral in which $AB = BC$. E is the point of intersection of the perpendicular bisectors of AD and CD. Then, $\angle ABC$ is

(a) $2\angle ABE$ (b) $2\angle AEB$
(c) $2AEC$ (d) $2\angle AED$

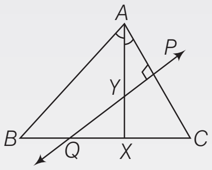
2. If a point moves so that the sum of its distances from two intersecting straight lines is always constant, then the locus of the moving point is

(a) $\angle FGO$
(b) $2\angle FGO$
(c) thrice of $\angle FGO$
(d) bisector of $\angle FGO$

3. The locus of the vertex of a triangle of given base AB and height h, is a pair of lines

(a) parallel to the base
(b) perpendicular to the base
(c) Both (a) and (b)
(d) None of the above

4. In the given figures, AX bisects $\angle BAC$ and PQ is perpendicular bisector of AC, which meets AX at point Y.



The following statements are given below

I. X is equidistant from AB and AC.
II. Y is equidistant from A and C.

(a) I is true
(b) II is true
(c) I and II are true
(d) None of the above

5. We construct a ΔABC, in which $AB = 4.2$ cm, $BC = 6.3$ cm and $AC = 5$ cm, and draw perpendicular bisector of BC, which meets AC at point D. The point D is equidistant from

(a) A and C (b) B and C
(c) A and B (d) None of these

6. We construct a right angled ΔPQR, in which $\angle Q = 90°$, hypotenuse $PR = 8$ cm and $QR = 4.5$ cm, and draw bisector of $\angle PQR$ which meets PR at point T. Then,

(a) $BT = AT$ (b) $BT > AT$
(c) $BT < AT$ (d) None of these

7. We construct a ΔABC, in which $\angle ABC = 75°$, $AB = 5$ cm and $BC = 6.4$ cm. Draw perpendicular bisector of side BC and also the bisector of $\angle ACB$. If these bisectors intersect each other at point P. Which of the following statements is/are true?

I. P is equidistant from B and C.
II. P is equidistant from AC and BC.

(a) I is true
(b) II is true
(c) Both I and II are true
(d) None of the above

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 408.

CHAPTER

15

Circles

Chapter Objectives

- Angle Properties of a Circle
- Cyclic Properties of Circles
- Tangent and Secant Properties of Circles

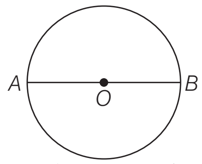
In class IX, we have already studied about circle and its related terms like chord, segments, sectors, arcs and chord properties. Here in this chapter, we will extend our knowledge to some more properties related to circle, like angle properties, cyclic properties, tangent and secant properties.

Circle

A circle is a collection of all points in a plane, which are at a constant distance from a fixed point. Here, fixed point is called **centre** and constant distance is called **radius** of circle.

Or

A circle is the locus of a point, which moves in a plane in such a way that its distance from a given fixed point in the plane is always constant.

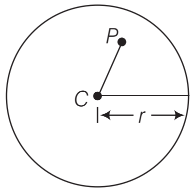


In the given figure, O is the centre of circle, OA and OB are the radii of the circle and AB is the diameter of the circle.

Some Basic Terms Related to Circle

There are various terms related to circle, which are as follow

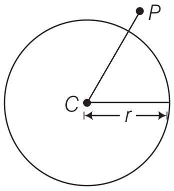
(i) **Interior of a circle** A point P lies inside a circle if and only if its distance from the centre of the circle is less than the radius of the circle.



i.e. $CP < r$

The set of all such points P forms the interior of the circle.

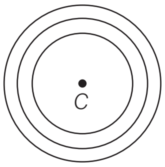
(ii) **Exterior of a circle** A point P lies outside a circle if and only if its distance from the centre of the circle is greater than the radius of the circle.



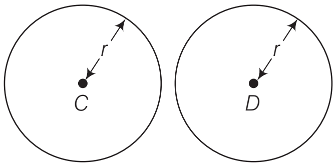
i.e. $\quad CP > r$.

The set of all such points P forms the exterior of the circle.

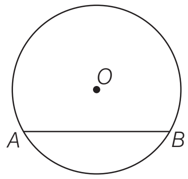
(iii) **Concentric circles** Two or more circles are called **concentric circles** if and only if they have same centre and different radii.



(iv) **Equal** (Congruent) **circles** Two or more circles are called **equal** (or congruent) **circles** if and only if they have same radii.

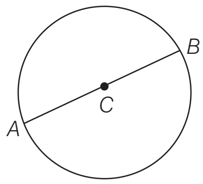


(v) **Chord of a circle** A line segment which join any two points on a circle, is called a **chord of the circle**.

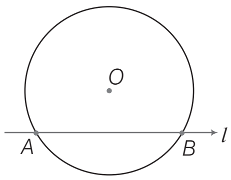


The maximum length of the chord is equal to the length of the diameter.

(vi) **Diameter of a circle** If a chord of a circle passing through its centre, then it is called a **diameter of a circle**.

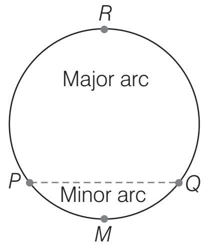


(vii) **Secant of a circle** A line which intersects the circle in two points, is called **secant of the circle**.



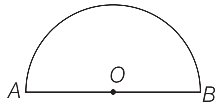
(viii) **Arc of a circle** A piece of a circle between two points is called **an arc.**

In figure, there are two pieces, one longer and the other smaller.

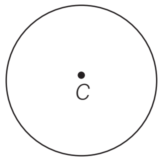


The longer one is called the **major arc** ($\overset{\frown}{PRQ}$) and the smaller one is called the **minor arc** ($\overset{\frown}{PMQ}$).

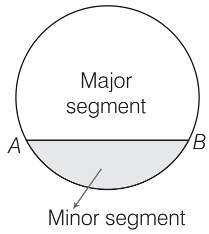
(ix) **Semi–circle** A diameter of a circle divides it into two equal parts, which are two equal arcs. Each of these two arcs is called **a semi–circle**.



(x) **Circumference of a circle** The whole arc of a circle is called the **circumference of a circle**.

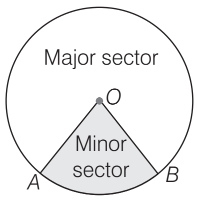


(xi) **Segment of a circle** The region between a chord and either of its arcs, is called a **segment of the circular** region or simply a **segment of the circle**.



The segment formed by minor arc alongwith chord, is called **minor segment** and the segment formed by major arc, is called the **major segment**.

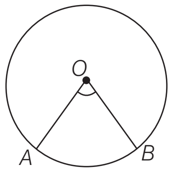
(xii) **Sector of a circle** The region between an arc and the two radii, is called a sector.



Minor sector is the sector formed by the minor arc and **major sector** is the sector formed by the major arc.

(xiii) **Angle subtended by an arc** The angle formed by the two bounding radii of an arc of a circle at the centre of the circle, is called the **angle subtended by an arc**.

In the given figure, $\angle AOB$ is the angle subtended by $\widehat{AB}$ of a circle with centre O.



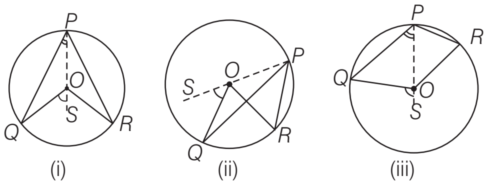
The measurement of $\angle AOB$ in degree is called degree measurement of $\widehat{AB}$.

Topic 1
Angle Properties of a Circle

The properties of angles of a circle in the form of theorems, are following

Theorem 1 *The angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the remaining part of the circle.*

Given A circle with centre O. Arc QR subtends $\angle QOR$ at the centre and $\angle QPR$ at any point P on the remaining part of the circle.



(i) (ii) (iii)

To prove $\angle QOR = 2\angle QPR$

Construction Join OP and produce it to any point S.

Proof In $\Delta POQ, OQ = OP$ [radii of same circle]

$\Rightarrow \quad \angle OQP = \angle QPO$ ...(i)

[$\because$ angles opposite to the equal sides of a triangle are equal]

Now, $\quad \angle QOS = \angle OQP + \angle QPO$

[$\because$ exterior angle = sum of opposite interior angles]

$\Rightarrow \quad \angle QOS = \angle QPO + \angle QPO$ [from Eq. (i)]

$\Rightarrow \quad \angle QOS = 2\angle QPO$ ...(ii)

Similarly, in ΔPOR, we have

$\angle SOR = 2\angle OPR$ ...(iii)

Case I From Fig. (i) and (iii), adding Eqs. (ii) and (iii), we get

$\Rightarrow \quad \angle QOS + \angle SOR = 2\angle QPO + 2\angle OPR$

$\Rightarrow \quad \angle QOR = 2(\angle QPO + \angle OPR)$

$\Rightarrow \quad \angle QOR = 2\angle QPR$

Case II From Fig. (ii), subtracting Eq. (ii) from Eq. (iii), we get

$\angle SOR - \angle QOS = 2\angle OPR - 2\angle QPO$

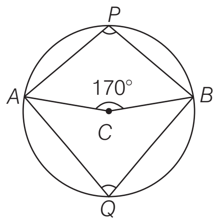
$\Rightarrow \quad \angle QOR = 2(\angle OPR - \angle QPO)$

$\Rightarrow \quad \angle QOR = 2\angle QPR$

In all the cases, we have

$\angle QOR = 2\angle QPR$ **Hence proved.**

Example 1. In the following figure, C be the centre of the circle and $\angle ACB = 170°$, then find the value of $\angle AQB$ and $\angle APB$.



Sol. Given, $\angle ACB = 170°$ ...(i)

Since, angle at the centre of a circle is twice the angle at the remaining part of the circumference of the circle.

$\therefore \quad \angle ACB = 2\angle AQB$

$\Rightarrow \quad 170° = 2\angle AQB$ [from Eq. (i)]

$\Rightarrow \quad \angle AQB = \dfrac{170°}{2} \Rightarrow \angle AQB = 85°$

Also, reflex $\angle ACB = 2\angle APB$

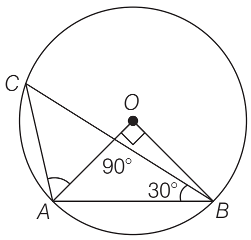
$\therefore \quad 360° - \angle ACB = 2\angle APB$

$\Rightarrow \quad 360° - 170° = 2\angle APB$ [from Eq. (i)]

$\Rightarrow \quad 190° = 2\angle APB \Rightarrow \angle APB = \dfrac{190°}{2} = 95°$

Hence, the required values of $\angle AQB$ and $\angle APB$ are 85° and 95°, respectively.

Example 2. In the given figure, O is the centre of the circle. If $\angle AOB = 90°$ and $\angle ABC = 30°$, then find the measure of $\angle CAO$.



Sol. In ΔOAB, $OA = OB$ [radii of same circle]

$\Rightarrow \quad \angle OAB = \angle OBA$

[$\because$ opposite equal sides of a triangle are equal]

In ΔOAB, $\angle OAB + \angle OBA + \angle AOB = 180°$

[sum of angles of a triangle]

$\Rightarrow \angle OAB + \angle OAB + 90° = 180° \Rightarrow \angle OAB = 45°$

Now, $\angle ACB = \frac{1}{2} \angle AOB$

[$\because$ angle subtended by an arc at the centre = double the angle subtended by it at any point on the remaining part of the circle]

$\Rightarrow \angle ACB = \frac{1}{2} \times 90° = 45°$

In ΔABC, $\angle CAB + \angle CBA + \angle ACB = 180°$

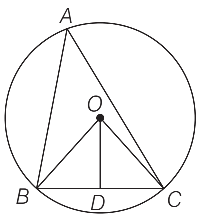
[angle sum property of a triangle]

$\Rightarrow \angle CAB + 30° + 45° = 180° \Rightarrow \angle CAB = 105°$

From the figure, $\angle CAO + \angle OAB = \angle CAB$

$\Rightarrow \angle CAO + 45° = 105° \Rightarrow \angle CAO = 60°$

Example 3. In the given figure, O is the circumcentre of ΔABC. If $OD \perp BC$, prove that $\angle BOD = \angle BAC$.



Sol. In ΔOBD and ΔOCD,

$OB = OC$ [radii of same circle]

$OD = OD$ [common]

$\angle ODB = \angle ODC$ [each angle = 90°, as $OD \perp BC$]

$\therefore \quad \Delta OBD \cong \Delta OCD$ [RHS rule of congruency]

$\Rightarrow \quad \angle BOD = \angle COD$ [by CPCT]

$\Rightarrow \quad \angle BOC = 2\angle BOD$ ...(i)

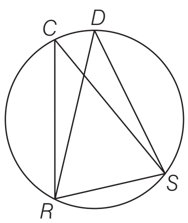
Now, $\angle BOC = 2\angle BAC$

[$\because$ angle subtended by an arc at the centre = double the angle subtended by it at any point on the remaining part of the circle]

$\Rightarrow \quad 2\angle BOD = 2\angle BAC$ [using Eq. (i)]

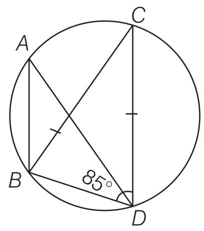
$\therefore \quad \angle BOD = \angle BAC$ **Hence proved.**

Theorem 2 *Angles in the same segment of a circle are equal,*



i.e. $\angle C = \angle D$ [$\because$ *$\angle C$ and $\angle D$ lie in the same segment*]

Example 4. In the following figure, $CB = CD$ and $\angle CDB = 85°$, then find $\angle BAD$.



Sol. Given, $CB = CD$ and $\angle CDB = 85°$.

Now, in ΔCBD, we have

$CB = CD$ [given]

$\therefore \quad \angle CBD = \angle CDB = 85°$ ...(i)

[$\because$ angles opposite to equal sides of a triangle are equal]

Also, $\angle CDB + \angle CBD + \angle BCD = 180°$

[by angle sum property of a triangle]

$\Rightarrow \quad 85° + 85° + \angle BCD = 180°$ [from Eq. (i)]

$\Rightarrow \quad 170° + \angle BCD = 180°$

$\Rightarrow \quad \angle BCD = 180° - 170°$

$\Rightarrow \quad \angle BCD = 10°$ ...(ii)

Now, $\angle BAD = \angle BCD$

[$\because$ angles of same segment are equal]

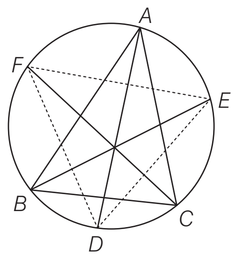
$\Rightarrow \quad \angle BAD = 10°$ [from Eq. (ii)]

Hence, the required value of $\angle BAD$ is 10°.

Example 5. Bisectors of angles A, B and C of a triangle ABC intersect its circumcircle at points D, E and F, respectively. Prove that the angles of the triangle DEF are $90° - \frac{1}{2}\angle A$, $90° - \frac{1}{2}\angle B$ and $90° - \frac{1}{2}\angle C$.

Sol. Here, $\angle ADE = \angle ABE$ [angles in same segment]

and $\angle ADF = \angle ACF$ [angles in same segment]



But $\angle ABE = \frac{1}{2}\angle B$ [$\because$ BE is bisector of $\angle B$]

and $\angle ACF = \frac{1}{2}\angle C$ [$\because$ CF is bisector of $\angle C$]

$\therefore \quad \angle ADE = \frac{1}{2}\angle B$ and $\angle ADF = \frac{1}{2}\angle C$

$\therefore \quad \angle EDF = \angle ADE + \angle ADF = \frac{1}{2}(\angle B + \angle C)$

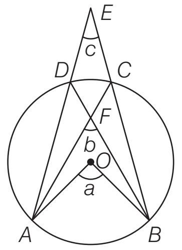
$= \frac{1}{2}(180° - \angle A)$ [$\because \angle A + \angle B + \angle C = 180°$]

$= 90° - \frac{1}{2}\angle A$

Similarly, $\angle DEF = 90° - \frac{1}{2}\angle B$ and $\angle EFD = 90° - \frac{1}{2}\angle C$.

Hence proved.

Example 6. In the given figure, O is the centre of the circle. Prove that $a = b + c$.



Sol. We have, $a = 2\angle ADB$

[∵ angle subtended by an arc at the centre is double the angle subtended by it at any other point on the remaining part of the circle]

$= 2\angle ACB$

[∵ angles in the same segment are equal]

$\Rightarrow \quad \angle ADB = \angle ACB = \dfrac{a}{2}$

Also, $\quad \angle BDE = 180° - \dfrac{a}{2} = \angle ACE$ [∵ linear pair axiom]

and $\quad \angle CFD = b$ [vertically opposite angles]

In quadrilateral $DECF$,

$\angle DEC + \angle ECF + \angle CFD + \angle FDE = 360°$

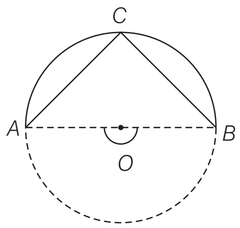
[∵ sum of all angles of a quadrilateral is 360°]

$\Rightarrow \quad c + 180° - \dfrac{a}{2} + b + 180° - \dfrac{a}{2} = 360°$

$\Rightarrow \quad c + b - a = 0 \Rightarrow a = b + c$ **Hence proved.**

Theorem 3 *The angle in a semi-circle is a right angle.*

Given A semi-circle ACB of a circle with centre O.



To prove $\angle ACB = 90°$

Proof Clearly, $\angle AOB = 2\angle ACB$

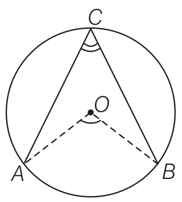
[∵ angle subtended by an arc at the centre is twice the angle subtended at the remaining circumference]

But $\quad \angle AOB = 180°$ [∵ AOB is a straight line]

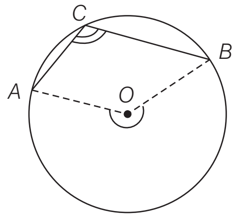
∴ $\quad 2\angle ACB = 180° \Rightarrow \angle ACB = 90°$

Hence, the angle in a semi-circle is 90°.

(i) In a circle, the angle in a segment greater than a semi-circle, is less than a right angle.



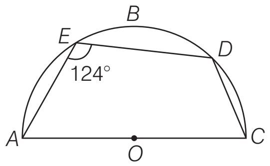
(ii) In a circle, the angle in a segment less than a semi-circle, is greater than a right angle.



Theorem 4 (**Converse of Theorem 3**) *If an arc of a circle subtends a right angle at any point on the remaining part of the circle, then the arc is a semi–circle.*

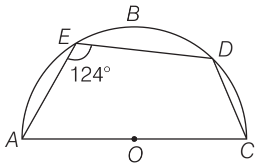
A circle drawn with hypotenuse of a right angled triangle as diameter, passes through its opposite vertex.

Example 7. In the given figure, ABC is a semi-circle with centre O. Find the values of $\angle DEC$ and $\angle DAC$, if $\angle AED = 124°$.



Sol. **Given** ABC is a semi-circle with centre O and $\angle AED = 124°$.

Construction Join AD and EC.



Since, angle in a semi-circle is 90°.

∴ $\quad \angle AEC = 90°$

Now, $\quad \angle DEC = \angle AED - \angle AEC$

$\Rightarrow \quad \angle DEC = 124° - 90°$ [given]

$\Rightarrow \quad \angle DEC = 34°$ ...(i)

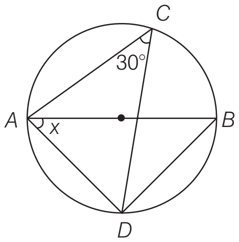
∴ $\quad \angle DAC = \angle DEC$

[∵ angles in the same segment are equal]

$\Rightarrow \quad \angle DAC = 34°$ [from Eq. (i)]

Hence, the required values of $\angle DEC$ and $\angle DAC$ are 34° and 34°, respectively.

Example 8. In the given circle with diameter AB, find the value of x. ***[2003]***



Sol. We have, $\angle ABD = \angle ACD$

[$\because$ angles in same segment are equal]

$\Rightarrow \quad \angle ABD = 30°$ [$\because \angle ACD = 30°$, given]

Since, AB is a diameter of the given circle, $\angle ADB = 90°$

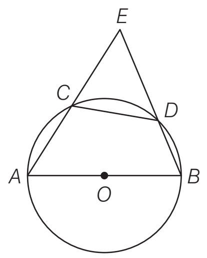
[$\because$ angle in a semi-circle = 90°]

In ΔABD, $\angle DAB + \angle ADB + \angle ABD = 180°$

[by angle sum property of a triangle]

$\Rightarrow x + 90° + 30° = 180° \Rightarrow x = 180° - 90° - 30° = 60°$

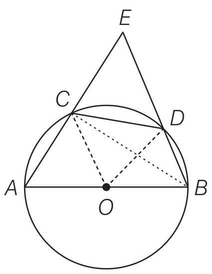
Example 9. In the adjoining figure, AB is a diameter of a circle with centre O and CD is a chord equal to the radius of the circle. AC and BD when extended intersect at E. Prove that $\angle AEB = 60°$.



Sol. Join OC, OD and CB.

As $CD = OC = OD$ [given]

$\therefore$ ΔOCD is an equilateral triangle.



$\Rightarrow \quad \angle COD = 60°$

$\therefore \quad \angle CBD = \frac{1}{2} \times \angle COD = \frac{1}{2} \times 60° = 30°$

[$\because$ the angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.]

As, AB is a diameter, $\angle ACB = 90°$ [$\because$ angle in a semi-circle]

Since, $\angle ACB$ is an exterior angle of ΔCBE.

$\therefore \quad \angle ACB = \angle CEB + \angle CBE$

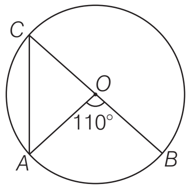
[$\because$ exterior angle = sum of two interior opposite angles]

$\Rightarrow \quad 90° = \angle CEB + 30°$ [$\because \angle CBE = \angle CBD = 30°$]

$\Rightarrow \angle CEB = 60° \Rightarrow \angle AEB = 60°$ **Hence proved.**

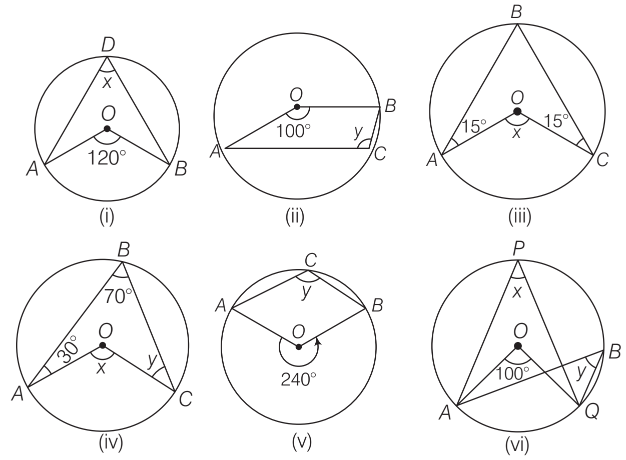
Topic Exercise 1

1. In the given figure, O is the centre of circle and $\angle AOB = 110°$. Calculate

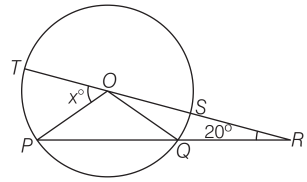


(i) $\angle ACO$ (ii) $\angle CAO$.

2. Find the value of unknown variables in the following.

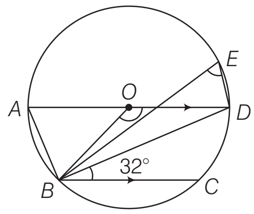


3. In the figure given below, 'O' is the centre of the circle. If $QR = OP$ and $\angle ORP = 20°$. Find the value of 'x' giving reasons.



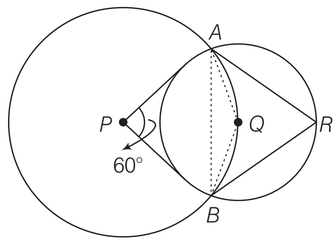
[2018]

4. In the given figure, AD is a diameter and O is the centre of the circle AD is parallel to BC and $\angle CBD = 32°$.

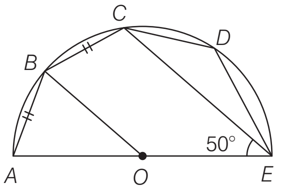


Find (i) $\angle OBD$ (ii) $\angle AOB$ (iii) $\angle BED$ **[2016]**

5. In the given figure, the two circles with centres P and Q intersect at A and B. Q lies on the circumference of the bigger circle. If $\angle APB = 60°$, then find $\angle ARB$.



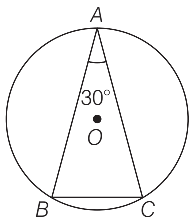
6. Given that $\angle AEC = 50°$, $AB = BC$, AOE is the diameter, find $\angle AOB$. Hence, show that $BO \parallel CE$.



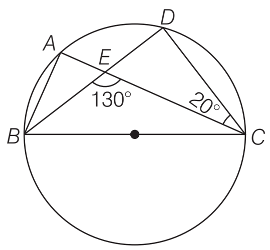
7. O is the circumcentre of the ΔABC and D is mid-point of the base BC. Prove that $\angle BOD = \angle A$.

8. AC and BD are chords of a circle that bisect each other. Prove that
(i) AC and BD are diameters. (ii) $ABCD$ is a rectangle.

9. In the given figure, ABC is a triangle in which $\angle BAC = 30°$. Show that BC equals the radius of the circumcircle of ΔABC, whose centre is O.

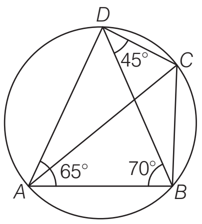


10. In the following figure, A, B, C and D are four points on a circle. AC and BD intersect at a point E. If $\angle BEC = 130°$ and $\angle ECD = 20°$, find $\angle BAC$.



(i) Show that AC is the diameter. (ii) Find $\angle ACB$.

11. In the given figure, $\angle BAD = 65°$, $\angle ABD = 70°$ and $\angle BDC = 45°$.



(i) Show that AC is the diameter.
(ii) Find $\angle ACB$.

Hints and Answers

1. (i) **Hint** The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

$$\angle ACO = \frac{1}{2}\angle AOB = \frac{1}{2} \times 110°. \textbf{ Ans. } 55°$$

(ii) **Hint** $OA = OC \Rightarrow \angle ACO = \angle CAO$ **Ans.** $55°$

2. (i) Do same as Q. 1. **Ans.** $60°$

(ii) **Hint** Reflex $\angle AOB = 360° - 100° = 260°$

$\therefore \quad y = \frac{1}{2}$ reflex $\angle AOB$ **Ans.** $130°$

(iii) **Hint** $\because$ Angle at centre $= 2 \times$ Angle at any point on the remaining part of circle

$\therefore \ \angle ABC = \frac{x}{2}$ and reflex $\angle AOC = (360° - x)$

In quadrilateral $ABCO$,

$$\angle A + \angle B + \angle C + \angle O = 360°$$

$$\Rightarrow \quad 15° + \frac{x}{2} + 15° + (360° - x) = 360°$$

$$\Rightarrow \quad 30° - \frac{x}{2} = 0$$ **Ans.** $60°$

(iv) Do same as part (iii). **Ans.** $140°, 40°$

(v) **Hint** $\because$ Angle at centre $= 2 \times$ Angle at any point on the remaining part of circle

$$\therefore \quad y = \frac{1}{2} \times 240°$$

Ans. $120°$

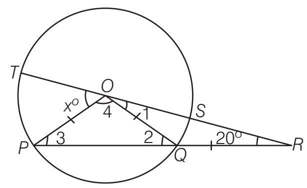
(vi) $\because$ Angle at centre $= 2 \times$ Angle at any point on the remaining part of circle

$$\therefore \quad x = \frac{1}{2} \times 100 = 50°$$

and $\quad x = y \quad$ [$\because$ angles in the same segment]

Ans. $50°$

3. **Hint**



$$OP = OQ = QR$$

$\therefore \quad \angle 1 = \angle ORP$

$\Rightarrow \quad \angle 1 = 20°$

Again, $\quad \angle 2 = \angle 1 + \angle ORP$

$\qquad = 20° + 20° = 40°$

Also, $\quad OP = OQ$

$\Rightarrow \quad \angle 3 = \angle 2 = 40°$

Now, $\angle 2 + \angle 3 + \angle 4 = 180°$

$\Rightarrow$ $40° + 40° + \angle 4 = 180° \Rightarrow \angle 4 = 100°$

Again, $x + \angle 1 + \angle 4 = 180°$

Ans. $x = 60°$

4. Hint $\angle ADB = \angle CBD = 32°$

[$\because AD \parallel BC$, so alternate interior angles]

and $\angle ODB = \angle OBD$

[$\because OB = OD$, so angles opposite to equal sides are equal]

(i) $\angle OBD = \angle ODB = \angle ADB$

(ii) $\angle AOB = 2 \times \angle ADB$

(iii) In ΔBOD, $\angle BOD = 180° - (\angle OBD + \angle ODB)$

$= 180° - (32° + 32°) = 116°$

$\therefore \angle BED = \dfrac{1}{2} \angle BOD$

Ans. (i) 32° (ii) 64° (iii) 58°

5. Hint Reflex $\angle APB = 360° - 60° = 300°$

$\angle AQB = \dfrac{1}{2} \cdot$ reflex $\angle APB$

[$\because$ angle at the centre = double the angle at any point on the remaining part of the circle]

$\Rightarrow \angle AQB = \dfrac{1}{2} \times 300° = 150° \Rightarrow \angle ARB = \dfrac{1}{2} \times 150°$

Ans. 75°

6. Hint Angle at centre = 2 × Angle at any point on the remaining part of circle

Join OC.

$\therefore$ $\angle AOC = 2 \times \angle AEC = 2 \times 50° = 100°$

Since, $AB = BC$

$\therefore$ $\angle AOB = \angle BOC$

[$\because$ equal chords of a circle subtends equal angles at the centre]

$\Rightarrow$ $\angle AOC = \angle AOB + \angle BOC = 100°$

$\Rightarrow \angle AOB + \angle AOB = 100° \Rightarrow 2\angle AOB = 100°$

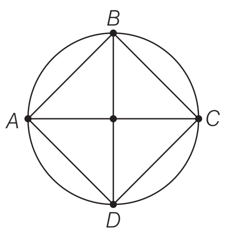
$\Rightarrow$ $\angle AOB = 50°$

Thus, $\angle AOB = \angle AEC = 50° \Rightarrow BO \parallel CE$

[$\because$ a pair of corresponding angles are equal]

7. Do same as Example 3.

8. (i) **Hint** AC and BD are chords of a circle that bisects each other.



$\Rightarrow$ Both are passing through the centre of the circle.

$\Rightarrow$ Both are the diameter of the circle.

(ii) **Hint** AC and BD are diameters of the circle

$\therefore$ $\angle ABC = 90°$; $\angle ADC = 90°$;

$\angle BAD = 90°$ and $\angle BCD = 90°$

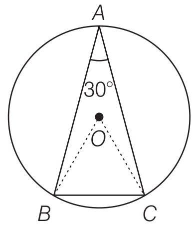
$\Rightarrow ABCD$ is quadrilateral in which each angle is 90°.

$\Rightarrow ABCD$ is rectangle.

9. Hint Join OB and OC.

Now, $\angle BOC = 2\angle BAC$

[$\because$ angle subtended by an arc at the centre = double the angle subtended by it at any point on the remaining part of the circle]



$\Rightarrow$ $\angle BOC = 2 \times 30°$

In ΔOBC, $OB = OC$ [radii of same circle]

$\therefore$ $\angle OCB = \angle OBC$

[$\because$ angles opposite to equal sides of a triangle are equal]

As the sum of angles of a triangle is 180°.

$\therefore$ $\angle OBC + \angle OCB + \angle BOC = 180°$

$\Rightarrow \Delta OBC$ is an equilateral triangle.

$\Rightarrow BC = OB$, i.e. BC is equal to the radius of circle.

10. Hint $\angle BEC$ is an exterior angle of ΔCDE.

$\therefore$ $\angle BEC = \angle ECD + \angle EDC$

[$\because$ exterior angle = sum of two interior opposite angles]

$\Rightarrow$ $130° = 20° + \angle EDC$

Now, $\angle BAC = \angle BDC$

Ans. 110°

11. (i) **Hint** In ΔABD, $\angle ADB + \angle BAD + \angle ABD = 180°$

[by angle sum property of a triangle]

$\Rightarrow \angle ADB + 65° + 70° = 180°$

From figure, $\angle ADC = \angle ADB + \angle BDC$

Thus, an arc of a circle subtends a right angle, so the arc is a semi-circle.

$\Rightarrow AC$ is a diameter of the circle.

(ii) **Hint** $\angle ACB = \angle ADB$

[angles in same segment of a circle]

Ans. 45°

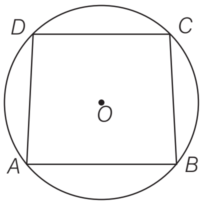
Topic 2
Cyclic Properties of Circles

Cyclic Quadrilateral

A quadrilateral inscribed in a circle, is called cyclic quadrilateral.

Or

A quadrilateral is called cyclic quadrilateral if and only if its all vertices lie on the circumference of a circle.



Concyclic Points

The points which lie on the circumference of the same circle, are called concyclic points.

In the above figure, A, B, C and D are the concyclic points.

Note The cyclic quadrilateral ABCD (say) has two pairs of opposite angles $\angle A$, $\angle C$ and $\angle B$, $\angle D$.

Properties of Cyclic Quadrilateral

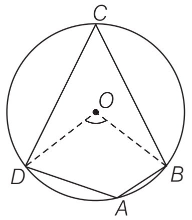
Its properties are shown in terms of theorems as follow

Theorem 1 *The opposite angles of a cyclic quadrilateral are supplementary.*

Given $ABCD$ is a cyclic quadrilateral in a circle with centre O.

To prove $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$.

Construction Join OB and OD.



Proof We know that, angle subtended by an arc at the centre is twice the angle subtended at the remaining part of the circumference of the circle.

$\therefore \quad \angle BOD = 2\angle C \quad$ ...(i)

and reflex $\angle BOD = 2\angle A \quad$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$\angle BOD + \text{Reflex } \angle BOD = 2\angle C + 2\angle A$$

$$\Rightarrow \quad 360° = 2(\angle A + \angle C)$$

[$\because$ sum of angles around a point is 360°]

$$\Rightarrow \quad \angle A + \angle C = \frac{360°}{2} \Rightarrow \angle A + \angle C = 180°$$

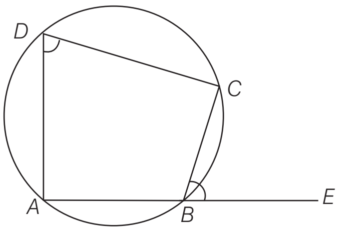
Similarly, join OA and OC and do as above, we get

$$\angle B + \angle D = 180°$$

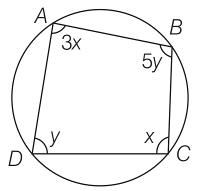
Hence, the each pair of opposite angles of a cyclic quadrilateral are supplementary.

Theorem 2 (**Converse of Theorem 1**) *If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is a cyclic quadrilateral.*

Theorem 3 *The exterior angle of a cyclic quadrilateral is equal to the opposite interior angle, i.e.* $\angle CBE = \angle ADC$.



Example 1. In the given figure, if $ABCD$ is a cyclic quadrilateral, then find the measure of each of its angles.



Sol. We know that the opposite angles of a cyclic quadrilateral are supplementary.

$\therefore \quad \angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$

$\Rightarrow \quad 3x + x = 180°$ and $5y + y = 180°$

[from the given figure]

$\Rightarrow \quad 4x = 180°$ and $6y = 180°$

$\Rightarrow \quad x = \dfrac{180°}{4}$ and $y = \dfrac{180°}{6}$

$\therefore \quad x = 45°$ and $y = 30°$

Now, $\angle A = 3x = 3 \times 45° = 135°$

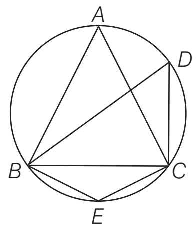
$\angle B = 5y = 5 \times 30° = 150°$

$\angle C = x = 45°$

and $\angle D = y = 30°$

Hence, the required angles of the cyclic quadrilateral are 135°, 150°, 45° and 30°.

Example 2. In the given figure, $\angle DBC = 58^\circ$. BD is a diameter of the circle.



Calculate

(i) $\angle BDC$ (ii) $\angle BEC$ (iii) $\angle BAC$. ***[2014]***

Sol. Given, $\angle DBC = 58^\circ$

(i) Since, BD is a diameter of a circle.

$\therefore \quad \angle BCD = 90^\circ$ [$\because$ angle in a semi-circle is 90°]

In ΔBCD, we have

$\angle DBC + \angle BCD + \angle BDC = 180^\circ$

[by angle sum property of a triangle]

$\Rightarrow \quad 58^\circ + 90^\circ + \angle BDC = 180^\circ$

$\therefore \quad \angle BDC = 180^\circ - (90^\circ + 58^\circ) = 180^\circ - 148^\circ = 32^\circ$

(ii) Since, $BECD$ is a cyclic quadrilateral.

$\therefore \quad \angle BEC + \angle BDC = 180^\circ$

[$\because$ sum of opposite angles in a cyclic quadrilateral is 180°]

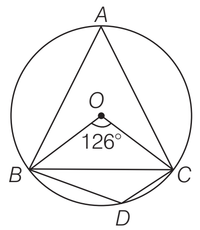
$\Rightarrow \quad \angle BEC = 180^\circ - 32^\circ = 148^\circ$

(iii) Now, $\angle BAC = \angle BDC$

[$\because$ angles in a same segment are equal]

$\therefore \quad \angle BAC = 32^\circ$ [$\because \angle BDC = 32^\circ$]

Example 3. In the given figure, O is the centre of the circle. If $\angle BOC = 126^\circ$, then find $\angle BAC$, $\angle BDC$ and $\angle OBC$.



Sol. Given, $\angle BOC = 126^\circ$

$\therefore \quad \angle BAC = \frac{1}{2}\angle BOC$

[$\because$ angle subtended by an arc at the remaining circumference is half the angle subtended at the centre]

$\Rightarrow \quad \angle BAC = \frac{1}{2} \times 126^\circ = 63^\circ$

Also, $\angle BDC + \angle BAC = 180^\circ$

[$\because$ sum of opposite angles of a cyclic quadrilateral is 180°]

$\Rightarrow \quad \angle BDC = 180^\circ - 63^\circ = 117^\circ$

In ΔOBC, we have $OB = OC$ [radii of same circle]

$\therefore \quad \angle OBC = \angle OCB$ ...(i)

Now, $\angle BOC + \angle OBC + \angle OCB = 180^\circ$

[by angle sum property of a triangle]

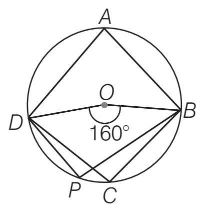
$\Rightarrow \quad \angle OBC + \angle OCB = 180^\circ - 126^\circ = 54^\circ$

$\Rightarrow \quad 2\angle OBC = 54^\circ$ [from Eq. (i)]

$\therefore \quad \angle OBC = 27^\circ$

Hence, $\angle BAC = 63^\circ$, $\angle BDC = 117^\circ$ and $\angle OBC = 27^\circ$.

Example 4. In the given figure, $ABCD$ is a cyclic quadrilateral, O is the centre of the circle. If $\angle BOD = 160^\circ$, then find the measure of $\angle BPD$ and $\angle BCD$.



Sol. Consider the arc BCD of the circle. This arc makes an angle $\angle BOD = 160^\circ$ at the centre of the circle and $\angle BAD$ at a point A on the circumference.

$\therefore \angle BAD = \frac{1}{2}\angle BOD = \frac{160^\circ}{2} = 80^\circ$

Since, $ABPD$ is a cyclic quadrilateral.

$\therefore \quad \angle BAD + \angle BPD = 180^\circ$

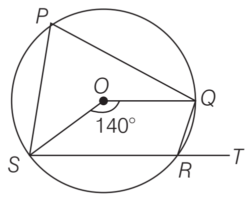
$\Rightarrow \quad 80^\circ + \angle BPD = 180^\circ$

$\Rightarrow \quad \angle BPD = 100^\circ$

$\Rightarrow \quad \angle BCD = 100^\circ$

[$\because \angle BPD$ and $\angle BCD$ are angles in the same segment, therefore $\angle BCD = \angle BPD$]

Example 5. In the given figure, O is the centre of the circle. The angle subtended by the arc SRQ at the centre is 140°. If SR is produced to T, then determine the values of $\angle SPQ$ and $\angle QRT$.



Sol. Given, $\angle SOQ = 140^\circ \Rightarrow \angle SPQ = \frac{1}{2}\angle SOQ$

[$\because$ angle subtended by an arc at the centre is twice the angle subtended at the remaining part of the circumference of the circle]

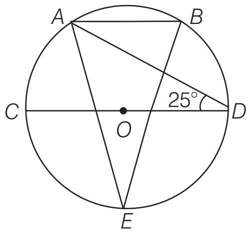
$\Rightarrow \quad \angle SPQ = \frac{1}{2} \times 140^\circ \Rightarrow \angle SPQ = 70^\circ$ ...(i)

We know that, exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

$\therefore \quad \angle QRT = \angle SPQ = 70^\circ$ [from Eq. (i)]

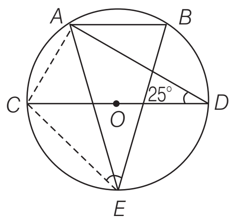
Hence, the required value of $\angle SPQ$ and $\angle QRT$ are 70° and 70°.

Example 6. In the given figure, $AB||CD$ and O is the centre of the circle. If $\angle ADC = 25°$, then find $\angle AEB$. Give reason in support of your answer.



Sol. Given, $AB || CD$, O is the centre of the circle and $\angle ADC = 25°$.

Join AC and CE.



We know that, angles in the same segment are equal.

$\therefore \quad \angle AEC = \angle ADC$

But $\quad \angle ADC = 25°$ [given]

$\therefore \quad \angle AEC = 25°$

$\because \quad AB || CD$ [given]

$\therefore \quad \angle BAD = \angle ADC$ [alternate interior angles]

But $\quad \angle ADC = 25°$ [given]

$\therefore \quad \angle BAD = 25°$

Also, angle in a semi-circle is a right angle.

$\therefore \quad \angle CAD = 90°$

In cyclic quadrilateral $ABEC$, $\angle CAB + \angle CEB = 180°$

[$\because$ sum of opposite angles in a cyclic quadrilateral is 180°]

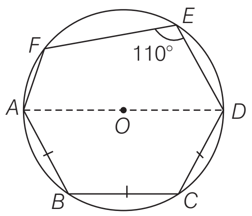
$\Rightarrow \angle CAD + \angle BAD + \angle AEC + \angle AEB = 180°$

$\Rightarrow \quad 90° + 25° + 25° + \angle AEB = 180°$

$\Rightarrow \quad 140° + \angle AEB = 180°$

$\therefore \quad \angle AEB = 180° - 140° = 40°$

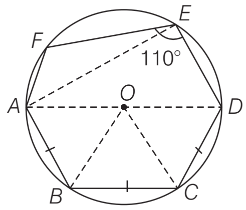
Example 7. In the given figure, AD is the diameter of the circle with centre O. Chords AB, BC and CD are equal.



If $\angle DEF = 110°$, then calculate

(i) $\angle AEF$ (ii) $\angle FAB$.

Sol. Join AE, OB and OC.



(i) Since, angle in a semi-circle is a right angle.

$\therefore \quad \angle AED = 90°$

Now, $\quad \angle AEF = \angle DEF - \angle AED = 110° - 90° = 20°$

(ii) As, $\quad AB = BC = CD$

$\therefore \quad$ arc AB = arc BC = arc CD

$\Rightarrow \quad \angle AOB = \angle BOC = \angle COD$

[$\because$ equal arcs subtend equal angles at centre]

But AOD is a straight line.

$\therefore \quad \angle AOB = \frac{1}{3}\angle AOD$

$\therefore \quad \angle AOB = \frac{1}{3} \times 180° = 60°$

$\because \quad OA = OB$ [radii of same circle]

$\therefore \quad \angle OAB = \angle OBA$

[$\because$ angles opposite to equal sides of a triangle are equal]

$\therefore \quad \angle OAB = \frac{1}{2}(180° - \angle AOB)$

[by angle sum property of a triangle]

$= \frac{1}{2}(180° - 60°) = \frac{1}{2} \times 120° = 60°$

In a cyclic quadrilateral $ADEF$,

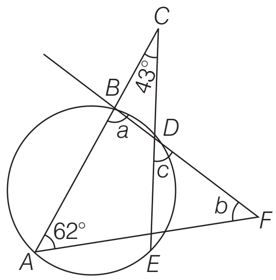
$\angle FAD + \angle DEF = 180° \Rightarrow \angle FAD + 110° = 180°$

$\Rightarrow \quad \angle FAD = 180° - 110°$

$\therefore \quad \angle FAD = 70°$

Now, $\quad \angle FAB = \angle FAD + \angle OAB = 70° + 60° = 130°$

Example 8. In the given figure, if $\angle ACE = 43°$ and $\angle CAF = 62°$, then find the values of a, b and c.



[2007]

Sol. In ΔACE, we have

$\angle AEC + \angle EAC + \angle ACE = 180°$

[by angle sum property of a triangle]

$\Rightarrow \quad \angle AEC + 62° + 43° = 180°$

$\Rightarrow \quad \angle AEC = 180° - 105° = 75°$

Since, points A, B, D and E are the points on a circle, so quadrilateral $ABDE$ is a cyclic quadrilateral.

We know that, sum of opposite angles of a cyclic quadrilateral is 180°.

$\therefore \quad \angle EAB + \angle EDB = 180°$

$\therefore \quad \angle EDB = 180° - 62° = 118°$

and $\quad \angle ABD + \angle AED = 180°$

$\Rightarrow \quad a = 180° - 75° = 105°$ [$\because \angle AEC = \angle AED = 75°$]

Since, BF is a straight line.

$\therefore \quad \angle BDE + \angle EDF = 180°$

$\Rightarrow \quad 118° + c = 180°$

$\Rightarrow \quad c = 180° - 118° = 62°$

In ΔDEF, we have

$\angle DEF + \angle EFD + \angle FDE = 180°$ [by angle sum property]

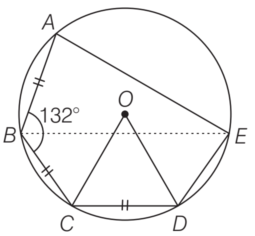
$\Rightarrow \quad 105° + b + 62° = 180°$

[$\because$ exterior angle of a cyclic quadrilateral is equal to the opposite interior angle, i.e. $\angle DEF = a = 105°$]

$\Rightarrow \quad b = 180° - (105° + 62°) = 180° - 167° = 13°$

Hence, $a = 105°$, $b = 13°$ and $c = 62°$.

Example 9. The given figure shows a pentagon, inscribed in a circle with centre O. Given $AB = BC = CD$ and $\angle ABC = 132°$.



Calculate the value of

(i) $\angle AEB$ (ii) $\angle AED$ (iii) $\angle COD$.

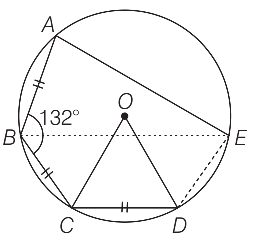
Sol.

(i) Join CE.

Since, $ABCE$ is a cyclic quadrilateral,

$$\angle AEC + 132° = 180°$$

[$\because$ sum of opposite angles of a cyclic quadrilateral is 180°]



$\Rightarrow \quad \angle AEC = 180° - 132° = 48°$

$\Rightarrow \quad \angle AEB + \angle BEC = 48° \quad$ ...(i) [from figure]

$\therefore \quad \angle AOB = \angle BOC$

[$\because AB = BC$ and equal chords subtend equal angles at the centre]

$\Rightarrow \quad \frac{1}{2}\angle AOB = \frac{1}{2}\angle BOC$

$\Rightarrow \quad \angle AEB = \angle BEC$

[$\because$ angle subtended by an arc at the center is double the angle subtended by it any point on remaining part of circle]

$\therefore \quad 2\angle AEB = 48°$

$\Rightarrow \quad \angle AEB = \frac{1}{2} \times 48° = 24°$

(ii) $\angle CED = \angle AEB$

[$\because CD = AB$ and equal chords subtend equal angles in the same segment]

$\Rightarrow \angle CED = 24°$

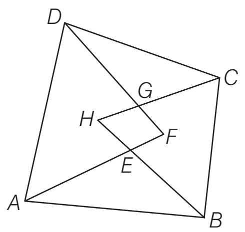
$\therefore \angle AED = \angle AEC + \angle CED = 48° + 24° = 72°$

(iii) $\angle COD = 2\,\angle CED$ [$\because$ angle at the centre = double the angle at the remaining part of the circle]

$\therefore \ \angle COD = 2 \times 24° = 48°$

Example 10. Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.

Sol. Given, a quadrilateral $ABCD$ in which internal angle bisectors AF, BH, CH and DF of angles A, B, C and D respectively form a quadrilateral $EFGH$.



We need to prove that quadrilateral $EFGH$ is cyclic.

$\angle HEF = \angle AEB \quad$ ...(i) [vertically opposite angles]

In ΔAEB,

$$\angle AEB + \frac{1}{2}\angle A + \frac{1}{2}\angle B = 180°$$

[$\because AE$ is bisector of $\angle A$ and BE is bisector of $\angle B$]

$\Rightarrow \quad \angle AEB = 180° - \frac{1}{2}(\angle A + \angle B) \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$\angle HEF = 180° - \frac{1}{2}(\angle A + \angle B) \quad ...(iii)$$

Similarly, $\angle HGF = 180° - \frac{1}{2}(\angle C + \angle D) \quad$ ...(iv)

On adding Eqs. (iii) and (iv), we get

$$\angle HEF + \angle HGF = 360° - \frac{1}{2}(\angle A + \angle B + \angle C + \angle D)$$

$= 360° - \frac{1}{2} \times 360°$ [sum of angles of a quadrilateral]

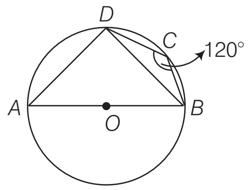
$= 360° - 180° = 180°$

Therefore, $EFGH$ is a cyclic quadrilateral.

[$\because$ sum of a pair of opposite angles is 180°]

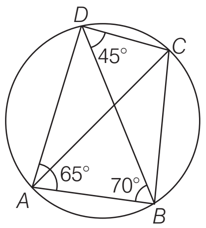
Topic Exercise 2

1. In the given figure, AB is the diameter of a circle with centre O and $\angle BCD = 120°$.



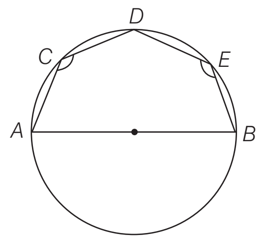
Find
(i) $\angle BAD$ (ii) $\angle DBA$.

2. In the given figure, **if** $\angle BAD = 65°$, $\angle ABD = 70°$ and $\angle BDC = 45°$. **[2013]**

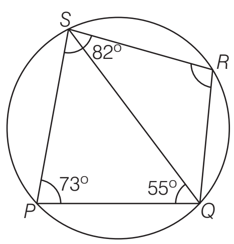


Prove that AC is a diameter of the circle and find $\angle ACB$.

3. In the given figure, AB is a diameter of the circle and C, D, E are any three points on the semi-circle. Find the value of $\angle ACD + \angle BED$.

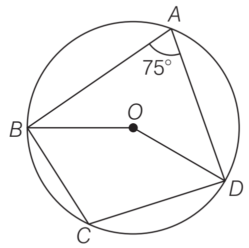


4. $PQRS$ is a cyclic quadrilateral. Given $\angle QPS = 73°$, $\angle PQS = 55°$ and $\angle PSR = 82°$, calculate



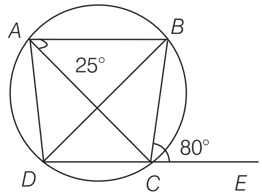
(i) $\angle QRS$ (ii) $\angle RQS$ (iii) $\angle PRQ$. **[2018]**

5. In the given figure, O is centre of the circle. If $\angle BAD = 75°$ and $BC = CD$, then find



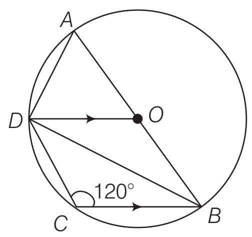
(i) $\angle BOD$ (ii) $\angle BCD$
(iii) $\angle BOC$ (iv) $\angle OBD$.

6. In the given figure, AB is parallel to DC, $\angle BCE = 80°$ and $\angle BAC = 25°$.



Find
(i) $\angle CAD$ (ii) $\angle CBD$ (iii) $\angle ADC$. **[2008]**

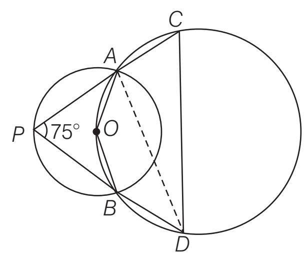
7. In the given figure, AB is a diameter of the circle with centre O, $DO \parallel CB$ and $\angle DCB = 120°$.



Calculate
(i) $\angle DAB$ (ii) $\angle DBA$ (iii) $\angle DBC$ (iv) $\angle ADC$.

Also, show that ΔAOD is an equilateral triangle.

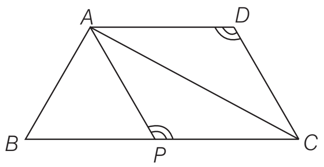
8. In the given figure, two circles intersect each other at A and B. The centre of the smaller circle is O and lies on the circumference of the larger circle. If PAC and PBD are straight lines and $\angle APB = 75°$, then find



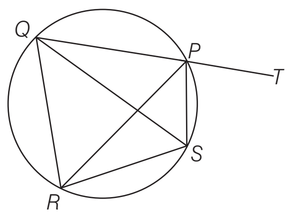
(i) $\angle AOB$ (ii) $\angle ACB$ (iii) $\angle ADB$

9. *ABCD* is a parallelogram. The circle through *A, B* and *C* intersect *CD,* produced at *E.* Prove that $AE = AD$.

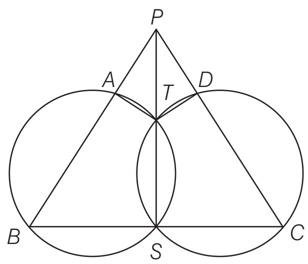
10. In the given figure, *P* is a point on the side *BC* of ΔABC, such that $AB = AP$. Through *A* and *C,* lines are drawn parallel to *BC* and *PA* respectively, so as to intersect at *D.* Show that *ABCD* is a cyclic quadrilateral.



11. In the given figure, *SP* is bisector of $\angle RPT$ and *PQRS* is a cyclic quadrilateral. Prove that $SQ = SR$.



12. In the given figure, two circles intersect at *S* and *T*. *STP*, *BSC* and *BAP* are straight lines. Prove that *PATD* is a cyclic quadrilateral.



13. Prove that, if two non-parallel sides of a trapezium are equal, then it is cyclic.

Or

Prove that isosceles trapezium is always cyclic.

14. Prove that, in a cyclic quadrilateral, if one pair of opposite sides is equal, the other pair is parallel.

15. If two sides of a cyclic quadrilateral are parallel, then prove that
(i) its other two sides are equal.
(ii) its diagonals are equal.

Hints and Answers

1. Hint (i) $\angle BAD + \angle DCB = 180°$

[$\because$ sum of opposite angles of a cyclic quadrilateral is 180°]

Ans. 60°

(ii) **Hint** $\angle ADB = 90°$ [$\because$ *AB* is the diameter]

In ΔABD, $\angle BAD + \angle ADB + \angle DBA = 180°$

[by angle sum property of a triangle]

$\Rightarrow \quad 60° + 90° + \angle DBA = 180°$ **Ans.** 30°

2. Hint $\angle BAD + \angle BCD = 180°$

[$\because$ sum of opposite angles of a cyclic quadrilateral is 180°]

$\Rightarrow \quad 65° + \angle BCD = 180° \Rightarrow \angle BCD = 115°$

In ΔADB, $\angle ADB + \angle DAB + \angle ABD = 180°$

[$\because$ angle sum property of a triangle]

$\Rightarrow \quad \angle ADB + 65° + 70° = 180°$

$\angle ACB = \angle ADB$

[$\because$ angle in the same segment are equal]

Here, $\angle ADC = \angle ADB + \angle BDC = 45° + 45° = 90°$

Since, angle in a semi-circle is 90°

$\therefore$ *AC* is diameter. **Ans.** 45°

3. Hint $\angle ACD + \angle AED = 180°$ ...(i)

[$\because$ sum of opposite angles of a cyclic quadrilateral is 180°]

$\because$ *AB* is a diameter of circle.

$\therefore \quad \angle AEB = 90°$ [angle in a semi-circle] ...(ii)

On adding Eqs. (i) and (ii), we get the required value.

Ans. 270°

4. Hint

(i) $\angle QPS + \angle QRS = 180°$ **Ans.** $\angle QRS = 107°$

(ii) In ΔPQS,

$\angle SPQ + \angle PQS + \angle PSQ = 180°$

Now, in ΔQRS,

$\angle QSR + \angle SRQ + \angle RQS = 180°$ **Ans.** $\angle RQS = 43°$

(iii) Here, $\angle PRQ = \angle PSQ$

[since, angles in same segment are equal]

Ans. $\angle PRQ = 52°$

5. (i) **Hint** $\angle BOD = 2 \times \angle BAD$

[$\because$ angle at centre $= 2 \times$ angle at any point on the remaining part of circle]

(ii) **Hint** $\angle BCD + \angle BAD = 180°$

[$\because$ sum of a pair of opposite angles of a cyclic quadrilateral is 180°]

(iii) **Hint** Join *OC*. As equal chords of a circle subtend equal angles at the centre and chord *BC* = chord *CD*, then $\angle BOC = \angle COD \Rightarrow \angle BOC = \frac{1}{2} \angle BOD$

(iv) **Hint** In ΔOBD, $OB = OD$ [$\because$ radii of same circle]

$\Rightarrow \quad \angle ODB = \angle OBD$

[$\because$ angle opposite to the equal sides are equal]

$\because \quad \angle OBD + \angle ODB + \angle BOD = 180°$

[by angle sum property of a triangle]

$\therefore \quad 2\angle OBD + 150° = 180°$

Ans. (i) 150° (ii) 105° (iii) 75° (iv) 15°

6. (i) **Hint** $\angle BAD = \angle BCE$

[$\because$ exterior angle of a cyclic quadrilateral is equal to the opposite interior angle]

$\Rightarrow \angle BAC + \angle CAD = 80° \Rightarrow \angle CAD = 80° - \angle BAC$

Ans. 55°

(ii) **Hint** $\angle CBD = \angle CAD$

[$\because$ angles in the same segment are equal]

Ans. 55°

(iii) **Hint** $\angle BAD + \angle ADC = 180°$ [$\because AB \parallel DC$]

$\Rightarrow (\angle BAC + \angle DAC) + \angle ADC = 180°$

$\Rightarrow \quad 25° + 55° + \angle ADC = 180°$ **Ans.** 100°

7. (i) **Hint** $\angle DAB + \angle DCB = 180°$

[$\because$ sum of a pair of opposite angles of cyclic quadrilateral is 180°]

Ans. 60°

(ii) **Hint** $\angle ADB = 90°$ [$\because$ angle in a semi-circle]

$\therefore \quad \angle DBA = 180° - (\angle ADB + \angle DAB)$ **Ans.** 30°

(iii) **Hint** $\angle BOD = 2 \times \angle DAB$

[$\because$ angle at centre $= 2 \times$ angle at any point on the remaining part of centre]

$OD = OB$ [radii]

$\Rightarrow \quad \angle ODB = \angle OBD$

[$\because$ angles opposite to equal sides]

$\therefore \quad \angle ODB = \frac{1}{2}(180° - \angle BOD)$

[by angle sum property of a triangle]

Since, $OD \parallel BC$

$\Rightarrow \quad \angle DBC = \angle ODB$ **Ans.** 30°

(iv) **Hint** $OA = OD$ [radii of same circle]

$\Rightarrow \angle ODA = \angle DAO = 60°$

In ΔBCD, $\angle BCD + \angle DBC + \angle CDB = 180°$

[angle sum property of triangle]

$\Rightarrow 120° + 30° + \angle CDB = 180° \Rightarrow \angle CDB = 30°$

$\therefore \angle ADC = \angle ODA + \angle ODB + \angle CDB$ **Ans.** 120°

Again, in ΔODA, $AO = OD$

$\Rightarrow \quad \angle OAD = \angle ODA = 60°$

[$\because$ angle opposite to equal sides]

$\therefore \quad \angle AOD = 180° - (60° + 60°) = 60°$

$\Rightarrow \Delta AOD$ is equilateral triangle.

8. (i) **Hint** $\angle AOB = 2 \times \angle APB$

[$\because$ angle at the centre $= 2 \times$ angle at any point on the remaining part of circle]

Ans. 150°

(ii) **Hint** First join BC.

$AOBC$ is a cyclic quadrilateral.

$\therefore \quad \angle AOB + \angle ACB = 180° \Rightarrow 150° + \angle ACB = 180°$

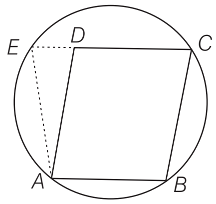
Ans. 30°

(iii) **Hint** $\angle ADB = \angle ACB$

[$\because$ angles in the same segment]

Ans. 30°

9. **Hint** Since, $ABCE$ is cyclic quadrilateral, therefore



$\angle AED + \angle ABC = 180°$ ...(i)

[$\because$ sum of pair of opposite angles of a cyclic quadrilateral is 180°]

$\because \quad \angle ADE + \angle ADC = 180°$

[$\because EDC$ is a straight line]

$\Rightarrow \quad \angle ADE + \angle ABC = 180°$ ...(ii)

[$\because \angle ADC = \angle ABC$, opposite angles of a parallelogram]

From Eqs. (i) and (ii), we get $\angle AED = \angle ADE$

10. **Hint** Prove sum of opposite angles is 180°.

11. Since, SP is a bisector of $\angle RPT$.

$\therefore \quad \angle RPS = \angle TPS$ ...(i)

$\angle RPS = \angle RQS$ [angle in the same segment] ...(ii)

and $\angle TPS = \angle QRS$ ...(iii)

[exterior angle of a cyclic quadrilateral]

From Eqs. (i), (ii) and (iii), we get

$\angle QRS = \angle RQS$

12. **Hint** We have, $\angle PAT = \angle PSB$ ...(i)

[$\because$ exterior angle of a cyclic quadrilateral is equal to the interior opposite angle]

and $\quad \angle PDT = \angle PSC$ ...(ii)

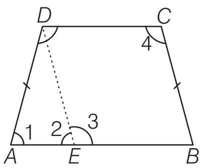
On adding Eqs. (i) and (ii), we get

$\angle PAT + \angle PDT = \angle PSB + \angle PSC$

But $\quad \angle PSB + \angle PSC = 180°$ [linear pair axiom]

$\Rightarrow \quad \angle PAT + \angle PDT = 180°$

13. **Hint** $AD = BC$ $\dots$(i) [given]



Draw $DE || CB$

$\Rightarrow$ $DE = BC$ [opposite sides of a parallelogram] $\dots$(ii)

From Eqs. (i) and (ii), we get $AD = DE \Rightarrow \angle 1 = \angle 2$

[$\because$ angles opposite to equal sides of triangle are equal]

Also, $\angle 3 = \angle 4$ [$\because$ opposite angles of a parallelogram]

Now, $\angle 2 + \angle 3 = 180°$ [by linear pair axiom]

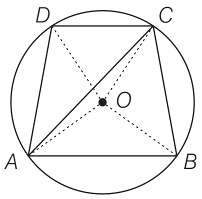
$\therefore$ $\angle 1 + \angle 4 = 180°$

14. **Hint** Join O to A, B, C, D and join A to C.

$AD = BC$ [given]

$\therefore$ $\angle AOD = \angle BOC$

[$\because$ equal chords of a circle subtends equal angles at the centre of the circle]



$\Rightarrow$ $2\angle ACD = 2\angle BAC$

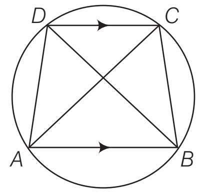
[$\because$ angle subtended by an arc at the centre is double the angle subtended by it at any other point on the remaining part of the circle]

or $\angle ACD = \angle BAC$

But they are alternate interior angles.

$\therefore$ $AB || DC$

15. Let $ABCD$ be a cyclic quadrilateral in which $AB || DC$.



Then, $\angle A + \angle D = 180°$ [sum of cointerior angles] $\dots$(i)

and $\angle D + \angle B = 180°$ $\dots$(ii)

[$\because$ sum of opposite angles in a cyclic quadrilateral is 180°]

From Eqs. (i) and (ii), we get

$\angle A = \angle B$ $\dots$(iii)

To show $\Delta ABC \cong \Delta BAD$ [by rule AAS]

By CPCT, $BC = AD$ and $AC = BD$.

Topic 3

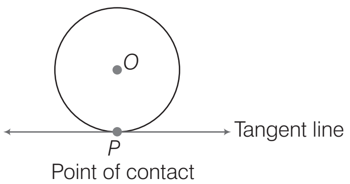
Tangent and Secant Properties of Circles

Tangent of a Circle

The word 'Tangent' has been derived from the Latin word 'Tangere', which means 'To touch' and it was introduced by the Danish Mathematician Thomas Fineke in 1583. Thus, a tangent to a circle is a line that intersects or touch the circle at only one point.

This point, i.e. the common point of the tangent and the circle is called the point of contact.

Also, we can say that the tangent to a circle is a special case of the secant, when two end points of its corresponding chord are coincide.



(i) There is only one tangent at a point of the circle.

(ii) A circle can have maximum two parallel tangents which can be drawn to the opposite side of the centre.

Properties of a Tangent

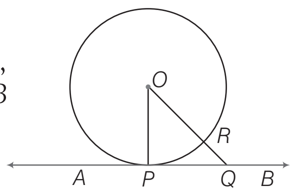
The properties of a tangent in the form of theorems are given below

Theorem 1 *The tangent at any point of a circle and the radius through the point of contact are perpendicular to each other.*

Given A circle with centre O and a tangent AB at a point P of the circle.

To prove $OP \perp AB$

Construction Take any point Q, other than P on the tangent AB and join OQ.



Proof Here, Q is a point on the tangent AB, other than the point of contact P. So, Q lies outside the circle [if Q lies inside the circle, then AB becomes a secant and not a tangent to the circle].
Let OQ intersects the circle at R.

$\because \quad OP = OR$ [radii of the same circle]

Now, $\quad OQ = OR + RQ$

$\Rightarrow \quad OQ > OR$

$\Rightarrow \quad OQ > OP$ $[\because OP = OR]$

or $\quad OP < OQ$

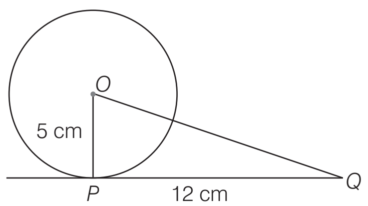
Thus, OP is shorter than any other segment joining O to any point of AB. Also, we know that, the shortest distance between a point and a line is perpendicular distance from the point to the line. So, OP is perpendicular to AB.

i.e. $OP \perp AB$. **Hence proved.**

Theorem 2 **(Converse of Theorem 1)** *A line drawn through the end of a radius and perpendicular to it, is a tangent to the circle.*

Example 1. **A line through the centre O of a circle of radius 5 cm cuts the tangent at a point P on the circle at Q such that $PQ = 12$ cm. Find the length of OQ.**

Sol. We know that, tangent at a point on a circle is perpendicular to the radius through the point of contact.



Therefore, OP is perpendicular to PQ.
In right angled ΔOPQ, we have

$OQ^2 = OP^2 + PQ^2$ [by Pythagoras theorem]

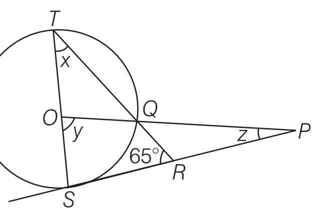
$\Rightarrow \quad OQ^2 = 5^2 + 12^2$

$\Rightarrow \quad OQ^2 = 25 + 144$

$\Rightarrow \quad OQ^2 = 169$

$\therefore \quad OQ = 13$ cm [taking positive square root]

Example 2. **In the given figure, O is the centre of the circle and SP is a tangent. If $\angle SRT = 65°$, then find the values of x, y and z.** ***[2015]***

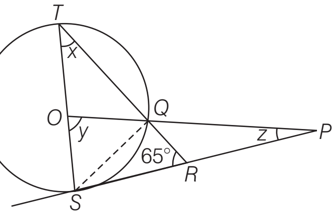


Sol. Given, $\angle TRS = 65°$

Join SQ. Here, SP is a tangent.

$\therefore \quad \angle OSR = 90°$

[$\because$ tangents at any point of a circle and radius through the point of contact are perpendicular to each other]



In ΔSTR, $\angle STR + \angle TRS + \angle TSR = 180°$

[by angle sum property of a triangle]

$\Rightarrow \quad x + 65° + 90° = 180° \Rightarrow x = 180° - 155°$

$\therefore \quad x = 25°$

We know that, angle made by a chord at centre is double the angle made by it at remaining part.

$\therefore \quad y = 2x = 2 \times 25° = 50°$

In ΔOPS, $\angle SOP + \angle OPS + \angle OSP = 180°$

[by angle sum property of a triangle]

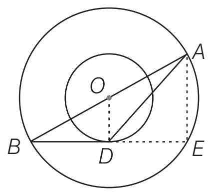
$\Rightarrow \quad 50° + z + 90° = 180°$

$\therefore \quad z = 180° - 140° = 40°$

Example 3. **The radii of two concentric circles are 13 cm and 8 cm. AB is a diameter of the bigger circle. BD is a tangent to the smaller circle touching it at D. Find the length of AD.**

Sol. Produce BD to meet the bigger circle at E. Join AE.

Then, $\quad \angle AEB = 90°$ [$\because$ angle in semi-circle]



Here, $OD \perp BE$ [$\because$ BE is tangent to the smaller circle at D and OD is its radius]

and $\quad BD = DE$

[$\because$ BE is chord of the bigger circle and $OD \perp BE$]

$\therefore \quad OD \parallel AE$ $[\because \angle AEB = \angle ODB = 90°]$

In ΔAEB, O and D are the mid-points of AB and BE.
Therefore, by mid-point theorem, we have

$$OD = \frac{1}{2}AE \Rightarrow AE = 2 \times 8 = 16 \text{ cm}$$

[$\because$ OD = radius of smaller circle = 8 cm]

In right angled ΔODB,

$OB^2 = OD^2 + BD^2$ [by Pythagoras theorem]

$\Rightarrow \quad (13)^2 = (8)^2 + (BD)^2$

$\Rightarrow \quad BD^2 = 169 - 64 = 105$

$\Rightarrow \quad BD = \sqrt{105}$ cm [taking positive square root]

$\Rightarrow \quad DE = \sqrt{105}$ cm $[\because BD = DE]$

Now, in ΔAED, we have

$AD^2 = AE^2 + ED^2$ [by Pythagoras theorem]

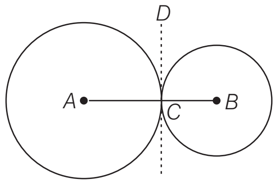
$\Rightarrow \quad AD^2 = 16^2 + (\sqrt{105})^2 = 256 + 105$

$\Rightarrow \quad AD^2 = 361 \Rightarrow AD = \sqrt{361} = 19$ cm

Hence, the length of AD is 19 cm.

Theorem 3 *If two circles touch each other, then the point of contact lies on the straight line through their centres.*

***Case* I** When two circles touch each other externally.



Given Two circles with centres A and B touching each other externally at C.

To prove C lies on the line AB.

Construction Through the point of contact C, draw a common tangent CD. Join AC and BC.

Proof Here, $\angle ACD = 90°$ ...(i)

[$\because$ radius through the point of contact is perpendicular to the tangent]

and $\angle BCD = 90°$ ...(ii)

[$\because$ radius through the point of contact is perpendicular to the tangent]

On adding Eqs. (i) and (ii), we get

$$\angle ACD + \angle BCD = 90° + 90°$$

$\Rightarrow$ $\angle ACD + \angle BCD = 180°$

$\Rightarrow$ $\angle ACB = 180°$

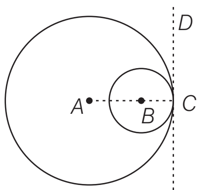
$\Rightarrow$ ACB is a straight line.

[$\because$ 180° is the angle of straight line]

$\therefore$ C lies on the line AB.

***Case* II** When the given two circles touch each other internally.

Given Two circles with centres A and B, touching each other internally at point C.



To prove C lies on the line AB produced.

Construction Through the point of contact C, draw a common tangent CD. Join AC and BC.

Proof Here, $\angle ACD = 90°$ ...(i)

[$\because$ radius through the point of contact is perpendicular to the tangent]

and $\angle BCD = 90°$ ...(ii)

[$\because$ radius through the point of contact is perpendicular to the tangent]

Thus, AC and BC both are perpendicular to the tangent CD at the same point C. [from Eqs. (i) and (ii)]

$\Rightarrow$ AC and BC lie in the same line AC.

[$\because$ only one perpendicular can be drawn, to a line through a point on it]

$\Rightarrow$ ABC is a straight line.

$\therefore$ C lies on the line AB produced. **Hence proved.**

Let r_1, r_2 be the radii of the bigger and smaller circles respectively, and d be the distance between their centres, i.e. $d = AB$, then

(i) the circles touch externally if and only if $r_1 + r_2 = d$.

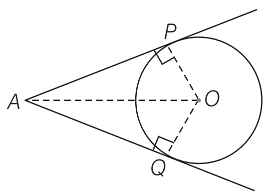
(ii) the circles touch internally if and only if $r_1 - r_2 = d$.

Theorem 4 *The lengths of two tangents drawn from an external point to a circle, are equal.*

Given AP and AQ are two tangents from a point A to a circle, with centre O.

To prove $AP = AQ$

Construction Join OP, OQ and OA.



Proof We know that, a tangent at any point of a circle is perpendicular to the radius through the point of contact.

Here, AP is a tangent and OP is the radius of the circle through P.

$\therefore$ $OP \perp AP$

Similarly, $OQ \perp AQ$

$\Rightarrow$ $\angle OPA = \angle OQA = 90°$ ...(i)

First method In $\Delta\, OPA$ and ΔOQA, we have

$OP = OQ$ [radii of a circle]

$\angle OPA = \angle OQA$ [from Eq. (i)]

$OA = OA$ [common side]

So, by RHS criterion of congruence, we get

$$\Delta OPA \cong \Delta OQA$$

Then, $AP = AQ$ [by CPCT]

Second method In right angled ΔOPA,

$OA^2 = OP^2 + AP^2$ [by Pythagoras theorem]

$\Rightarrow$ $AP^2 = OA^2 - OP^2 \Rightarrow AP^2 = OA^2 - OQ^2$

[$\because OP = OQ =$ radii of a circle]

$\Rightarrow$ $AP^2 = AQ^2$

[$\because$ in $\Delta\, OQA$, $OA^2 = OQ^2 + AQ^2$ $\Rightarrow AQ^2 = OA^2 - OQ^2$]

$\Rightarrow$ $AP = AQ$ **Hence proved.**

From the above theorem, $\angle OAP = \angle OAQ$

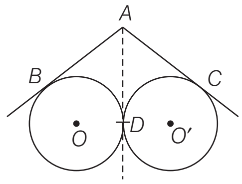
$[\because \Delta OPA \cong \Delta OQA]$

$\Rightarrow$ OP is the angle bisector of $\angle PAQ$.

Thus, the centre of the circle lies on the bisector of the angle between the two tangents.

Example 4. Two circles with centres O and O' touch each other externally at point D. If A is a point on the common tangent through D, then prove that the tangents AB and AC are equal.

Sol. **Given** Two circles with centres O and O' touch each other externally at point D. AB and AC are the tangents to the circles with centres O and O' respectively from point A, which lie on the common tangent AD.



To prove $AB = AC$

Proof AB and AD are the tangents to the circle (with centre O) from point A.

$\therefore \quad AB = AD$ ...(i)

[$\because$ lengths of two tangents drawn from an external point to a circle are equal]

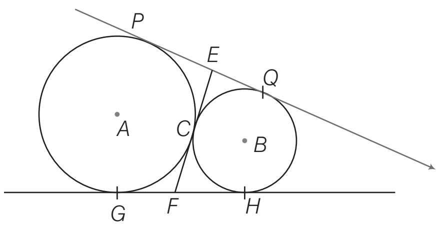
Again, AC and AD the tangents to the circle (with centre O') from point A.

$\therefore \quad AC = AD$ ...(ii)

[$\because$ lengths of two tangents drawn from an external point to a circle are equal]

From Eqs. (i) and (ii), we get $AB = AC$ **Hence proved.**

Example 5. In the given figure, if two circles touch each other externally at C, then prove that common tangent at C bisects the other two common tangents.



Sol. **Given** PQ and GH are the common tangents to both circles and common tangent at C meets PQ at E and GH at F.

To prove EF bisects PQ and GH.

Proof We know that, tangents drawn from an external point to a circle are equal in length.

$\therefore \quad EP = EC$ and $EQ = EC \Rightarrow EP = EQ$

$\Rightarrow$ PQ is bisected by EF at E.

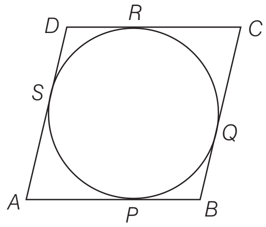
Similarly, $GF = FC$ and $HF = FC \Rightarrow GF = HF$

$\Rightarrow$ GH is bisected by EF at F.

Hence, the common tangent EF drawn at point C bisects the other two common tangents. **Hence proved.**

Example 6. If the sides of a parallelogram touch a circle, then prove that the parallelogram is a rhombus.

Sol. **Given** A parallelogram $ABCD$ in which a circle touches the sides at P, Q, R and S, respectively.



To prove $ABCD$ is a rhombus.

Proof Since, tangents from an external point to a circle are equal in length.

$\therefore \quad AP = AS$

Similarly, $\quad BP = BQ, CR = CQ$

and $\quad DR = DS$

On adding these, we get

$$AP + BP + CR + DR = AS + BQ + CQ + DS$$

$\Rightarrow (AP + BP) + (CR + DR) = (AS + DS) + (BQ + CQ)$

$\Rightarrow \quad AB + CD = AD + BC$

But $ABCD$ is a parallelogram.

$\therefore \quad AB = CD$ and $AD = BC$

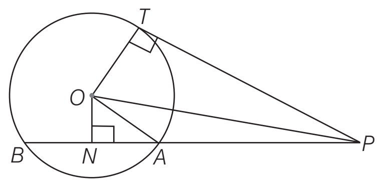
$\therefore \quad AB + AB = BC + BC$

$\Rightarrow \quad 2AB = 2BC \Rightarrow AB = BC$

$\Rightarrow \quad AB = BC = CD = AD$

Hence, $ABCD$ is a rhombus. **Hence proved.**

Example 7. In the given figure, from an external point P, a tangent PT and a line segment PAB drawn to a circle with centre O. ON is perpendicular on the chord AB.



Prove that

(i) $PA \cdot PB = PN^2 - AN^2$

(ii) $PN^2 - AN^2 = OP^2 - OT^2$

(iii) $PA \cdot PB = PT^2$.

Sol. **Given** PT is a tangent drawn from an external point P and a line segment PAB is drawn to a circle with centre O. ON is perpendicular on the chord AB.

To prove

(i) $PA \cdot PB = PN^2 - AN^2$

(ii) $PN^2 - AN^2 = OP^2 - OT^2$

(iii) $PA \cdot PB = PT^2$

Proof

(i) $PA \cdot PB = (PN - AN)(PN + BN)$

$= (PN - AN)(PN + AN)$

[$\because AN = BN$ as $ON \perp AB$ and bisect it]

$= PN^2 - AN^2$

(ii) $PN^2 - AN^2 = (OP^2 - ON^2) - AN^2$

[$\because$ in right angled ΔONP by Pythagoras theorem $OP^2 = ON^2 + PN^2 \Rightarrow PN^2 = OP^2 - ON^2$]

$= OP^2 - ON^2 - AN^2$

$= OP^2 - (ON^2 + AN^2)$

$= OP^2 - OA^2$ [$\because$ in ΔONA, $OA^2 = ON^2 + AN^2$]

$= OP^2 - OT^2$ [$\because OA = OT =$ radii]

(iii) From parts (i) and (ii), we get

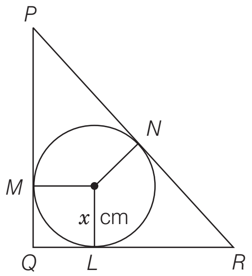
$PA \cdot PB = OP^2 - OT^2$

$\Rightarrow PA \cdot PB = PT^2$

[$\because$ in ΔOTP, $OP^2 = OT^2 + PT^2 \Rightarrow PT^2 = OP^2 - OT^2$]

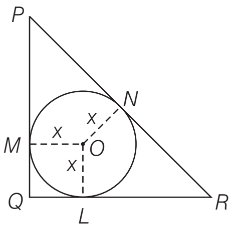
Hence proved.

Example 8. In the given figure, $PQ = 24$ cm, $QR = 7$ cm and $\angle PQR = 90°$. Find the radius of the inscribed circle.



[2012]

Sol. Given, $PQ = 24$ cm, $QR = 7$ cm and $\angle PQR = 90°$.



In right angled ΔPQR,

$PR^2 = PQ^2 + QR^2$ [by Pythagoras theorem]

$= (24)^2 + (7)^2 = 576 + 49 = 625$

$\Rightarrow PR^2 = (25)^2 \Rightarrow PR = 25$ cm

[taking positive square root as length cannot be negative]

Here, PQ, QR and PR are the tangents of inscribed circle.

The tangent at any point of a circle and the radius through the point are perpendicular to each other.

$\therefore \quad \angle OMQ = \angle OLQ = \angle ONR = 90°$

Since, all the angles of a quadrilateral $OLQM$ are $90°$ and $QL = QM$.

[$\because$ tangents drawn from an external point to the circle are equal in length]

So, $OLQM$ is a square.

$\Rightarrow \quad QL = QM = x$ (say)

Now, $\quad LR = QR - QL = 7 - x = RN$

[$\because$ tangents drawn from an external point to the circle are equal in length]

and $\quad MP = PQ - QM = 24 - x = PN$

$\because \quad PR = PN + NR$

$\therefore \quad 25 = 24 - x + 7 - x \Rightarrow 25 = 31 - 2x$

$\Rightarrow \quad 2x = 31 - 25 \Rightarrow 2x = 6 \Rightarrow x = \frac{6}{2}$

Hence, the radius of the inscribed circle is 3 cm.

Example 9. If a, b and c are the sides of a right triangle, where c is hypotenuse, then prove that the radius r of the circle which touches the sides of the triangle, is given by $r = \frac{a+b-c}{2}$.

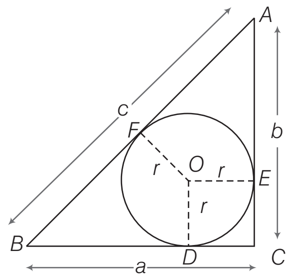
Sol. Let a, b and c be the sides of right angled ΔABC, such that $BC = a$, $CA = b$ and $AB = c$.

Let the circle touches the sides BC, CA, AB at D, E and F, respectively.

Then, $\quad AE = AF$ and $BD = BF$

[$\because$ tangents drawn from an external point to the circle are equal in length]

Also, $\quad CE = CD = r$



$\therefore \quad AF = AE = AC - EC = b - r$ and $BF = a - r$

Now, $\quad AB = c$

$\Rightarrow AF + BF = (b - r) + (a - r)$

$\Rightarrow \quad AB = b - r + a - r$

$\Rightarrow \quad c = a + b - 2r$

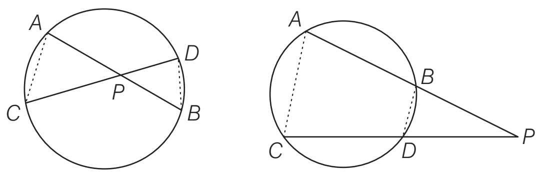
$\Rightarrow \quad 2r = a + b - c$

$\therefore \quad r = \frac{a+b-c}{2}$ **Hence proved.**

Theorem 5 *If two chords of a circle intersect internally or externally, then the product of the lengths of segments are equal.*

[2015]

Given Two chords AB and CD of a circle intersecting at P internally [Fig. (i)] or externally [Fig. (ii)].



(i) (ii)

To prove $PA \cdot PB = PC \cdot PD$

Construction Join AC and BD.

Proof

***Case* I** When AB and CD intersect internally as shown in Fig. (i).

In ΔACP and ΔDBP,

$$\angle APC = \angle BPD \quad \text{[vertically opposite angles]}$$
$$\angle CAP = \angle PDB \quad \text{[angles in same segment]}$$
$$\therefore \quad \Delta ACP \sim \Delta DBP \quad \text{[by AA similarity criterion]}$$
$$\Rightarrow \quad \frac{PA}{PD} = \frac{PC}{PB}$$

[$\because$ corresponding sides of similar triangles are proportional]

$$\Rightarrow \quad PA \cdot PB = PC \cdot PD$$

***Case* II** When AB and CD intersect externally as shown in Fig. (ii).

In ΔACP and ΔDBP, $\angle CAB = \angle BDP$

[exterior angle of a cyclic quadrilateral = interior opposite angle]

$$\angle P = \angle P \quad \text{[common angle]}$$
$$\therefore \quad \Delta ACP \sim \Delta DBP \quad \text{[by AA similarity criterion]}$$
$$\Rightarrow \quad \frac{PA}{PD} = \frac{PC}{PB}$$

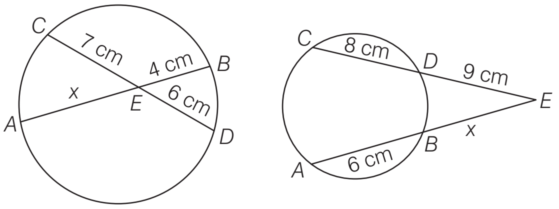
[$\because$ corresponding sides of similar triangles are proportional]

$$\Rightarrow \quad PA \cdot PB = PC \cdot PD$$

Hence, in both the cases,

$$PA \cdot PB = PC \cdot PD$$ **Hence proved.**

Example 10. Find the length of the unknown part of the chord in the given figures.



(i) (ii)

Sol. We know that, when two chords of a circle intersect internally or externally, then the products of the lengths of segments are equal.

(i) $\because EA \cdot EB = EC \cdot ED \Rightarrow x \cdot 4 = 7 \cdot 6$

$\Rightarrow \quad x = \dfrac{7 \times 6}{4} = \dfrac{42}{4} = \dfrac{21}{2}$

Therefore, $EA = x = \dfrac{21}{2}$ cm.

(ii) $\because EA \cdot EB = EC \cdot ED$

$\Rightarrow \quad (6 + x)\, x = 8 \cdot 9 \Rightarrow 6x + x^2 = 72$

$\Rightarrow \quad x^2 + 6x - 72 = 0 \quad \Rightarrow x^2 + 12x - 6x - 72 = 0$

$\Rightarrow \quad x\,(x + 12) - 6\,(x + 12) = 0 \quad \Rightarrow (x - 6)\,(x + 12) = 0$

$\Rightarrow \quad x = 6$ or -12

$\therefore \quad x = 6$

[$\because x = -12$ (not possible) as length cannot be negative]

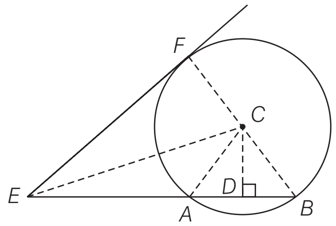
Therefore, $EB = x = 6$ cm.

Theorem 6 *If a chord and a tangent intersect externally, then the product of the lengths of the segments of the chord is equal to the square of the length of the tangent from the point of contact to the point of intersection.*

Given Let AB be the chord of a circle with centre C and tangent to the circle at F meets the chord AB produced at E.

To prove $EA \times EB = (EF)^2$

Construction Draw $CD \perp AB$ and join AC, BC, CE and CF.



Proof Here, $AD = DB$ ...(i)

[$\because$ perpendicular from the centre bisects the chord]

Now, $EA \times EB = (ED - AD)(ED + DB)$

$= (ED - AD)(ED + AD)$ [from Eq. (i)]

$EA \times EB = (ED)^2 - (AD)^2$

[$\because (a - b)(a + b) = a^2 - b^2$]

$= (ED)^2 - [(AC)^2 - (CD)^2]$

[by Pythagoras theorem in ΔADC, $(AD)^2 = (AC)^2 - (CD)^2$]

$= (ED)^2 - (AC)^2 + (CD)^2$

$= (ED)^2 + (CD)^2 - (AC)^2 = (EC)^2 - (AC)^2$

[by Pythagoras theorem in ΔEDC, $(ED)^2 + (CD)^2 = (EC)^2$]

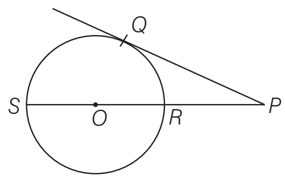
$= (EC)^2 - (CF)^2$

[$\because$ AC = CF, radii of the same circle]

$= (EF)^2$ [by Pythagoras theorem in ΔEFC, $(EC)^2 - (CF)^2 = (EF)^2$]

$\therefore \quad (EA) \times (EB) = (EF)^2$ **Hence proved.**

Example 11. In the given figure, O is the centre of the circle and PQ is a tangent at Q. If $PQ = 20$ cm and $PR = 8$ cm. Calculate the radius of the circle.



Sol. Given, PQ is the tangent to the circle with centre O.

Also, $PQ = 20$ cm and $PR = 8$ cm ...(i)

Since, $(PQ)^2 = PR \times PS$

[$\because$ product of the lengths of the segments of the chord = square of the length of the tangent]

$\Rightarrow \quad (20)^2 = 8 \times PS$ [from Eq. (i)]

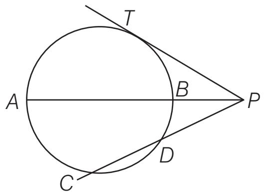
$\Rightarrow \quad 400 = 8 \times PS \Rightarrow PS = \dfrac{400}{8} = 50$ ...(ii)

$\therefore$ Diameter, $SR = PS - PR = 50 - 8$ [from Eqs. (i) and (ii)]
$= 42\,\text{cm}$

$\therefore$ Radius of a circle $= \dfrac{\text{Diameter}}{2} = \dfrac{42}{2} = 21\,\text{cm}$

Hence, the required radius of a circle is 21 cm.

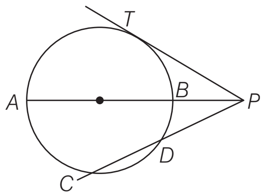
Example 12. In the given figure, diameter AB and chord CD of a circle meet at P. PT is a tangent to the circle at T and $CD = 7.8$ cm, $PD = 5$ cm, $PB = 4$ cm.



Find

(i) AB. (ii) the length of tangent PT. ***[2014]***

Sol. Given, $CD = 7.8\,\text{cm}$, $PD = 5\,\text{cm}$, $PB = 4\,\text{cm}$.



(i) Since, the chord CD and tangent at point T intersect each other at P.

$\therefore \quad PC \times PD = PT^2$ ...(i)

Since, the chord AB and tangent at point T intersect each other at P.

$\therefore \quad PA \times PB = PT^2$ ...(ii)

From Eqs. (i) and (ii), we get

$PC \times PD = PA \times PB$ ...(iii)

Now, $\quad PA = PB + BA = 4 + AB$

and $\quad PC = PD + DC = 5 + 7.8 = 12.8\,\text{cm}$

From Eq. (iii), we get

$12.8 \times 5 = (4 + AB) \times 4 \Rightarrow 4 + AB = \dfrac{12.8 \times 5}{4}$

$\Rightarrow \quad AB = 16 - 4 = 12\,\text{cm}$

(ii) From Eq. (ii), we get

$PT^2 = PA \times PB = (PB + BA) \times PB$
$= (4 + 12) \times 4 = 16 \times 4 = 64$

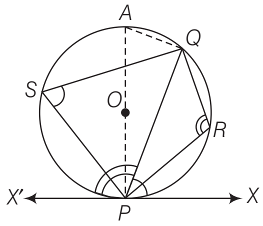
$\Rightarrow \quad PT = 8$ cm [taking positive square root]

Theorem 7 *If a line touches a circle and from the point of contact, a chord is drawn then the angles between the tangent and the chord are respectively, equal to the angle in the corresponding alternate segments.*

Given A tangent to a circle at a point P and PQ is a chord. S and R are the points in the alternate segments.

To prove (i) $\angle QPX \equiv \angle QSP$ (ii) $\angle QPX' = \angle QRP$

Construction Let O be the centre of circle. Join OP and produce it to meet the circle at A. Join AQ.



Proof (i) Clearly, $\angle AQP = 90°$ [$\because$ angle in a semi-circle]

and in ΔAPQ, $\angle QAP + \angle APQ + \angle AQP = 180°$

$\Rightarrow \quad \angle QAP + \angle APQ = 90°$ [$\because \angle AQP = 90°$] ...(i)

Also, $\angle QPX + \angle APQ = 90°$ ...(ii)

[$\because$ radius through the point of contact is perpendicular to the tangent]

Now, from Eqs. (i) and (ii), we get

$\angle QAP + \angle APQ = \angle QPX + \angle APQ$

$\Rightarrow \quad \angle QAP = \angle QPX$ ...(iii)

$\Rightarrow \quad \angle QPX = \angle QSP$

[$\because \angle QAP = \angle QSP$; angles in same segment]

(ii) Now, $PQRS$ is a cyclic quadrilateral, therefore

$\angle QSP + \angle QRP = 180°$ ...(iv)

Also, $\quad \angle QPX + \angle QPX' = 180°$ ...(v)

[by linear pair axiom]

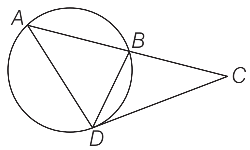
From Eqs. (iii) and (iv), we get

$\angle QSP + \angle QRP = \angle QPX + \angle QPX'$

$\Rightarrow \quad \angle QRP = \angle QPX'$ [$\because \angle QPX = \angle QSP$]

Hence proved.

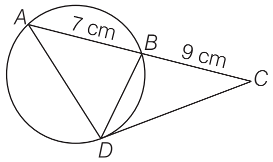
Example 13. In the given figure, $AB = 7$ cm and $BC = 9$ cm.



(i) Prove that $\Delta ACD \sim \Delta DCB$.

(ii) Find the length of CD. ***[2009]***

Sol. Given, $AB = 7\,\text{cm}$ and $BC = 9\,\text{cm}$



(i) In ΔACD and ΔDCB,

$\angle C = \angle C$ [common angle]

$\Rightarrow \quad \angle CDB = \angle BAD$ [angles in alternate segment]

$\therefore \quad \Delta ACD \sim \Delta DCB$ [by AA similarity criterion]

Hence proved.

(ii) Since, the chord AB and tangent at D intersect each other at point C.

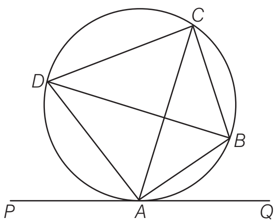
$\therefore \quad AC \times BC = CD^2$ [By Theorem 6]

$\Rightarrow \quad 16 \times 9 = CD^2 \Rightarrow CD^2 = 144 = (12)^2$

$\Rightarrow \quad CD = 12\text{ cm}$ [taking positive square root]

Hence, the required length of $CD = 12$ cm.

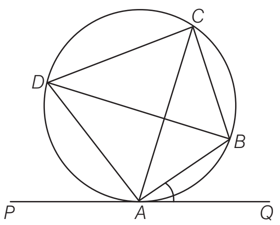
Example 14. In the given figure, PQ is a tangent to the circle at A. AB and AD are the bisectors of $\angle CAQ$ and $\angle PAC$. If $\angle BAQ = 30°$, prove that



(i) BD is a diameter of the circle.

(ii) ABC is an isosceles triangle. ***[2017]***

Sol. Given, $\angle BAQ = 30°$ and AB is the bisector of $\angle CAQ$.



$\therefore \quad \angle CAB = 30°$

$\therefore \quad \angle CAQ = 2\angle BAQ = 2 \times 30° = 60°$

$\because$ PAQ is a straight line.

$\therefore \quad \angle PAC + \angle CAQ = 180°$

$\Rightarrow \quad \angle PAC = 180° - \angle CAQ = 180° - 60° = 120°$

Now, AD is bisector of $\angle PAC$.

$\therefore \quad \angle DAC = \frac{1}{2}\angle PAC = \frac{1}{2} \times 120° = 60°$

and $\quad \angle DAB = \angle DAC + \angle CAB = 60° + 30° = 90°$

(i) We know that, angle in a semi-circle is 90°, i.e. angle made by diameter to any point on the circle 90°.

$\therefore$ BD is a diameter of the circle. **Hence proved.**

(ii) We know that, if a chord is drawn through the point of contact of a tangent to a circle, then the angles made by the chord with the given tangent are respectively equal to the angle formed in the corresponding alternate segment.

$\therefore \quad \angle CDA = \angle CAQ \Rightarrow \angle CDA = 60°$ [$\because \angle CAQ = 60°$]

Here, $ABCD$ is cyclic quadrilateral.

$\therefore \quad \angle CDA + \angle CBA = 180°$

[$\because$ sum of opposite angle of a cyclic quadrilateral is 180°]

$\therefore \quad \angle CBA = 180° - \angle CDA = 180° - 60° = 120°$

Now, in ΔABC, $\angle BCA + \angle CAB + \angle CBA = 180°$

[by angle sum property of a triangle]

$\therefore \quad \angle BCA + 30° + 120° = 180°$

$\Rightarrow \quad \angle BCA = 180° - 150° = 30°$

Here, $\angle BCA = \angle CAB$

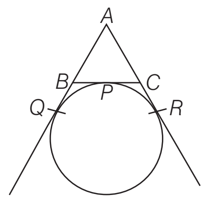
$\therefore$ ΔABC is an isosceles triangle. **Hence proved.**

Topic Exercise 3

1. There are two concentric circles, each with centre O and of radii 10 cm and 26 cm, respectively. Find the length of the chord AB of the outer circle, which touches the inner circle at P.

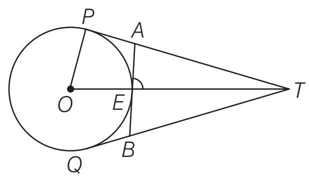
2. A circle is touching the side BC of ΔABC at point P and touching AB and AC produced at Q and R, respectively. Prove that

$AQ = \frac{1}{2}$ (perimeter of ΔABC)

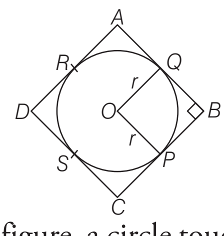


3. TA and TB are tangents to a circle with centre O from an external point T. OT intersects the circle at point P. Prove that TP bisects $\angle ATB$.

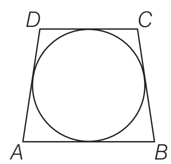
4. In the given figure, O is the centre of a circle of radius 5 cm, T is a point such that $OT = 13$ cm and OT intersects the circle at E. If AB is the tangent to the circle at E, then find the length of AB.



5. In the given figure, a circle is inscribed in a quadrilateral $ABCD$ in which $\angle B = 90°$. If $AD = 23$ cm, $AB = 29$ cm and $DS = 5$ cm, then find the radius of the circle.

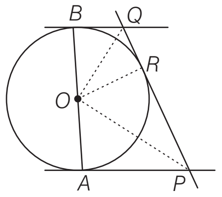


6. In the given figure, a circle touches all the four sides of a quadrilateral $ABCD$, whose three sides are $AB = 6$ cm, $BC = 7$ cm and $CD = 4$ cm. Find AD.



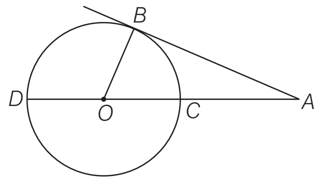
7. PA and PB are tangents drawn from an external point P to a circle with centre C. Prove that $\angle APB = 2\angle CAB$.

8. In the given figure, AB is a diameter of the circle with centre O. AP, BQ and PRQ are tangents. Prove that $\angle POQ = 90°$.

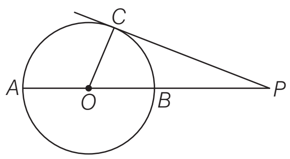


9. Let s denotes the semi-perimeter of ΔABC in which $BC = a, CA = b, AB = c$. If a circle touches the sides BC, CA, AB at D, E, F respectively. Prove that $BD = s - b$.

10. In the given figure, O is the centre of the circle and AB is a tangent at B. If $AB = 15$ cm and $AC = 7.5$ cm, then calculate the radius of the circle.

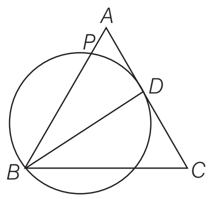


11. In the given figure, AB is a diameter of a circle with centre O and CP is the tangent to the circle at the point C.



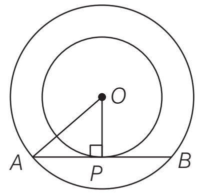
(i) If $AP = 20$ cm and $CP = 10$ cm, then find the radius of the circle.
(ii) If $CP = 15$ cm and $BP = 7.5$ cm, then find the radius of the circle.

12. In the given figure, ABC is a triangle in which $AB = AC$. A circle through B touches AC at D and intersects AB at P. If D is the mid-point of AC, then show that $4\,AP = AB$.



Hints and Answers

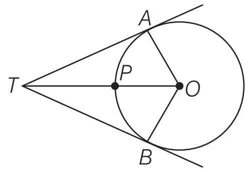
1. **Hint** Radius of outer circle, $AO = 26$ cm and radius of inner circle, $AP = 10$ cm
In ΔACP, $AO^2 = AP^2 + OP^2$
[By Pythagoras Theorem] [$\because OP \perp AB$]



$\Rightarrow \quad 26^2 = AP^2 + 10^2$
$\Rightarrow \quad AP^2 = 576 \Rightarrow AP = 24$ cm
$\therefore \quad AB = 2AP$ **Ans.** 48 cm

2. **Hint** $AQ = AR$ [tangents from A] ...(i)
$BP = QB$ [tangents from B] ...(ii)
$CP = CR$ [tangents from C] ...(iii)
$\therefore$ Perimeter of ΔABC
$= AB + BC + AC$
$= AB + BP + CP + AC$
$= AQ + AR$ [using Eqs. (ii) and (iii)]
$= 2AQ$ [using Eq. (i)]

3. **Hint** To show that
$\Delta TOA \cong \Delta TOB$ [by SSS congruency rule]



By CPCT, $\angle ATO = \angle BTO$
$\Rightarrow TP$ bisects $\angle ATB$.

4. **Hint** Since, $OP \perp PT$
In right angled ΔOPT,
$OT^2 = OP^2 + PT^2$ [by Pythagoras theorem]
$\Rightarrow \quad PT^2 = (13)^2 - (5)^2 = 169 - 25 = 144$
$\Rightarrow \quad PT = 12$ cm
Now, $OE + ET = 13$ cm
$\Rightarrow \quad ET = 13 - 5 = 8$ cm [$\because OE = 5$ cm]
Let $\quad AP = x \Rightarrow AT = 12 - x$
In right angled ΔAET, $AT^2 = AE^2 + ET^2$
$AT^2 = PA^2 + ET^2$ [$\because PA = AE$]
$\Rightarrow (12 - x)^2 = x^2 + (8)^2 \Rightarrow x = \dfrac{10}{3}$
$\therefore \quad AB = 2AE = 2x$
Ans. 20/3 cm

5. **Hint** We know that, the tangents drawn from an external point to a circle are equal.

$\therefore \quad AQ = AR, BP = BQ, CS = CP$ and $DR = DS$

Now, $DR = DS = 5$ cm

$AR = (AD - DR) = (23 - 5)$ cm $= 18$ cm

$\therefore \quad AQ = AR = 18$ cm

$BQ = (AB - AQ) = (29 - 18)$ cm $= 11$ cm

Since, tangent to a circle is perpendicular to the radius through the point of contact.

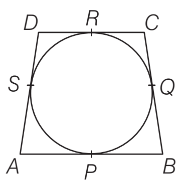
$\therefore \quad \angle OQB = \angle OPB = 90°$

and $\angle B = 90°$ [given]

$\Rightarrow$ $OQBP$ is a square $\Rightarrow OP = OQ = BQ$

Ans. 11cm

6. Hint Since, the tangents from an external point being equal.



$\therefore AP = AS, BP = BQ, CR = CQ$ and $DR = DS$

$\therefore \quad AP + BP + CR + DR = AS + BQ + CQ + DS$

$\Rightarrow (AP + BP) + (CR + DR) = (AS + DS) + (BQ + CQ)$

$\Rightarrow \quad AB + CD = AD + BC$

$\Rightarrow \quad AD = (AB + CD) - BC$

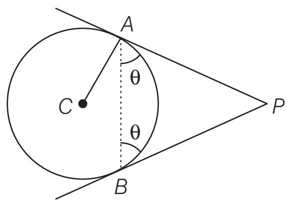
Ans. 3 cm

7. Hint $PA = PB$

$\Rightarrow \angle PAB = \angle PBA = \theta$ (let)

In ΔPAB, $\angle PAB + \angle PBA + \angle APB = 180°$

[By angle sum property of a triangle]



$\Rightarrow \quad \theta + \theta + \angle APB = 180°$

$\Rightarrow \quad \angle APB = 180° - 2\theta$ ...(i)

Now, $\angle CAB + \angle PAB = 90°$

[$\because$ tangent at a point to a circle is perpendicular to the radius]

$\Rightarrow \quad \angle CAB = 90° - \theta$ ...(ii)

From Eqs. (i) and (ii), we get

$\angle APB = 2\angle CAB$

8. Hint In ΔOAP and ΔORP,

$PA = PR$ [tangents from point P]

$OA = OR$ [radii]

$OP = OP$ [common]

$\therefore \quad \Delta OAP \cong \Delta ORP$ [by SSS congruency rule]

By CPCT, $\angle AOP = \angle POR$ ...(i)

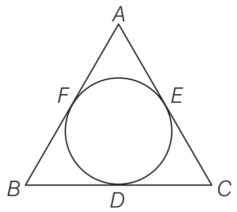
Similarly, $\angle BOQ = \angle ROQ$ ...(ii)

Now, $\angle AOP + \angle POR + \angle ROQ + \angle BOQ = 180°$

[$\because$ AOB is a straight line]

$\therefore \quad 2(\angle POR + \angle ROQ) = 180°$

9. Hint Since, tangent are drawn from an external point to the circle are equal length.



Let $BD = BF = x$

$CD = CE = y$ and $AE = AF = z$

Now, $BC + CA + AB = a + b + c$

$\Rightarrow (BD + DC) + (CE + EA) + (AF + FB) = a + b + c$

$\Rightarrow \quad (x + y) + (y + z) + (z + x) = a + b + c$

$\Rightarrow \quad 2(x + y + z) = 2s$

$\Rightarrow \quad x + y + z = s \Rightarrow x = s - (y + z)$

10. Do same as Example 11.

Ans. 11.25 cm

11. Do same as Example 11.

Ans. (i) 75 cm (ii) 11.25 cm

12. Hint Since, chord PB and tangent at point D intersect each other at A.

$\therefore \quad AD^2 = AP \times AB$ ...(i)

Since, D is the mid-point of AC.

$\therefore \quad AD = \frac{1}{2}AC = \frac{1}{2}AB$ [$\because AB = AC$]

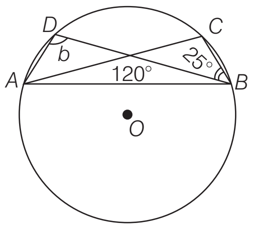
$\Rightarrow \quad AD^2 = \frac{1}{4}AB^2$

$\Rightarrow \quad AP \times AB = \frac{1}{4}AB^2$ [from Eq. (i)]

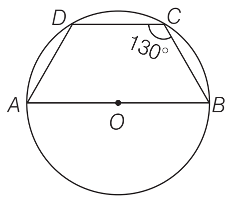
CHAPTER EXERCISE

a 3 Marks Questions

1. In the given figure, O is the centre of a circle. Find the value of b.

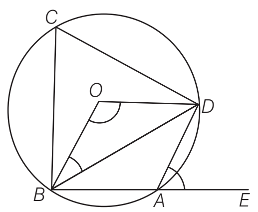


2. In the given figure, AB is the diameter of a circle with centre O and $\angle BCD = 130°$. Find



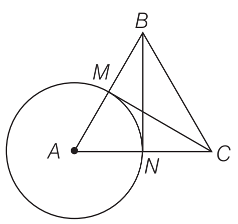
(i) $\angle DAB$ (ii) $\angle DBA$ ***[2012]***

3. In the given figure, O is the centre of the circle and $\angle DAE = 70°$. Giving suitable reasons, find the measure of



(i) $\angle BCD$ (ii) $\angle BOD$ (iii) $\angle OBD$ ***[2017]***

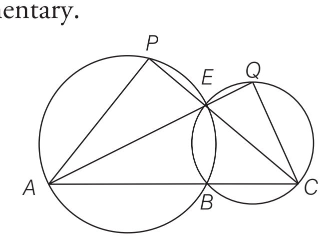
4. ABC is an equilateral triangle. A circle is drawn with centre A, so that it cuts AB and AC at points M and N, respectively. Prove that $BN = CM$.



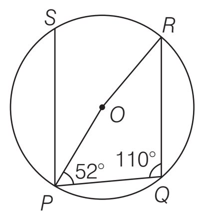
5. If two opposite sides of a cyclic quadrilateral are equal, then prove that its diagonals are equal.

6. ABC is an isosceles triangle in which $AB = AC$. If D and E are the mid-points of AB and AC respectively, then prove that B, C, D and E are concyclic.

7. In the given figure, ABC, AEQ and CEP are straight lines. Show that $\angle APE$ and $\angle CQE$ are supplementary.

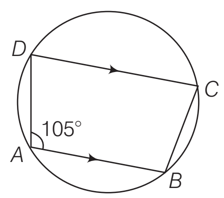


8. In the given figure, $\angle OPQ = 52°$, $\angle PQR = 110°$, then find



(i) $\angle PSR$ (ii) $\angle POR$ (iii) $\angle ORQ$

9. In the given figure, $\angle DAB = 105°$ and AB is parallel to CD.



Find (i) $\angle BCD$ (ii) $\angle ADC$ (iii) $\angle ABC$

10. Two circles intersect each other at P and Q. Through P, a straight line APB is drawn to meet the circles in A and B. Through Q, a straight line is drawn to meet the circles at C and D. Prove that AC is parallel to BD.

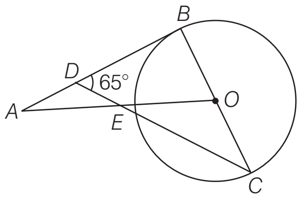
11. PQR is a right angled triangle with $PQ = 3$ cm and $QR = 4$ cm. A circle which touches all the sides of the triangle is inscribed in the triangle. Calculate the radius of the circle. ***[2005]***

12. Two chords AB and CD of a circle intersect each other at P outside the circle. If $AB = 5$ cm, $BP = 3$ cm and $PD = 2$ cm, then find the value of CD.

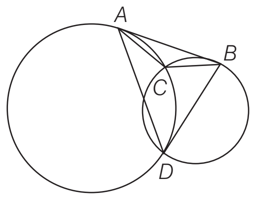
13. Two chords AB and CD of a circle intersect each other at a point E inside the circle. If $AB = 9$ cm, $AE = 4$ cm and $ED = 6$ cm, then find CE.

14. If PA and PB are two tangents drawn from a point P to a circle with centre O touching it at A and B, then prove that OP is perpendicular bisector of AB.

15. In the given figure, O is the centre of circle and AB is a tangent to it at point B. If $\angle BDC = 65°$, then find $\angle BAO$.

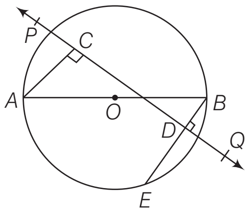


16. In the given figure, AB is a common tangent to two circles intersecting at C and D. Write down the measure of $(\angle ACB + \angle ADB)$. Justify your answer.



[2000]

17. In the given figure, AB is diameter of a circle with centre O. AC and BD are perpendiculars on a line PQ. BD meets the circle at E. Prove that $AC = ED$.

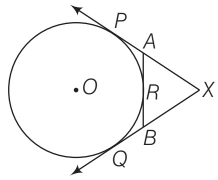


18. $ABCD$ is a cyclic quadrilateral. Sides AB and CD produced meet at a point E, whereas sides BC and AD produced meet at a point F. If $\angle DCF : \angle F : \angle E = 3:5:4$, then find the angles of the cyclic quadrilateral $ABCD$.

19. Find the length of the tangent drawn to a circle of radius 8 cm from a point which is at a distance of 10 cm from the centre of the circle.

20. Two circles of radii 5 cm and 3 cm are concentric. Calculate the length of a chord of the outer circle which touches the inner.

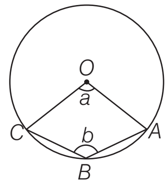
21. In the given figure, XP and XQ are tangents from X to the circle with centre O and R is a point on the circle. Prove that $XA + AR = XB + BR$.



22. Let P be the mid-point of an arc APB of a circle. Prove that the tangent drawn at P will be parallel to the chord AB.

b 4 Marks Questions

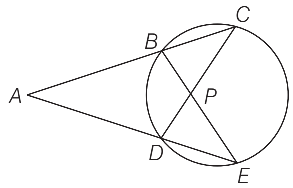
23. The given figure shows a circle with centre O. Given, $\angle AOC = a$ and $\angle ABC = b$.



(i) Find the relationship between a and b.
(ii) Find the measure of $\angle OAB$, if $OABC$ is a parallelogram.

24. AB and CD are two chords of a circle, intersecting each other at P such that $AP = CP$. Show that $AB = CD$.

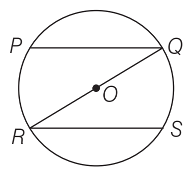
25. In the given figure, $AC = AE$.



Show that (i) $CP = EP$ (ii) $BP = DP$

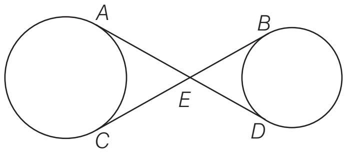
26. P and Q are centres of circles of radii 9 cm and 2 cm, respectively, $PQ = 17$ cm. R is the centre of a circle of radius x cm, which touches the above circles externally. Given that $\angle PRQ = 90°$, write an equation in x and solve it. **[2004]**

27. In the given figure, O is the centre of the circle, chord PQ is parallel and equal to the chord RS and QR is the diameter.



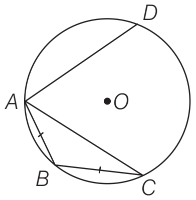
Prove that (i) arc PR = arc QS (ii) arc PQ = arc RS.

28. In the given figure, common tangents AD and BC to two circles intersect at E. Prove that $AD = BC$.



29. If the sides of a quadrilateral $ABCD$ touch a circle, then prove that $AB + CD = BC + AD$.

30. In the given figure, chord AB = chord BC.



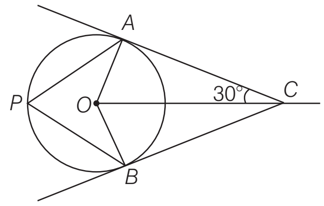
(i) What is the relation between arcs AB and BC?

(ii) What is the relation between $\angle AOB$ and $\angle BOC$?

(iii) If arc AD is greater than arc ABC, then what is the relation between chords AD and AC?

(iv) If $\angle AOB = 50°$, then find the measure of $\angle BAC$.

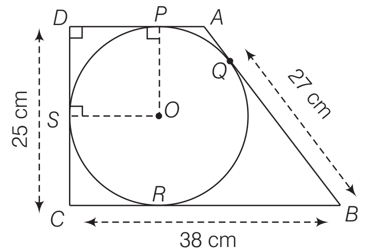
31. In the given figure, O is the centre of the circle. Tangents at A and B meet at C.



If $\angle ACO = 30°$, find

(i) $\angle BCO$ (ii) $\angle AOB$ (iii) $\angle APB$. ***[2011]***

32. In the given figure, a circle is inscribed in the quadrilateral $ABCD$. If $BC = 38$ cm, $QB = 27$ cm, $DC = 25$ cm and AD is perpendicular to DC, then find the radius of the circle.



Hints and Answers

1. Hint $\because \angle ACB = 180° - (60° + 25°) = 95°$

$\therefore \quad \angle ADB = b$ [$\because$ angles in the same segment are equal]

Ans. 95°

2. (i) **Hint** Now, $\angle DAB + \angle BCD = 180°$

[$\because$ sum of opposite angles of cyclic quadrilateral is 180°]

$\Rightarrow \quad \angle DAB + 130° = 180°$

Ans. 50°

(ii) **Hint** $\angle ADB = 90°$ [$\because$ angle in semi-circle]

In ΔADB, $\angle DBA + \angle DAB + \angle ADB = 180°$

[by angle sum property of a triangle]

$\Rightarrow \angle DBA + 50° + 90° = 180°$ [from part (i)]

Ans. 40°

3. Hint $\angle DAB = 180° - \angle DAE$ [by linear pair axiom]

$= 180° - 70° = 110°$

(i) $\angle BCD + \angle DAB = 180°$

[$\because$ sum of opposite angles of cyclic quadrilateral is 180°]

(ii) $\angle BOD = 2\angle BCD = 2 \times 70° = 140°$

(iii) $\because$ $OB = OD$ = Radius of circle

$\therefore \quad \angle OBD = \angle ODB$ ...(i)

$\Rightarrow \angle OBD + \angle BDO + \angle DOB = 180°$

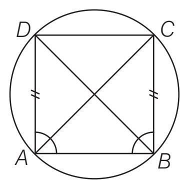
$\Rightarrow \angle OBD + \angle ODB + 140° = 180°$ [from part (ii)]

$\Rightarrow \quad 2\angle OBD = 180° - 140° = 40°$

Ans. (i) 70° (ii) 140° (iii) 20°

4. Hint To show $\Delta BMC \cong \Delta CNB$ by CPCT, $BN = CM$

5. Hint Given, two opposite sides of a cyclic quadrilateral are equal.



Here, to show $\Delta DAB \cong \Delta CBA$ [by rule SAS]

By CPCT, $AC = BD$

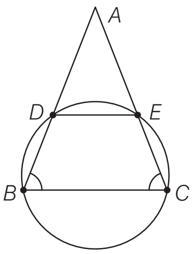
6. Hint Here, $AB = AC$

$\Rightarrow \quad \angle ABC = \angle ACB$ ...(i)

$AD = AE$

[$\because$ D and E are the mid-points of AB and AC]

$\angle AED = \angle ADE$ ...(ii)



Now, $\angle A + \angle ABC + \angle ACB = 180°$
[by angle sum property of a triangle]

and $\angle A + \angle ADE + \angle AED = 180°$

$\Rightarrow$ $\angle A + 2\angle ACB = 180°$ and $\angle A + 2\angle ADE = 180°$
[using Eqs. (i) and (ii)]

$\therefore$ $\angle A + 2\angle ACB = \angle A + 2\angle ADE$

$\Rightarrow$ $\angle ACB = \angle ADE$

$\Rightarrow$ $\angle ECB = \angle ADE$

$\Rightarrow$ $\angle ECB + \angle EDB = \angle ADE + \angle EDB = 180°$

7. Hint $\angle APC = \angle EBC$
[$\because$ exterior angle of a cyclic quadrilateral equals the opposite interior angle]

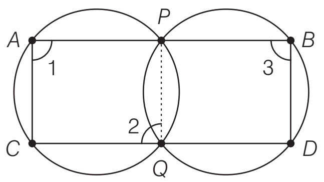
$\angle EBC + \angle EQC = 180°$
[$\because$ sum of opposite angles of a cyclic quadrilateral is 180°]

$\therefore$ $\angle APC + \angle EQC = 180°$

8. Hint Use property of cyclic quadrilateral.
Ans. (i) 70° (ii) 220° (iii) 58°

9. Hint Use property of cyclic quadrilateral and parallel lines.
Ans. (i) 75° (ii) 75° (iii) 105°

10. Hint Join PQ.



Here, $APQC$ is a cyclic quadrilateral.

$\therefore$ $\angle 1 + \angle 2 = 180°$ ...(i)

Again, $BPQC$ is a cyclic quadrilateral.

$\therefore$ $\angle 2 = \angle 3$ ...(iii)
[$\because$ exterior angle of a cyclic quadrilateral equals the opposite interior angle]

From Eqs. (i) and (ii), we get

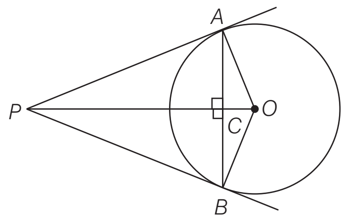
$\angle 1 + \angle 3 = 180°$

11. Do same as Example 8 of Topic 3.
Ans. 1 cm

12. Do same as Example 10 of Topic 3. **Ans.** 10 cm

13. Do same as Example 10 of Topic 3. **Ans.** $\frac{10}{3}$ cm

14. Hint Given, PA and PB are the two tangents from a point P lying outside the circle to the circle with centre O.



We show, $\Delta PAO \cong \Delta PBO$, then use CPCT.

15. Hint $\angle ABO = 90°$
[$\because$ AB is tangent at point B to the circle]
[by angle sum property of triangle]

In ΔBCD, $\angle DBC + \angle BCD + \angle CDB = 180°$

$\Rightarrow$ $90° + \angle BCD + 65° = 180°$

$\Rightarrow$ $\angle BCD = 25°$

$\therefore$ $\angle BOE = 2\angle BCE$

In ΔABO, $\angle BAO = 180° - (\angle ABO + \angle AOB)$

Ans. 40°

16. Hint $\angle CAB = \angle ADC$ ...(i)
[by alternate segment theorem]

and $\angle CBA = \angle BDC$ ...(ii)
[by alternate segment theorem]

$\therefore$ $\angle ACB = 180° - (\angle CAB + \angle CBA)$
$= 180° - (\angle ADC + \angle BDC)$
[using Eqs. (i) and (ii)]

Ans. 180°

17. Hint Join AE

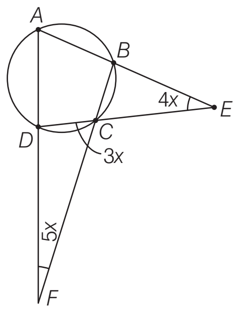
$\Rightarrow$ $\angle CDE = 90°, \angle ACD = 90°$

Also, $\angle AED = 90°$

$\Rightarrow$ $ACDE$ is a rectangle.

$\therefore$ $AC = ED$

18. Hint Here, $\angle DCF = 3x$
$\angle F = 5x$ and $\angle E = 4x$



Now, $\angle A = 3x$

[$\because$ exterior angle of cyclic quadrilateral equals the opposite interior angle]

$\angle ADC = 3x + 5x = 8x$

[$\because$ exterior angle of triangle equals the sum of two interior opposite angles]

In ΔADE, $\angle A + \angle D + \angle E = 180°$

[by angle sum property of triangle]

$\Rightarrow$ $3x + 8x + 4x = 180°$

$\Rightarrow$ $15x = 180° \Rightarrow x = 12°$

$\therefore$ $\angle A = 3x, \angle E = 4x$

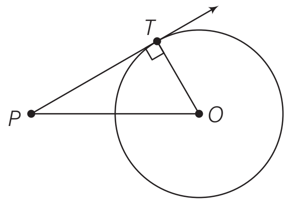
$\angle C = 180 - \angle A, \angle D = 8x$

and $\angle B = 180° - \angle D$

Ans. $36°, 144°, 96°, 84°$

19. Hint $PO^2 = PT^2 + TO^2$

$\Rightarrow$ $PT^2 = PO^2 - TO^2 = 10^2 - 8^2$



Ans. 6 cm

20. Do same as Q. 1 of Topic Exercise 3. **Ans.** 8 cm

21. Hint $XP = XQ$ [tangents from X]

$AP = AR$ [tangents from A] ...(i)

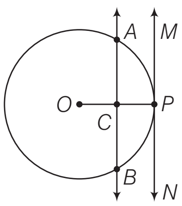
$BR = BQ$ [tangents from B]

Now, $XA + AR = XA + AP = XP$ ...(ii)

and $XB + BR = XB + BQ = XQ$ ...(iii)

From Eqs. (i), (ii) and (iii), we get the result.

22. Hint Since, P is the mid-point of arc APB and $OP \perp MN$.



$\Rightarrow$ C is the mid-point of AB.

$\Rightarrow$ OC is the perpendicular bisector of AB.

23. (i) **Hint** Reflex $\angle COA = 360° - \angle COA = 360° - a$

Since, angle made by an arc at the centre of the circle $= 2 \times$ (angle made by the arc at any point on the remaining part)

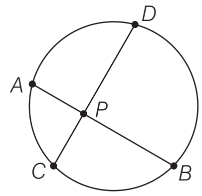
$\therefore$ $360° - a = 2b$

(ii) **Hint** Here, $OA = OC$ and $OABC$ is a parallelogram.

$\Rightarrow$ $OABC$ is a square.

Ans. (i) $a + 2b = 360°$ (ii) $90°$

24. Hint $AP \times PB = PC \times PD$



25. Hint From the figure, $AC \times AB = AE \times AD$

But $AC = AE$ [given]

$\therefore$ $AB = AD$ ... (i)

Now, $CB = AC - AB$ ...(ii)

and $DE = AE - AD$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$CB = DE$

In ΔCBP and ΔDPE,

$\angle BPC = \angle DPE$ [vertically opposite angles]

$\Rightarrow$ $\angle BCP = \angle DEP$ [angles in the same segment]

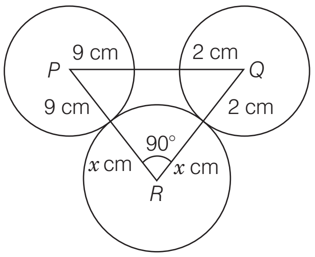
$CB = DE$ [proved above]

$\therefore$ $\Delta CBP \cong \Delta DPE$

$\Rightarrow$ $CP = EP$ and $BP = DP$ [by CPCT]

26. Hint Given, P and Q are the centres of circles, whose radii are 9 cm and 2 cm, respectively.

Also, $PQ = 17$ cm



A circle with centre R and radius x cm touches above circles.

$\therefore$ $PR = 9 + x$ cm and $QR = 2 + x$ cm

Given, PRQ is a right angled triangle.

$\therefore$ $PQ^2 = PR^2 + QR^2$ [by Pythagoras theorem]

$\Rightarrow$ $(17)^2 = (9 + x)^2 + (2 + x)^2$

$\Rightarrow$ $(x - 6)(x + 17) = 0$

$\Rightarrow$ $x = -17$ and $x = 6$

$\therefore$ $x = 6$ cm [$\because$ radius cannot be negative]

27. Since, $PQ \parallel RS$ and $PQ = RS$

Join OP and OS.

(i) **Hint** $\angle PQR = \angle SRQ$ [alternate angles]

Also, $\angle QOS = 2\angle SRQ$

and $\angle POR = 2\angle PQR$

$\Rightarrow \quad \angle POR = \angle QOS$

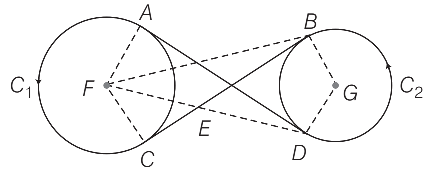
$\Rightarrow \quad \text{arc } PR = \text{arc } QS$

(ii) **Hint** $PQ = RS \Rightarrow \widehat{PQ} = \widehat{RS}$

28. **Given** AD and BC are two common tangents of circles C_1 and C_2 (say).

To prove $AD = BC$

Proof Let F and G be the centres of the circles C_1 and C_2, respectively and join AF, FC, FB and FD, BG and GD.



Now, tangents FB and FD are drawn from an external point F to the circle C_2.

$\therefore \quad FD = FB \quad$ ...(i)

Since, the radius is perpendicular to the tangent at the point of contact.

$\therefore \quad AF \perp AD$ and $FC \perp BC$

$\Rightarrow \quad \angle FAD = \angle FCB = 90°$

In right angled ΔFAD and ΔFCB,

$AF = FC$ [radii of same circle]

$\angle A = \angle C = 90°$

and $FD = FB$ [from Eq. (i)]

$\therefore \quad \Delta AFD \cong \Delta CFB$ [by SAS congruence rule]

Then, $AD = BC$ [by CPCT]

29. Do same as Example 6 of Topic 3.

30. (i) **Hint** Since chord AB = chord BC

$\Rightarrow$ arc AB = arc BC

[$\because$ equal chords in any circle having equal ars]

(ii) **Hint** $\angle AOB = \angle BOC$ [$\because$ chord AB = chord BC]

(iii) **Hint** Since, arc AD > arc ABC

$\Rightarrow \quad \angle AOD > \angle BOC$

$\Rightarrow$ chord AD > chord AC

(iv) **Hint** Since $\angle AOB = 50°$

$\therefore \quad \angle AOC = 2 \times \angle AOB = 100°$

reflex $\angle AOC = 360° - 100° = 260°$

$\angle ABC = \frac{1}{2}$ reflex $\angle AOC = \frac{1}{2} \times 260° = 130°$

$AB = BC$ [given]

$\angle ACB = \angle BAC$

In ΔABC, $\angle BAC + \angle BCA + \angle ABC = 180°$

[by angle sum property of a triangle]

$\Rightarrow \quad 2\angle BAC + 130° = 180°$

$\Rightarrow \quad 2\angle BAC = 50°$

Ans. 25°

31. (i) **Hint** $\angle BCO = \angle ACO$

[$\because$ tangents are equally inclined to the line joining the point and the centre of the circle]

Ans. 30°

(ii) **Hint** $\angle AOB = 180° - 60°$ **Ans.** 120°

(iii) **Hint** $\angle APB = \frac{1}{2} \angle AOB$ **Ans.** 60°

32. **Hint** Since, tangents to a circle is perpendicular to the radius through the point.

$\therefore \quad \angle OPD = \angle OSD = 90°$, $\angle D = 90°$ and $PO = OS$

Therefore, $PQSD$ is square.

Since, tangents from an exterior point to a circle are equal in length.

$\therefore \quad BR = BQ$, $AQ = AP$ and $DP = DS$

Now, $BC = 38$ cm

$\Rightarrow \quad CR + BR = 38$

$\Rightarrow \quad CR + BQ = 38 \quad$ [$\because BR = BQ$]

$\Rightarrow \quad CR = 38 - 27 = 11$

$\because \quad DS = CD - CS$

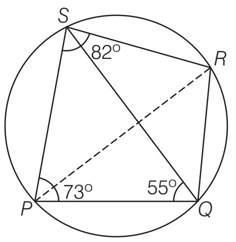
Ans. 14 cm

ARCHIVES* (Last 8 Years)

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

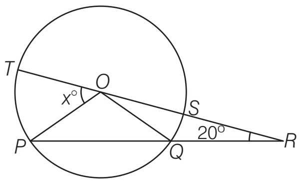
2018

1 $PQRS$ is a cyclic quadrilateral. Given $\angle QPS = 73°$, $\angle PQS = 55°$ and $\angle PSR = 82°$, calculate



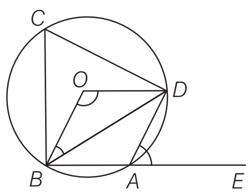
(i) $\angle QRS$ (ii) $\angle RQS$ (iii) $\angle PRQ$.

2 In the figure given below, 'O' is the centre of the circle. If $QR = OP$ and $\angle ORP = 20°$. Find the value of 'x' giving reasons.



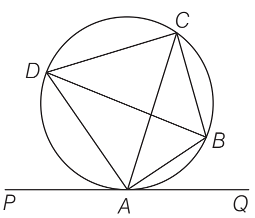
2017

3 In the given figure, O is the centre of the circle and $\angle DAE = 70°$. Giving suitable reasons, find the measure of



(i) $\angle BCD$ (ii) $\angle BOD$ (iii) $\angle OBD$

4 In the given figure, PQ is a tangent to the circle at A. AB and AD are bisectors of $\angle CAQ$ and $\angle PAC$.



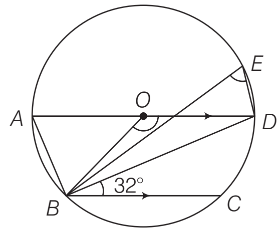
If $\angle BAQ = 30°$, then prove that

(i) BD is a diameter of the circle.

(ii) ABC is an isosceles triangle.

2016

5 In the given figure, AD is a diameter and O is centre of the circle. AD is parallel to BC and $\angle CBD = 32°$.

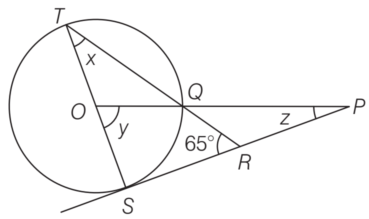


Find

(i) $\angle OBD$ (ii) $\angle AOB$ (iii) $\angle BED$

2015

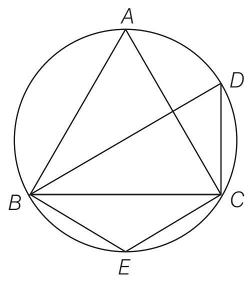
6 In the given figure, O is the centre of the circle and SP is a tangent. If $\angle SRT = 65°$, then find the values of x, y and z.



7 If two chords of a circle intersect internally or externally, then prove that the product of the lengths of segments are equal.

2014

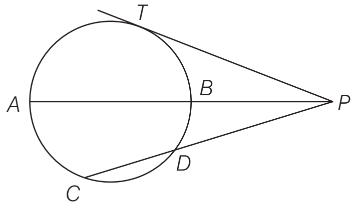
8 In the given figure, $\angle DBC = 58°$. BD is a diameter of the circle.



Calculate

(i) $\angle BDC$ (ii) $\angle BEC$ (iii) $\angle BAC$

9 In the given figure, diameter AB and chord CD of a circle meet at P. PT is a tangent to the circle at T and $CD = 7.8$ cm, $PD = 5$ cm, $PB = 4$ cm.

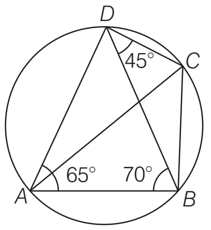


Find

(i) AB. (ii) the length of tangent PT.

2013

10 In the given figure, $\angle BAD = 65°$, $\angle ABD = 70°$, $\angle BDC = 45°$.

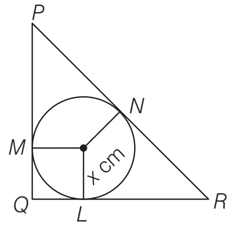


(i) Prove that AC is a diameter of the circle

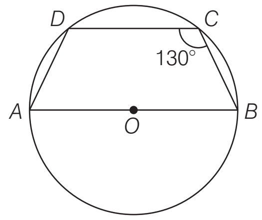
(ii) Find $\angle ACB$.

2012

11 In the given figure, $PQ = 24$ cm, $QR = 7$ cm and $\angle PQR = 90°$. Find the radius of the inscribed circle.



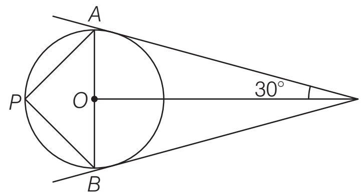
12 In the given figure, AB is the diameter of a circle with centre O and $\angle BCD = 130°$.



Find (i) $\angle DAB$ (ii) $\angle DBA$

2011

13 In the given figure, O is the centre of the circle. Tangents at A and B meet at C.



If $\angle ACO = 30°$, then find

(i) $\angle BCO$ (ii) $\angle AOB$ (iii) $\angle APB$

* All these questions are completely covered in chapter either as solved examples or in topic/chapter exercise.

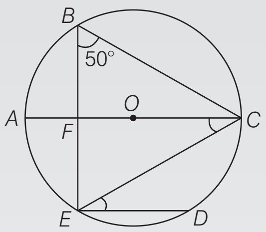
CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. If P, Q, S and R are points on the circumference of a circle of radius r, such that PQR is an equilateral triangle and PS is a diameter of the circle. Then, the perimeter of the quadrilateral $PQSR$ will be

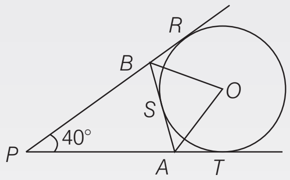
(a) $2\sqrt{3}r$ (b) $2r$
(c) $2(\sqrt{3}+1)r$ (d) $2\sqrt{3}+r$

2. In the given figure, if the chord ED is parallel to the diameter AC. Then, the measure of $\angle CED$ will be



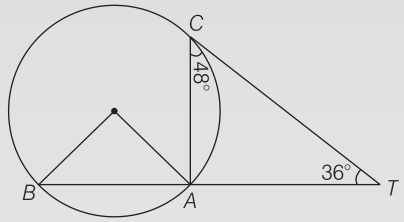
(a) 50° (b) 70°
(c) 40° (d) 30°

3. As shown in the figure, ΔPAB is formed by three tangents to circle with centre O and $\angle APB = 40°$. Then, the measure of $\angle AOB$ will be



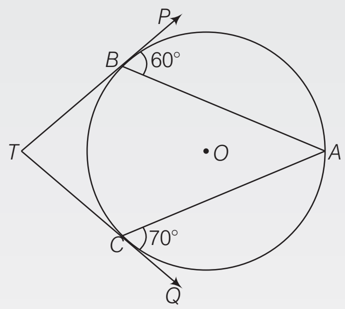
(a) 50° (b) 40°
(c) 80° (d) 70°

4. A, B and C are three points on a circle. The tangents at C meets BA produced at T. Given that $\angle ATC = 36°$ and $\angle ACT = 48°$. Then, the angle subtended by AB at the centre of the circle will be



(a) 69° (b) 96°
(c) 72° (d) 82°

5. In the given figure, TBP and TCQ are tangents to the circle whose centre is O. Also, $\angle PBA = 60°$ and $\angle ACQ = 70°$. Then, $\angle BAC$ and $\angle BTC$ are respectively.



(a) 80°, 50° (b) 50°, 80°
(c) 60°, 50° (d) 70°, 80°

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 410.

CHAPTER

16

Constructions

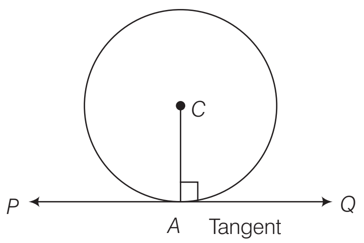
In earlier classes, we have already studied construction by using a ruler and a pair of compasses to draw angle bisector and perpendicular bisector of line segment, triangles etc. In this chapter, we will study some more constructions, such as tangent to a circle, circumscribing and inscribing a circle on a triangle and regular hexagon.

Chapter Objectives

- Construction of a Tangent to a Circle from an External Point
- Circumscribed Circle of a Polygon
- Inscribed Circle of a Polygon

Tangent to a Circle

A line which meets a circle at one and only one point, is called a tangent to a circle.



In the above figure, *PQ* is a tangent to a circle, as it meet the circle at only one point, i.e. *A*. Here, the point *A* at which the tangent line meets the circle, is called the **point of contact**.

Note The tangent is always perpendicular to the radius of the circle at the point of contact.

Construction of a Tangent to a Circle from an External Point

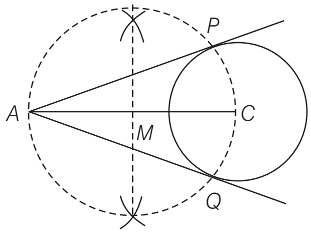
From an external point, only two tangents can be drawn. These tangents can be drawn in two ways, such as when centre of a circle is known and when centre of a circle is unknown. These two ways are explained below

When Centre of a Circle is Known

To understand, how to construct tangents to a given circle from an exterior point when centre of a circle is known, the following steps are given

***Step* I** Draw a circle with centre C and take a point *A* exterior to it.

Step II Join CA and draw the perpendicular bisector of CA. Let it meets CA at M, which is the mid–point of CA.



Step III Draw a dotted circle with centre M and radius AM = MC, which cuts the given circle at the points *P* and *Q*.

Step IV Now, join *PA* and *QA*.

Hence, *PA* and *QA* are the required tangents from exterior point A to the given circle.

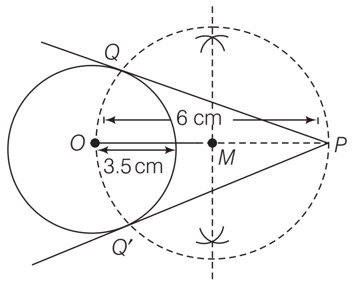
Note We observe that the lengths of two tangents drawn from an exterior point of a circle are equal.

Example 1. Draw a circle of radius 3.5 cm. Mark a point *P* outside the circle at a distance of 6 cm from the centre. Construct two tangents from *P* to the given circle. Measure and write down the length of one tangent. ***[2011]***

Sol. Given, a circle of radius 3.5 cm and distance between the point and centre of a circle is 6 cm.

Steps of construction

1. Draw a circle with centre *O* and radius 3.5 cm and take a point *P* outside the circle at a distance of 6 cm from its centre.
2. Join *OP*.
3. Draw a perpendicular bisector of *OP*, which intersects *OP* at (the mid-point) *M*.



4. Taking *M* as centre and *OM* = *MP* as radius, draw a dotted circle, which intersects the given circle at *Q* and *Q′*.
5. Join *PQ* and *PQ′*, which are the required tangents.

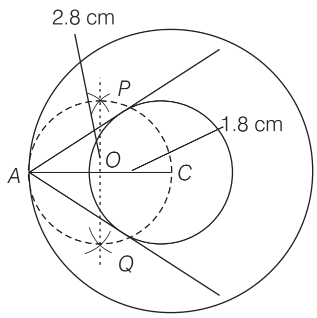
   On measuring the length of tangent with the help of ruler, we get *PQ* = *PQ′* = 4.9 cm.

Example 2. Construct a tangents to a circle of radius 1.8 cm from a point on the concentric circle of radius 2.8 cm and measure its length.

Sol. Given, two concentric circles of radii 1.8 cm and 2.8 cm with common centre *C*.

Steps of construction

1. Draw two circles with common centre *C* and radii 1.8 cm and 2.8 cm.
2. Take a point *A* on the circumference of the outer circle and join *CA*.
3. Now, draw the perpendicular bisector of *CA*. Let it meets *CA* at point *O*, which is the mid-point of *CA*.



4. Draw a dotted circle with centre *O* and radius *OC* = *OA*, which cuts inner circle at two points, say *P* and *Q*.
5. Join *AP* and *AQ*.

   Hence, *AP* and *AQ* are the required tangents.

   On measuring the lengths, we get

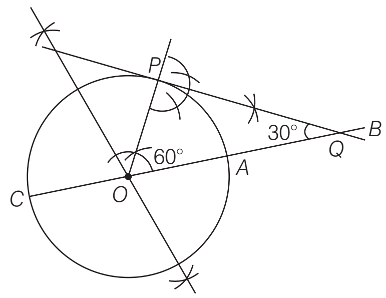
   $$AP = AQ = 2.14\text{ cm}$$

Example 3. Draw a circle of radius 7 cm. Draw a tangent to this circle making an angle of 30° with a line passing through the centre of the circle.

Sol. Given, a circle of radius 7 cm.

Steps of construction

1. Draw a circle with centre *O* and radius 7 cm.
2. Draw the diameter *CA* of this circle and produce it to *B*.
3. Construct an $\angle AOP$ (where, *P* is on the circumference) equal to the complement of 30°, i.e. equal to 60°.



4. Draw perpendicular on *OP* at point *P*, which intersects *OB* at point *Q*.

   Hence, *PQ* is the desired tangent, such that $\angle OQP = 30°$.

When Centre of a Circle is Unknown

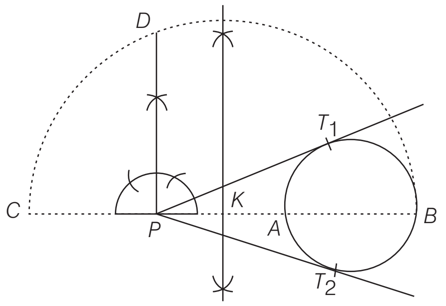
To understand how to construct tangents to a given circle from an exterior point when centre of a circle is unknown, the following steps are given below

***Step* I** Draw a circle of given radius and take a point P outside it.

***Step* II** Through P, draw a line (i.e. secant) intersecting the given circle at points A and B respectively and produce it to C in opposite direction of AB such that $AP = CP$.

***Step* III** Now, bisect the segment CB. Let its mid-point be K. Then, take K as centre and KB ($= KC$) as radius, draw a dotted semi-circle.

***Step* IV** At point P, draw $PD \perp CB$ which cuts the semi-circle at point D.



***Step* V** Take P as centre and PD as radius draw two arcs which intersect the given circle at points T_1 and T_2.

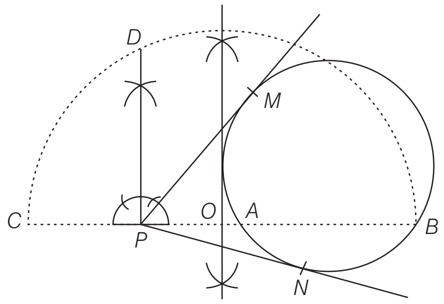
***Step* VI** Join PT_1 and PT_2, which are the required tangents.

Example 4. **Draw a circle of radius 2.8 cm. From an external point P, draw tangents to the circle without using the centre of the circle.**

Sol. Given, a circle of radius 2.8 cm and we have to draw tangents without using the centre.

Steps of construction

1. First, draw a circle of radius 2.8 cm and take a point P outside the circle.
2. Through P, draw a secant PAB which intersects the circle at points A and B and extend it to C in opposite direction of AB such that $PC = PA$.

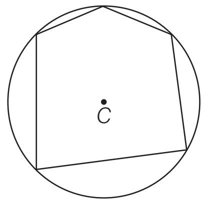


3. Now, bisect BC and take its mid-point as O. Draw a dotted semi-circle with centre O and radius OB ($= OC$).
4. Draw $PD \perp BC$ which intersects the semi-circle at point D.
5. With centre P and radius PD draw two arcs which intersects the given circle at points M and N.
6. Join PM and PN.

   Thus, PM and PN are the required tangents to the given circle.

Circumscribed Circle of a Polygon

A circle passing through all the vertices of a polygon (triangle or hexagon) is called a circumscribed circle of the polygon, and its centre is called **circumcentre**.

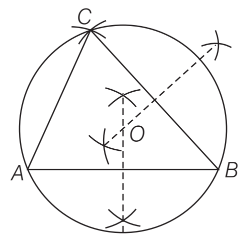


Construction of Circumcircle of a Triangle

To understand how to construct the circumscribed circle of a given triangle, the following steps are given below

***Step* I** Draw a ΔABC with the given data.

***Step* II** Draw the perpendicular bisectors of any two sides, say AB and BC. Let these bisectors meet at point O.



***Step* III** Draw a circle with centre O and radius OA, which passes through the points A, B and C. It is the required circumcircle of ΔABC.

Note (i) The point O of the above construction is called the circumcentre of triangle.

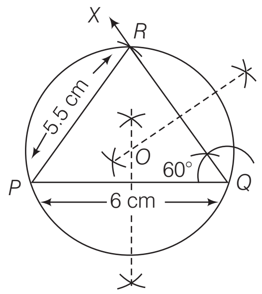
(ii) The circumcentre of the triangle is always equidistant from the vertices of the triangle.

Example 5. Construct a ΔPQR, in which $PQ = 6$ cm, $PR = 5.5$ cm and $\angle Q = 60°$. **Draw the circumcircle of ΔPQR. Also, write the steps of construction.**

Sol. Given, $PQ = 6$ cm, $PR = 5.5$ cm and $\angle Q = 60°$.

Steps of construction

1. First, draw the base $PQ = 6$ cm.
2. Taking Q as centre, draw a ray QX making an angle of 60° with PQ.
3. Now, taking P as centre, draw an arc of 5.5 cm, which cuts QX at point R.
4. Join PR. Thus, PQR is the required triangle.
5. Draw the perpendicular bisectors of PQ and QR, which meet at point O.



6. Draw a circle with centre O and radius OP, which passes through all the vertices of ΔPQR.

   Hence, it is the required circumcircle of ΔPQR.

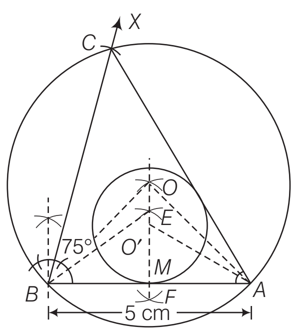
Example 6. Using a ruler and a pair of compasses only, construct a ΔABC, such that $AB = 5$ cm, $\angle ABC = 75°$ and the radius of the circumcircle of ΔABC is 3.5 cm. On the same diagram, construct a circle, touching AB at its middle point and also touching the side AC. ***[2004]***

Sol. Given, in ΔABC, $AB = 5$ cm, $\angle ABC = 75°$ and radius of a circumcircle of ΔABC is 3.5 cm.

Steps of construction

1. Draw a line segment $AB = 5$ cm.
2. From point B, draw a ray BX, making $\angle ABX = 75°$.
3. Taking A and B as centres one-by-one, mark two arcs of radius 3.5 cm, which intersect at the point O.
4. Taking O as the centre and OB as radius, draw a circumcircle, which passes through the points A and B. It also intersects BX at point C. Join AC and BC.

   Hence, ΔABC is the required triangle.
5. Draw perpendicular bisector EF of AB, which intersects AB at point M and draw an angle bisector of $\angle CAB$, which intersects EF at O'.



6. Taking O' as centre and $O'M$ as radius, draw a circle, which touches AB and AC.

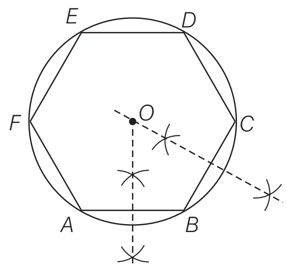
Construction of a Circle Circumscribing a Given Regular Hexagon

To understand how to construct a circle about a given regular hexagon, the following steps are given below

***Step* I** Construct a regular hexagon $ABCDEF$ with given data.

***Step* II** Draw perpendicular bisectors of any two sides of regular polygon, say AB and BC. Let these bisectors intersect each other at O.

***Step* III** Draw a circle with centre O and radius OA, which passes through all the vertices of the regular hexagon $ABCDEF$.



Hence, it is the required circle about the regular hexagon.

Note (i) Whenever a circle circumscribes a given regular hexagon, its radius is always equal to the length of the side of the regular hexagon.

(ii) Each interior angle of the regular hexagon = 120°, where n denotes the number of sides.

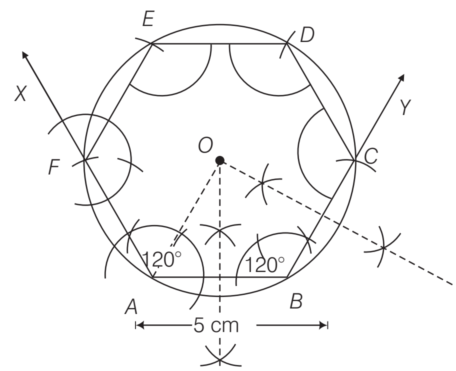
Example 7. Construct a regular hexagon of side 5 cm. Construct a circle circumscribing the hexagon. All traces of construction must be clearly shown. ***[2015]***

Sol. Given, each side of a regular hexagon is 5 cm.

We know that each interior angle of regular hexagon = 120°

Steps of construction

1. First, draw a line segment $AB = 5$ cm.
2. Taking A and B as centres, draw two rays AX and BY, making an angle of 120° with AB.



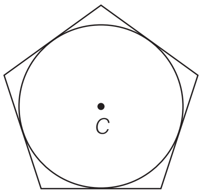
3. Cut $AF = 5$ cm from AX and $BC = 5$ cm from BY.

4. Now, take F, E, D and C respectively, as centres to repeat steps 2 and 3. Then, we get a regular hexagon $ABCDEF$ with each side equal to 5 cm and each angle equal to 120°.
5. Draw the perpendicular bisectors of sides AB and BC, which intersect each other at point O.
6. Draw a circle with O as centre and OA as radius, which passes through all the vertices of the regular hexagon $ABCDEF$.

   Hence, $ABCDEF$ is the required circumcircle of the regular hexagon.

Inscribed Circle of a Polygon

A circle touching all the sides of a triangle or hexagon is called an inscribed circle of the polygon, and its centre is called **incentre**.



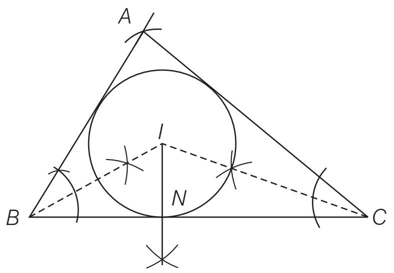
Construction of Incircle of a Triangle

To understand how to construct the inscribed circle of a given triangle, the following steps are given below

***Step* I** Draw a ΔABC with the given data.

***Step* II** Draw the internal bisectors of any two angles of a triangle, say $\angle B$ and $\angle C$. Let these bisectors meet at point I.

***Step* III** Draw perpendicular to BC from I, which intersects BC at N.



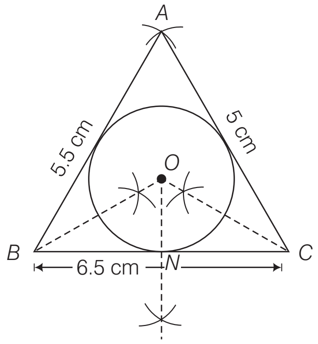
***Step* IV** Now, draw a circle with centre I and radius NI, which touches all sides of ΔABC.
Hence, it is the required incircle of ΔABC.

Example 8. Construct a ΔABC with $BC = 6.5$ cm, $AB = 5.5$ cm and $AC = 5$ cm. Construct the incircle of the triangle. Measure and record the radius of the incircle. ***[2014]***

Sol. Given, $BC = 6.5$ cm, $AB = 5.5$ cm and $AC = 5$ cm.

Steps of construction

1. Draw the base $BC = 6.5$ cm.
2. Taking B and C as centres, draw two arcs of radius 5.5 cm and 5 cm respectively, which intersect each other at A.
3. Join AB and AC. Thus, ABC is the required triangle.
4. Now, draw the internal bisectors of $\angle B$ and $\angle C$. Let these bisectors meet at point O.
5. From point O, draw a perpendicular on side BC, which cuts BC at N.



6. Taking O as centre and radius equal to ON, draw a circle which touches all the sides of the ΔABC.

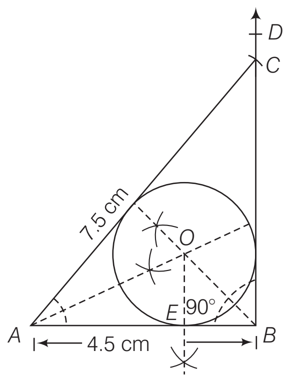
   Hence, it is the required incircle of ΔABC.
7. Measure ON, which is the required radius of the incircle, i.e. $ON = 1.55$ cm.

Example 9. Draw a right angled ΔABC with $AB = 4.5$ cm, $AC = 7.5$ cm and $\angle B = 90°$ and also draw its incircle.

Sol. Given, $AB = 4.5$ cm, $AC = 7.5$ cm and $\angle B = 90°$.

Steps of construction

1. First, draw the base $AB = 4.5$ cm.
2. Taking B as centre, draw a ray BD making an angle of 90° with AB.
3. Draw an arc with centre A and radius 7.5 cm, which cuts BD at C.
4. Join AC to obtain ΔABC.
5. Draw the internal bisectors of $\angle A$ and $\angle B$, which meet each other at O.
6. Draw a perpendicular to AB from O, which intersects AB at E.



7. Now, draw a circle with centre O and radius OE, which touches all the sides of the ΔABC.

   Hence, it is the required incircle of ΔABC.

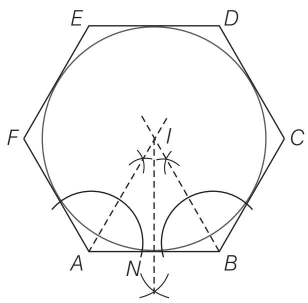
Construction of a Circle Inscribing a Given Regular Hexagon

To understand how to construct a circle inscribing a given regular hexagon, the following steps are given below

***Step* I** Construct a regular hexagon *ABCDEF* with given data.

***Step* II** Draw the internal bisectors of any two adjacent angles, say $\angle A$ and $\angle B$. Let these bisectors intersect each other at I.

***Step* III** Draw perpendicular to AB from point I, which intersects AB at N.



***Step* IV** Draw a circle with centre I and radius NI, which touches all the sides of regular hexagon *ABCDEF*.

Hence, it is the required incircle of regular hexagon.

Example 10. Draw a regular hexagon of side 3 cm. Draw its incircle.

Sol. Given, each side of regular hexagon is 3 cm.

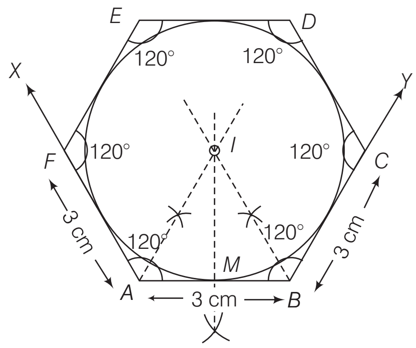
We know that, each interior angle of a regular hexagon

$= 120^\circ$

Steps of construction

1. Draw a line segment $AB = 3$ cm.
2. Taking A and B as centres, draw two rays AX and BY, making an angle of 120° with AB.
3. Cut $AF = 3$ cm from AX and $BC = 3$ cm from BY. Now, take F, E, D and C respectively as centres to repeat steps 2, 3.

   Then, we get a required regular hexagon *ABCDEF* with each side equal to 3 cm and each angle is equal to 120°.
4. Draw the internal bisectors of $\angle A$ and $\angle B$, which intersect each other at I.
5. Draw the perpendicular to AB from I, which intersects AB at M.



6. Draw a circle with centre I and radius IM, which touches all the sides of regular hexagon *ABCDEF*.

   Hence, it is the required incircle of regular hexagon *ABCDEF*.

CHAPTER EXERCISE

a 3 Marks Questions

1. Draw a circle of radius 4 cm, mark its centre as O. Take a point A at a distance of 7 cm from O. Construct a pair of tangents to this circle from point A.

2. Draw a circle of radius 6 cm. From a point 10 cm away from its centre, construct the pair of tangents to the circle and measure their lengths.

3. Construct a pair of tangents to a circle of radius 5.5 cm from a point on the circumference of the concentric circle of radius 7.5 cm and measure the lengths of tangents.

4. Construct tangents to a circle of radius 4.5 cm from a point on the circumference of the concentric circle of radius 6 cm.

5. Draw concentric circle with the help of big and small circular bangles. Construct a pair of tangents from a point A on the circumference of the big circle.

6. Draw a circle of radius 5 cm with centre. Draw two tangents to it, inclined at an angle of 60° to each other.

7. Draw a circle of radius 3.3 cm. From an external point Q, draw tangents to the circle without using the centre of the circle.

8. Draw a circle of radius 4 cm. Mark a point A outside the circle. Draw the tangents to the circle from point A, without using the centre of the circle.

9. Using a ruler and a pair of compasses only, construct

(i) a ΔABC, given $AB = 4$ cm, $BC = 6$ cm and $\angle ABC = 90°$.

(ii) a circle that passes through the points A, B and C and mark its centre as O. **[2008]**

10. Construct the circumcircle of a triangle whose sides are 5 cm, 6 cm and 8 cm.

11. Draw a triangle with sides 5.5 cm, 6 cm and 6.5 cm. Construct its circumcircle.

12. Construct a circumscribing circle of an equilateral triangle of side 4 cm.

13. Use a ruler and a pair of compasses to construct a ΔABC, in which $BC = 4.2$ cm, $\angle ABC = 60°$ and $AB = 5$ cm. Construct a circle of radius 2 cm to touch both the arms of $\angle ABC$ of ΔABC. **[2006]**

14. Construct a ΔABC, such that $BC = 4$ cm, $\angle C = 75°$ and radius of circumcircle of ΔABC is 3 cm.

15. Inscribe a circle of an equilateral triangle having side 2.5 cm.

16. Inscribe the equilateral triangle in the circle of radius 3.5 cm.

17. Construct a triangle with sides 6.1 cm, 7 cm and 8 cm. Draw its incircle.

18. ABC is a right angled triangle, right angled at B with $AB = 8$ cm and $BC = 6$ cm. Construct an inscribed circle which touches all the sides of the triangle.

19. Using a ruler and a pair of compasses only, construct a ΔABC with $BC = 6.4$ cm, $CA = 5.8$ cm and $\angle ABC = 60°$. Draw its incircle. Measure and record the radius of the incircle. **[2007]**

b 4 Marks Questions

20. Construct a ΔABC, in which $AB = 5$ cm, $BC = 7$ cm and $AC = 6$ cm.

(i) Mark D, the mid-point of AB.

(ii) Construct the circle, which touches BC at C and passes through D. **[2001]**

21. Using a ruler and a pair of compasses only, construct

(i) a ΔABC, in which $AB = 9$ cm, $BC = 10$ cm and $\angle ABC = 45°$.

(ii) a circle of radius 2 cm to touch the arms of $\angle ACB$. **[2002]**

22. Draw a line $AB = 5$ cm. Mark a point C on AB such that $AC = 3$ cm. Using a ruler and a compasses only, construct

(i) a circle of radius 2.5 cm, passing through A and C.

(ii) two tangents to the circle from the external point B. Measure and record the length of the tangents. **[2016]**

23. Construct an $\angle PQR = 45°$. Mark a point S on QR, such that $QS = 4.5$ cm. Construct a circle to touch PQ at Q and also to pass through S. **[2003]**

24. Using a ruler and a pair of compasses only, construct

(i) a ΔABC, such that $AB = 6.5$ cm, $\angle A = 30°$ and $\angle B = 60°$.

(ii) in the same figure, draw a circle passing through the points A, B and C and mark its circumcentre as O.

25. Using a ruler and a pair of compasses only, construct

(i) a ΔEFG, such that $EF = 7$ cm, $\angle E = 45°$ and $\angle F = 75°$.

(ii) in the same figure, draw a circle which touches all the sides of the triangle.

26. Using a ruler and a compasses only, construct

(i) a ΔABC in which $AB = 3.5$ cm, $BC = 6$ cm and $\angle ABC = 120°$.

(ii) in the same diagram, draw a circle with BC as diameter. Find a point P on the circumference of the circle which is equidistant from AB and BC.

(iii) Measure $\angle BCP$. **[2013]**

27. Construct a ΔABC in which base $BC = 6$ cm, $AB = 5.5$ cm and $\angle ABC = 120°$.

(i) Construct circle circumscribing the ΔABC.

(ii) Draw a cyclic quadrilateral $ABCD$, so that D is equidistant from B and C. **[2012]**

28. Using ruler and compass only, construct a ΔABC such that $BC = 5$ cm and $AB = 6.5$ cm and $\angle ABC = 120°$.

(i) Construct a circumcircle of ΔABC.

(ii) Construct a cyclic quadrilateral $ABCD$, such that D is equidistant from AB and BC. **[2018]**

29. Construct a regular hexagon of side 4 cm. Construct a circle circumscribing the hexagon. **[2010]**

30. Construct a regular hexagon of side 6 cm and construct a circle in the regular hexagon. Mark its incentre as I.

Hints and Answers

1. Do same as Example 1.

2. Do same as Example 1. **Ans.** 8 cm

3. Do same as Example 2. **Ans.** 5.1 cm

4. Do same as Example 2.

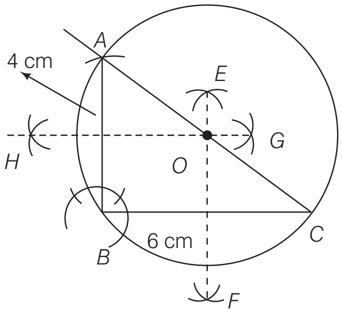
5. Do same as Example 2.

6. Do same as Example 3.

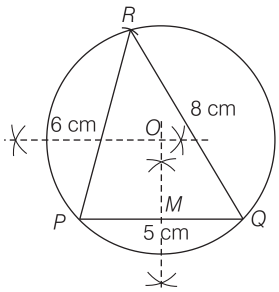
7. Do same as Example 4.

8. Do same as Example 4.

9. Do same as Example 5.



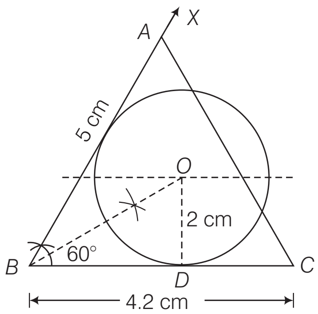
10. Hint



11. Do same as Q. 10.

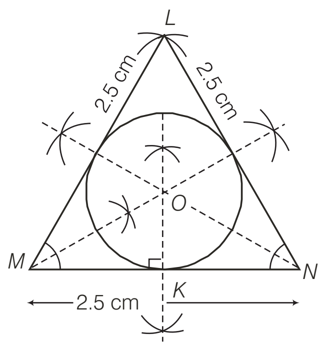
12. Do same as Q. 10.

13. Hint



14. Do same as Example 6.

15. Hint



16. Do same as Q. 15.

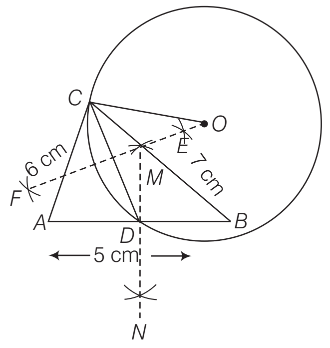
17. Do same as Example 8.

18. Do same as Example 9.

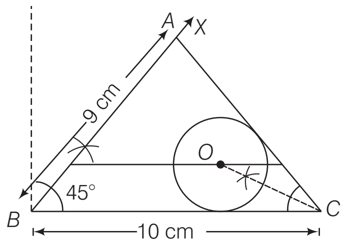
19. Do same as Example 9.

Ans. Radius of incircle = 1.5 cm

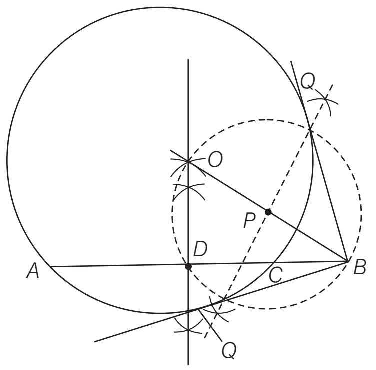
20. Hint



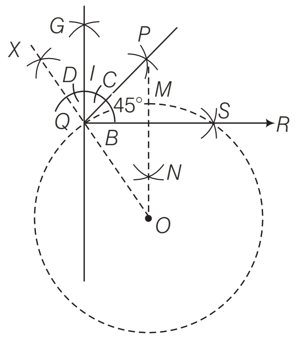
21. Hint



22. Hint



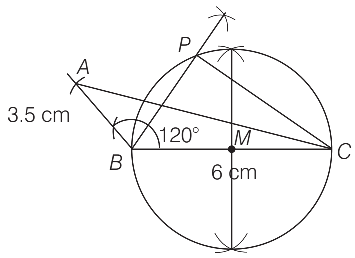
23. Hint



24. Do same as Example 5.

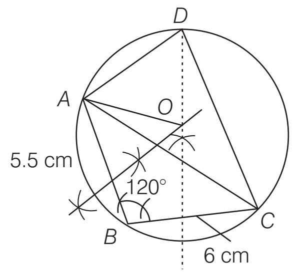
25. Do same as Example 9.

26. Hint

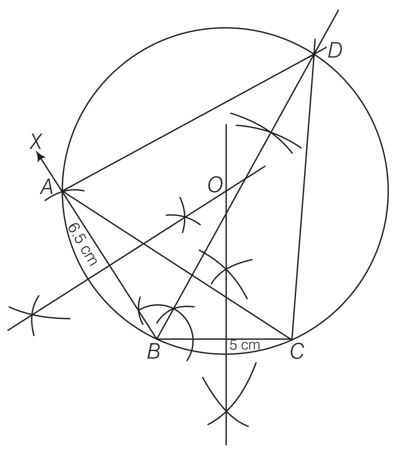


Ans. (iii) $\angle BCP = 30°$

27. Hint



28. Hint



29. Do same as Example 7.

30. Do same as Example 10.

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1 Using ruler and compass only, construct a ΔABC such that $BC = 5$ cm and $AB = 6.5$ cm and $\angle ABC = 120°$.

(i) Construct a circumcircle of ΔABC.

(ii) Construct a cyclic quadrilateral $ABCD$, such that D is equidistant from AB and BC.

2016

2 Draw a line $AB = 5$ cm. Mark a point C on AB such that $AC = 3$ cm. Using a ruler and a compasses only, construct

(i) a circle of radius 2.5 cm, passing through A and C.

(ii) two tangents to the circle from the external point B. Measure and record the length of the tangents.

2015

3 Construct a regular hexagon of side 5 cm. Construct a circle circumscribing the hexagon. All traces of construction must be clearly shown.

2014

4 Construct a ΔABC with $BC = 6.5$ cm, $AB = 5.5$ cm and $AC = 5$ cm. Construct the incircle of the triangle. Measure and record the radius of the incircle.

2013

5 Using a ruler and a compasses only, construct

(i) a ABC in which $AB = 3.5$ cm, $BC = 6$ cm and $\angle ABC = 120°$.

(ii) In the same diagram, draw a circle with BC as diameter. Find a point P on the circumference of the circle which is equidistant from AB and BC.

(iii) Measure $\angle BCP$.

2012

6 Construct a ΔABC in which base $BC = 6$ cm, $AB = 5.5$ cm and $\angle ABC = 120°$.

(i) Construct a circle circumscribing the ΔABC.

(ii) Draw a cyclic quadrilateral $ABCQ$, so that D is equidistant from B and C.

2011

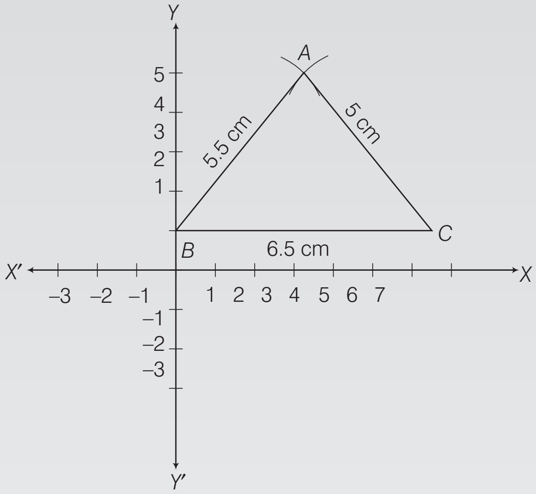
7 Draw a circle of radius 3.5 cm. Mark a point P outside the circle at a distance of 6 cm from the centre. Construct two tangents from P to the given circle. Measure and write down the length of one tangent.

* All these questions are completely covered in chapter either as solved examples or in chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

Directions (Q. Nos. 1-4) *Given, a $\triangle ABC$ with $BC = 6.5$ cm, $AB = 5.5$ cm and $AC = 5$ cm as shown below*



1. Construct a incircle of a $\triangle ABC$.

2. The radius of incircle is

(a) 1.5 cm (b) 2 cm (c) 3 cm (d) 4.5 cm

3. The coordinates of centre of circle is

(a) (1.25, 2.5) (b) (1.75, 3.25) (c) (3.25, 4.5) (d) (3.5, 2.5)

4. The distance from centre of the circle to vertex B of $\triangle ABC$ is

(a) 1.50 units (b) 3.80 units (c) 4.25 units (d) 5 units

5. Draw a circle of radius 3 cm. Take two points P and Q on one of its extended diameter, each at a distance of 7 cm from its centre. Draw tangents to the circle from these two points P and Q.

6. Let ABC be a right angled triangle, in which $AB = 6$ cm, $BC = 8$ cm, $\angle B = 90^\circ$ and BD is the perpendicular from B on AC. The circle through B, C and D is drawn. Construct the tangents from A to this circle.

7. Draw a circle with the help of circular solid ring construct a pair of tangents from a point P outside the circle.

8. Construct a tangent to a circle of radius 4 cm from a point on the concentric circle of radius 6 cm and measure its length.

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 412.

CHAPTER

17

Surface Area and Volume

In earlier classes, we have learnt to find the surface area and volume of some solid figures like; cuboid and cube. In this chapter, we will learn to find the surface area and volume of some new solid figures, like cylinder, cone and sphere.

Chapter Objectives

- Surface Area and Volume of a Cylinder
- Surface Area and Volume of a Cone
- Surface Area and Volume of a Sphere
- Conversion of Solid and Combination of Two Solids

Topic 1

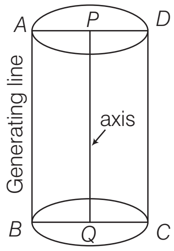
Surface Area and Volume of a Cylinder

Right Circular Cylinder

A right circular cylinder has two plane congruent circular ends parallel to each other and joined by a curved surface at right angles to both the ends.

e.g. A round glass jar, a circular oil container, a gas cylinder, a circular pillar of a hall, a piece of a circular pipe, etc., are all right circular cylinder.

In other words, a solid generated by the revolution of a rectangle about one of its sides which is kept fixed, is called a **right circular cylinder**. The fixed side of the rectangle about which it rotates, is called the **axis of the cylinder**.

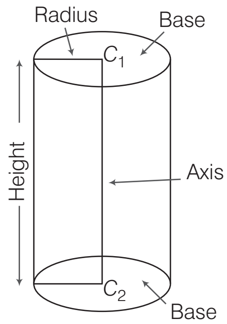


In the figure, line AB when moving on the circumference of the circle BC generates a **right circular cylinder**. Here, the line AB is the generating line and PQ joining the centres of the circular base and top is called the **axis**. Also, circle BC is the **base** of the cylinder.

Important Terms Related to Right Circular Cylinder

There are various terms related to right circular cylinder, which are as follow

(i) **Base** Each of the circular ends on which the cylinder rests, is called its base.



(ii) **Axis** The line segment joining the centres of two circular bases is called the axis of cylinder and the axis of cylinder is always perpendicular to the bases of it.

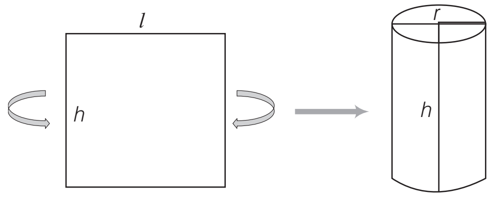
(iii) **Radius** The radius of circular bases is called the radius of cylinder (radius of both ends will be always same).

(iv) **Height** The length of the axis of the cylinder is called the height of the cylinder.

(v) **Lateral surface** The curved surface joining the two bases of a right circular cylinder is called its lateral surface.

Surface Area of a Right Circular Cylinder

We know that right circular cylinder is generated by revolution of a rectangular sheet about one of its side, so the area of the rectangular sheet gives us the curved surface area of the cylinder and length of the rectangular sheet is equal to the circumference of the circular base.



$\therefore$ Curved surface area of the cylinder
= Area of rectangular sheet
= Length of rectangular sheet $\times$ Height
= Circumference of the base of the cylinder $\times$ Height
[$\because$ length of rectangular sheet = circumference of the base]

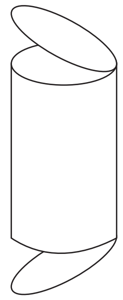
$\therefore$ $\boxed{\textbf{Curved surface area of cylinder} = 2\pi rh \textbf{ sq units}}$

where, circumference of the base of the cylinder $= 2\pi r$, r is the radius of the base and h is the height of the cylinder.

Total Surface Area of a Right Circular Cylinder

For total surface area of a cylinder, we also take area of both circular bases with curved surface area of the cylinder. So,

Total surface area of cylinder
= Curved surface area of cylinder
+ Area of both circular ends



$$= 2\pi rh + \pi r^2 + \pi r^2 = 2\pi rh + 2\pi r^2$$

$\therefore$ $\boxed{\textbf{TSA of cylinder} = 2\pi r\,(h + r) \textbf{ sq units}}$

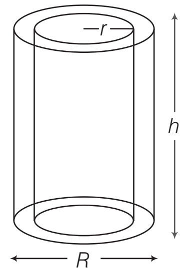
where, r = radius of base and h = height of cylinder.

Surface Area of a Hollow Cylinder

A solid bounded by two coaxial cylinders of the same height and different radii is called a hollow cylinder.

Let a hollow cylinder whose external and internal radii are R and r, respectively with h as height. Then,

$\boxed{\textbf{Surface area of each base} = \pi(R^2 - r^2) \textbf{ sq units}}$



Curved surface area = External surface area
+ Internal surface area

$$= 2\pi Rh + 2\pi rh$$
$$= 2\pi h(R + r) \text{ sq units}$$

$\boxed{\textbf{Curved surface area of hollow cylinder} = 2\pi h(R + r) \textbf{ sq units}}$

Total surface area = Curved surface area
+ 2(Surface area of each base)

$$= 2\pi h(R + r) + 2\pi(R^2 - r^2)$$

$\therefore$ $\boxed{\textbf{TSA of hollow cylinder} = 2\pi(R + r)(h + R - r) \textbf{ sq units}}$

Example 1. The ratio between the curved surface area and the total surface area of a right circular cylinder is 1 : 3. Find the ratio between the height and radius of the cylinder.

Sol. Let radius of the cylinder = r and height = h.

Then, curved surface area = $2\pi rh$

and total surface area = $2\pi r(h+r)$

According to the question,

Curved surface area : Total surface area = 1 : 3

$$\therefore \quad \frac{2\pi rh}{2\pi r(h+r)} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{h}{h+r} = \frac{1}{3}$$

$$\Rightarrow \quad 3h = h + r \Rightarrow 2h = r$$

$$\Rightarrow \quad \frac{h}{r} = \frac{1}{2} \quad \text{or} \quad h : r = 1 : 2$$

Hence, the required ratio between height and radius of the cylinder is 1 : 2.

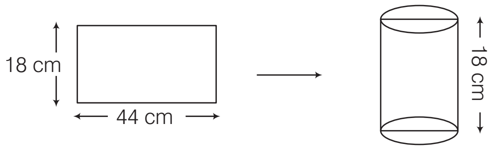
Example 2. A rectangular sheet of paper 44 cm × 18 cm is rolled along its length and a cylinder is formed. Find the radius of the cylinder.

Sol. Given, measures of a rectangular sheet = 44 cm × 18 cm

Here, length = 44 cm and breadth = 18 cm

Let radius of base of cylinder = r

On revolving it about its length, a cylinder is formed.



$\therefore$ Circumference of base of cylinder = Length of sheet

$$\Rightarrow \quad 2\pi r = 44$$

$$\Rightarrow \quad 2 \times \frac{22}{7} \times r = 44$$

$$\Rightarrow \quad r = \frac{44 \times 7}{2 \times 22} = 7$$

Hence, the radius of the cylinder is 7 cm.

Example 3. A cylindrical roller 2.5 m in length, 1.75 m in radius when rolled on a road was found to cover an area of 5500 m^2. How many revolutions did it make?

Sol. Given, length of cylindrical roller = 2.5 m = $\frac{5}{2}$ m,

and radius = 1.75 m = $\frac{7}{4}$ m

$\therefore$ Area covered by the roller in one revolution

= Curved surface area of roller

$$= 2\pi rh = \left(2 \times \frac{22}{7} \times \frac{7}{4} \times \frac{5}{2}\right)$$

$$= \frac{55}{2}\,m^2$$

Let the number of revolutions made by the roller be n, then

Area covered in one revolution × n = Total area covered

$$\Rightarrow \quad \frac{55}{2} \times n = 5500 \Rightarrow n = 200$$

Hence, the number of revolutions made by the roller are 200.

Example 4. An iron pipe 20 cm long has exterior diameter equal to 25 cm. If the thickness of the pipe is 1 cm, then find the whole surface area of the pipe.

Sol. Given, length of the pipe (h) = 20 cm

and exterior diameter = 25 cm

Now, exterior radius (R) = $\frac{25}{2}$ = 12.5 cm

Thickness of the pipe = 1 cm

$\therefore$ Internal radius (r) = External radius − Thickness

= 12.5 − 1 = 11.5 cm

Now, whole surface area of the pipe

= (External curved surface) + (Internal curved surface) + 2 (Area of the base of the pipe)

$= 2\pi Rh + 2\pi rh + 2\pi(R^2 - r^2)$

$= 2\pi h(R+r) + 2\pi(R-r)(R+r)$ $[\because a^2 - b^2 = (a-b)(a+b)]$

$= 2\pi(R+r)(h+R-r)$

$= 2 \times \frac{22}{7} \times (12.5 + 11.5) \times (20 + 12.5 - 11.5)$

$= 2 \times \frac{22}{7} \times 24 \times 21 = 3168\ cm^2$

Example 5. The cost of painting the outer curved surface of a cylinder at ₹ 1.50 per cm^2 is ₹ 660. If the height of the cylinder is 2 m. Then, find the curved surface of the base of cylinder.

Sol. Given, height of the cylinder = 2 m = 200 cm

[$\because$ 1 m = 100 cm]

Total cost of painting = ₹ 660

and rate of painting the outer curved surface area of the cylinder = ₹ 1.50 per cm^2

$\because$ Total cost of painting = Curved surface area of cylinder × Rate of painting the curved surface area

$\Rightarrow$ 660 = Curved surface area × 1.50

$\Rightarrow$ Curved surface area = $\frac{660}{1.50}$

$$\Rightarrow \quad 2\pi rh = 440$$

[$\because$ curved surface area of cylinder = $2\pi rh$]

$$\Rightarrow \quad 2 \times \frac{22}{7} \times r \times 200 = 440$$

$$\Rightarrow \quad r = \frac{440 \times 7}{2 \times 22 \times 200} = \frac{35}{100}\ \text{cm}$$

$\therefore$ Curved surface of the base of cylinder

= Circumference of the base of cylinder

$= 2\pi r = 2 \times \frac{22}{7} \times \frac{35}{100} = 2.2$ cm

Volume of a Right Circular Cylinder

We know that a right circular cylinder can be built up using circles of the same radius. Let h be the height and r be the radius of the right circular cylinder, then

Volume of the cylinder

= Measure of the space occupied by the cylinder

= Area of circular base × Height

$\therefore$ $\boxed{\textbf{Volume of the cylinder} = \pi r^2 h \textbf{ cu units}}$

Volume of a Hollow Cylinder

For volume of the material used for making a hollow cylinder will be obtain by subtracting the interior volume from exterior volume, i.e.

Volume of the material = Exterior volume − Interior volume

$$= \pi R^2 h - \pi r^2 h$$

$\therefore$ $\boxed{\textbf{Volume of hollow cylinder} = \pi(R^2 - r^2)h}$

where, R and r are exterior and interior radii respectively, h is the height of that hollow cylinder.

Example 6. The circumference of the base of a cylindrical vessel is 132 cm and its height is 25 cm. Find the

(i) radius of the cylinder.

(ii) volume of cylinder. $\left[\text{use } \pi = \dfrac{22}{7}\right]$ ***[2018]***

Sol. Given, circumference of base of a cylindrical vessel = 132 cm and height = 25 cm

(i) We know that base of a cylinder is in the shape of a circle.

$\therefore$ Circumference of base of a cylinder $= 2\pi r$

$\Rightarrow\ 2\pi r = 132 \Rightarrow 2\times\dfrac{22}{7}\times r = 132 \Rightarrow r = 3\times 7 \Rightarrow r = 21\text{ cm}$

(ii) Volume of cylinder $= \pi r^2 h = \dfrac{22}{7}\times 21\times 21\times 25$

$= 66\times 21\times 25 = 34650\text{ cm}^3$

Example 7. A cylindrical bucket of diameter 28 cm and height 12 cm is full of water. The water is emptied into a rectangular tub of length 66 cm and breadth 28 cm, then find the height of water rises in the tub.

Sol. Given, diameter of bucket = 28 cm

$\therefore$ Radius $(r) = \dfrac{28}{2} = 14\text{ cm}$

and height of water in bucket $(h) = 12$ cm

Volume of water in the bucket $= \pi r^2 h$

$= \dfrac{22}{7}\times 14\times 14\times 12 = 7392$ cu cm

Let h be the height of water rises in the tub.

Then, volume of water in the tub $= (66\times 28\times h)$ cu cm

According to the question,

Volume of water in bucket = Volume of water in tub

$\Rightarrow\ 7392 = 66\times 28\times h \Rightarrow h = \dfrac{7392}{66\times 28} = 4\text{ cm}$

Hence, the water rises to a height of 4 cm in the tub.

Example 8. 30 circular plates, each of radius 14 cm and thickness 3 cm are placed one above the another to form a cylindrical solid. Find

(i) the total surface area.

(ii) the volume of the cylinder so formed.

Sol. $\because$ Height of the cylinder formed, h

= Number of plates × Thickness of one plate

$= (30\times 3)$ cm = 90 cm

Let radius of each circular plate, $r = 14$ cm

(i) Total surface area of cylinder $= 2\pi r(h + r)$

$= 2\times\dfrac{22}{7}\times 14\times(90 + 14) = (88\times 104)\text{ cm}^2 = 9152\text{ cm}^2$

(ii) Volume of the cylinder $= \pi r^2 h = \dfrac{22}{7}\times(14)^2\times 90$

$= (22\times 28\times 90) = 55440\text{ cm}^3$

Example 9. The difference between outside and inside surfaces of a cylindrical metallic pipe 14 cm long is 44 cm^2. If the pipe is made of 88 cm^3 of metal. Find outer and inner radii of the pipe.

Sol. Let R cm and r cm be the external and internal radii of the metallic pipe, respectively.

Given, height/length of the pipe $(h) = 14$ cm

According to the question,

Outside surface area − Inside surface area = 44 cm^2

$\Rightarrow\ 2\pi Rh - 2\pi rh = 44$ [given]

$\Rightarrow\ 2\times\dfrac{22}{7}(R - r)\times 14 = 44 \Rightarrow R - r = \dfrac{1}{2}$ ...(i)

$\because$ Volume of the metal used for making the pipe = 88 cm^3

$\pi(R^2 - r^2)h = 88$

$\Rightarrow\ \dfrac{22}{7}\times(R + r)(R - r)\times 14 = 88$ $[\because a^2 - b^2 = (a + b)(a - b)]$

$\Rightarrow\ \dfrac{22}{7}\times(R + r)\times\dfrac{1}{2}\times 14 = 88$ [from Eq. (i)]

$\Rightarrow\ R + r = \dfrac{88}{22} \Rightarrow R + r = 4$ ...(ii)

On adding Eqs. (i) and (ii), we get

$R - r + R + r = \dfrac{1}{2} + 4 = 0.5 + 4 = 4.5$

$\Rightarrow\ 2R = 4.5 \Rightarrow R = \dfrac{4.5}{2} = 2.25\text{ cm}$

On putting $R = 2.25$ cm in Eq. (ii), we get

$r = 4 - 2.25 = 1.75\text{ cm}$

Hence, the outer and inner radii of the pipe are 2.25 cm and 1.75 cm, respectively.

Topic Exercise 1

1. If radius and height of the cylinder are 2 cm and 14 cm, respectively. Then, find
   (i) its lateral surface area (ii) its total surface area
   (iii) its volume
2. If the curved surface area of a cylinder is 1760 cm^2 and its base radius is 14 cm. Then, find its height.
3. The curved surface area of a right circular cylinder is 660 cm^2 and base radius is 5 cm. Find its height.

4. The heights of two cylinders are in the ratio 5 : 3 and their radii are in the ratio 2 : 3. Then, find the ratio of their curved surface area.

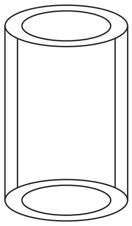
5. The area of the curved surface of a right circular cylinder is 4400 cm^2 and the circumference of its base is 110 cm. Find the height of the cylinder.

6. A cylinder is formed by rolling a sheet whose dimension is $8.8 \text{ cm} \times 2.8$ cm. Find the radius of the cylinder.

7. A roadroller (in the shape of a cylinder) has a diameter 0.7 m and its width is 1.2 m. Find the least number of revolutions that the roller must make in order to level a playground of size 120 m by 44 m.

8. An iron pipe 10 cm long has exterior diameter equal to 15 cm. If the thickness of the pipe is 2 cm, find the whole surface of the pipe.

9. The adjoining figure shows a metal pipe 77 cm long. The inner diameter of a cross-section is 4 cm and the outer one is 4.4 cm. Find its



(i) inner curved surface area.
(ii) outer curved surface area.
(iii) total surface area.

10. If the diameter of the base and height of a cylinder are 6 cm and 14 cm, respectively. Then, find its volume.

11. A rectangular paper 11 cm by 8 cm can be exactly wrapped to cover the curved surface of a cylinder of height 8 cm. Find the volume of the cylinder.

12. The internal and external diameters of steel pipe of length 140 cm are 8 cm and 10 cm, respectively. Then, find the volume of steel.

13. In a cylinder, if radius is halved and height is doubled, then find its volume.

14. Calculate the volume of a right circular cylinder with base radius 14 cm and height 14 cm.

15. The radius and the height of a cylinder are in the ratio 5 : 7 and its volume is 550 cm^3. Find its radius.

16. A lead pencil consists of a cylinder of wood with a solid cylinder of graphite filled in the interior. The diameter of the pencil is 7 mm and the diameter of the graphite is 1 mm. If the length of the pencil is 14 cm, find the volume of the wood and that of the graphite.

Hints and Answers

1. (i) **Hint** Lateral surface area of cylinder $= 2\pi rh$
Ans. 176 cm^2
(ii) **Hint** Total surface area of cylinder $= 2\pi r(r+h)$
Ans. 201.14 cm^2
(iii) **Hint** Volume of cylinder $= \pi r^2 h$. **Ans.** 176 cm^3

2. **Hint** Curved surface area $= 2\pi rh = 1760 \Rightarrow h = \dfrac{1760}{2\pi r}$
Ans. 20 cm

3. Do same as Q. 2. **Ans.** 21 cm

4. **Hint** Curved surface area $= 2\pi rh$
$\therefore$ Required ratio $= \dfrac{2\pi \cdot 2r \cdot 5h}{2\pi \cdot 3r \cdot 3h}$ **Ans.** 10 : 9

5. **Hint** Circumference of base, $2\pi r = 110$
and curved surface area, $2\pi rh = 4400$
$\Rightarrow \quad 110 \times h = 4400 \Rightarrow h = 40$ cm **Ans.** 40 cm

6. Do same as Example 2. **Ans.** 1.4 cm

7. Do same as Example 3. **Ans.** 2000

8. Do same as Example 4. **Ans.** 980.57 cm^2

9. **Hint**
(i) Inner curved surface area $= 2\pi r_1 h \quad [\because r_1 = 2 \text{ cm}]$
Ans. 968 cm^2
(ii) Outer curved surface area $= 2\pi r_2 h \; [\because r_2 = 2.2 \text{ cm}]$
Ans. 1064.8 cm^2
(iii) Total surface area $= 2\pi(r_1 + r_2)\,[h + r_2 - r_1]$
Ans. 2038.08 cm^2

10. Do same as Example 6. **Ans.** 396 cm^3

11. **Hint** Here, height $(h) = 8$ cm
$\therefore$ Circumference of base = Length of sheet $\Rightarrow 2\pi r = 11$
$\Rightarrow \quad r = \dfrac{11 \times 7}{2 \times 22} = \dfrac{7}{4}$
Now, volume $= \pi r^2 h = \dfrac{22}{7} \times \dfrac{7}{4} \times \dfrac{7}{4} \times 8$ **Ans.** 77 cm^3

12. **Hint** Volume of steel = Volume of hollow cylinder
$= \dfrac{22}{7}[(5)^2 - (4)^2] \times 140$ **Ans.** 3960 cm^3

13. **Hint** $\because$ Volume of cylinder $= \pi r^2 h$.
$\therefore$ Volume of new cylinder $= \pi\left(\dfrac{r}{2}\right)^2 2h = \pi \dfrac{r^2}{2} h$
$= \dfrac{1}{2} \times$ Volume of cylinder. **Ans.** Halved

14. **Hint** Volume $= \pi r^2 h$ **Ans.** 8624 cm^3

15. **Hint** Let radius $= 5x$, height $= 7x$.
Then, $\pi r^2 h = 550 \Rightarrow \dfrac{22}{7}(5x)^2(7x) = 550$
$\Rightarrow \quad x^3 = 1 \Rightarrow x = 1$ **Ans.** 5 cm

16. **Hint** Volume of hollow wood $= \pi h(R^2 - r^2)$ and volume of graphite $= \pi r^2 h$. **Ans.** 5.28 cm^3, 0.11 cm^3

Topic 2

Surface Area and Volume of a Cone

Right Circular Cone

If a right angled triangle is revolved about one of the two sides forming a right angle, keeping the other sides fixed in position, then the solid so obtained by revolving the line segments, is called a **right circular cone**.

In our daily life, we see many objects having conical shape; an ice–cream, conical tent, birthday cap, conical vessel etc.

Important Terms Related to Right Circular Cone

There are various terms related to right circular cone, which are as follow

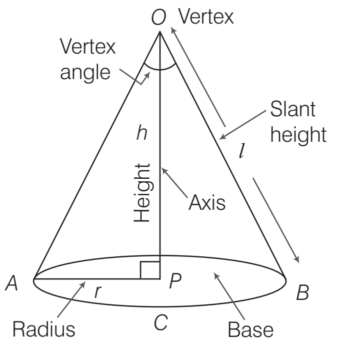
(i) **Vertex** O is called the vertex of the cone.

(ii) **Axis** The fixed line OP is called the axis of the cone.

(iii) **Base** The circle ABC is called the base of the cone.

(iv) **Slant height** The length of the line segment joining the vertex O to any point on the circular edge of the base, is called the slant height of the cone. Here, $OA = OB$ are the slant heights of the cone.

(v) **Vertical angle** $\angle AOB$ is called the vertex angle of the cone.

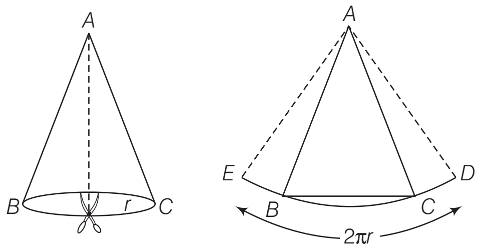


(vi) **Height** The length of the line segment joining the vertex to the centre of the base, is called the height of the cone. Here, OP is the height of the cone.

(vii) **Radius** The radius AP of the base circle is called the radius of the cone.

Curved Surface Area of a Cone

When a cone is cut along by any slant height and spread out on plane surface, it forms a shape of sector of a circle. The area of this sector is called **curved surface area of the cone.**



$\therefore$ $\boxed{\textbf{Curved surface area of a cone} = \pi r l \textbf{ sq units}}$

where, r = radius of the base of cone

and l = slant height of the cone, in which

$l = \sqrt{h^2 + r^2}$, here h = vertical height of the cone.

Surface Area of a Cone

If the base of the cone is to be closed, then a circular piece of paper of radius r is also required.

Surface (Total surface) area of a cone

$= $ Curved surface area of a cone + Area of the base

$= \pi r l + \pi r^2$

$\therefore$ $\boxed{\textbf{SA (TSA) of a cone} = \pi r (l + r) \textbf{ sq units}}$

Volume of a Cone

The quantity of liquid contained in a cone, is called the volume of a cone.

$\therefore$ $\boxed{\textbf{Volume of a cone} = \frac{1}{3}\pi r^2 h \textbf{ cu units}}$

where, r is the radius and h is the height of a cone.

Example 1. A solid cone of radius 5 cm and height 7 cm is given. Calculate

(i) the curved surface area of a cone.

(ii) the volume of a cone. [Take $\pi = 3.14$ and $\sqrt{74} = 8.6$]

Sol. Given, radius of a cone, $r = 5$ cm

and height of a cone, $h = 7$ cm

(i) Curved surface area of a cone $= \pi r l$

$= 3.14 \times 5 \times \sqrt{5^2 + 7^2}$ $\quad [\because l = \sqrt{r^2 + h^2}]$

$= 3.14 \times 5 \times \sqrt{25 + 49} = 3.14 \times 5 \times \sqrt{74}$

$= 15.7 \times 8.6 = 135.02 \text{ cm}^2$

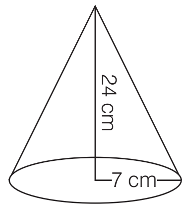
(ii) Volume of a cone $= \frac{1}{3}\pi r^2 h = \frac{3.14}{3} \times (5)^2 \times 7$

$= \frac{549.5}{3} = 183.17 \text{ cm}^3$

Example 2. A joker's cap is in the form of a right circular cone of base radius 7 cm and height 24 cm. Find the area of the sheet required to make 10 such caps.

Sol. Given, radius of a cone, $r = 7$ cm

and height of a cone, $h = 24$ cm



We know that, $l = \sqrt{h^2 + r^2} = \sqrt{(24)^2 + (7)^2}$

$= \sqrt{576 + 49} = \sqrt{625} = 25$ cm

$\therefore$ Curved surface area of a joker's cap

= Curved surface area of a cone $= \pi rl$

$= \frac{22}{7} \times 7 \times 25 = 22 \times 25 = 550 \text{ cm}^2$

$\because$ Sheet required to make 1 cap $= 550 \text{ cm}^2$

$\therefore$ Sheet required to make 10 such caps $= 550 \times 10$

$= 5500 \text{ cm}^2$

Example 3. If the height of a cone is doubled, then find its increased volume in percentage.

Sol. Let r and h be the radius and height of a cone, respectively.

Then, original volume of a cone $= \frac{1}{3}\pi r^2 h = V$ (say)

New radius $= r$ and new height $= 2h$

Then, new volume of a cone

$= \frac{1}{3}\pi r^2 \times 2h = 2 \times \left(\frac{1}{3}\pi r^2 h\right) = 2V$

$\therefore$ Increase in volume $= 2V - V = V$

Increase percentage of volume

$= \left(\frac{\text{Increase in volume}}{\text{Original volume}} \times 100\right)\%$

$= \left(\frac{V}{V} \times 100\right)\% = 100\%$

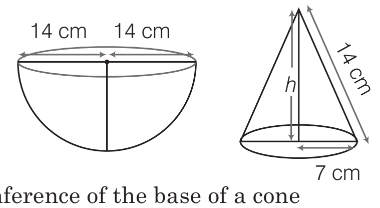
Example 4. A semi-circular sheet of metal of diameter 28 cm is bent to form an open conical cup. Find the capacity of the cup. [Take $\pi = 22/7$]

Sol. Given, diameter of a semi-circular sheet = 28 cm

$\therefore$ Radius of a semi-circular sheet, $r = \frac{28}{2} = 14$ cm

Let the radius of a conical cup be R.

Since, a semi-circular sheet of metal is bent to form an open conical cup.



$\therefore$ Circumference of the base of a cone

= Circumference of a semi-circle

$\Rightarrow \quad 2\pi R = \pi r \Rightarrow 2\pi R = \pi \times 14 \Rightarrow R = 7$ cm

Now, $\quad h = \sqrt{l^2 - R^2} = \sqrt{14^2 - 7^2} \quad [\because l^2 = h^2 + R^2]$

$= \sqrt{196 - 49} = \sqrt{147} = 12.12$ cm

$\therefore$ Volume or capacity of conical cup

$= \frac{1}{3}\pi R^2 h = \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 12.12 = 622.16 \text{ cm}^3$

Hence, the capacity of an open conical cup is 622.16 cm^3.

Example 5. Monika has a piece of canvas whose area is 551 m^2. She uses it to made conical tent with a base radius of 5 m. Assuming that all the stitching margins and the wastage incurred while cutting, amounts to approximately 1 m^2, then find the volume of the tent that can be made with it.

Sol. Let l m be the slant height and h m be the height of cone.

Here, base radius of cone = 5 m,

Total area of canvas $= 551 \text{ m}^2$ and wastage $= 1 \text{ m}^2$.

$\therefore$ The actual area of the conical tent

$= (551 - 1) \text{ m}^2 = 550 \text{ m}^2$

Now, area of tent = curved surface area of cone $= \pi rl$

$\Rightarrow \quad \frac{22}{7} \times 7 \times l = 550 \Rightarrow l = 25$

Also, $\quad l^2 = h^2 + r^2 \Rightarrow 25^2 = h^2 + 7^2$

$\Rightarrow \quad h^2 = 625 - 49 = 576 \Rightarrow h = 24$

[taking positive square root]

$\therefore$ Volume of the conical tent

$= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times 7^2 \times 24 = (22 \times 7 \times 8) = 1232 \text{ m}^3$

Example 6. A conical tent is 10 m high and the radius of its base is 24 m. Find

(i) the slant height of the tent.

(ii) the cost of the canvas required to make the tent, if the cost of 1 m^2 canvas is ₹ 70.

(iii) the volume of air contained in the conical tent.

Sol. Given, height of a conical tent, $h = 10$ cm

and radius, $r = 24$ m

(i) We know that, $l = \sqrt{r^2 + h^2} = \sqrt{(24)^2 + (10)^2}$

$= \sqrt{576 + 100} = \sqrt{676} = 26$ m

Hence, the slant height of the tent is 26 m.

(ii) Canvas required to make the tent

= Curved surface area of the tent

$= \pi rl = \pi \times 24 \times 26 = 624\pi \text{ m}^2$

$\because$ Cost of 1 m^2 canvas = ₹ 70

$\therefore$ Cost of $624\pi \text{ m}^2$ canvas $= 70 \times 624\pi = 70 \times 624 \times \frac{22}{7}$

$= 10 \times 624 \times 22 =$ ₹ 137280

Hence, the required cost of the canvas is ₹137280.

(iii) Volume of air contained in the conical tent

$= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times (24)^2 \times 10 = 6034.29 \text{ m}^3$

Example 7. The volume of a conical tent is 1232 m^3 and the area of the base floor is 154 m^2. Calculate

(i) radius of the floor. (ii) height of the tent.

(iii) length of the canvas required to cover this conical tent, if its width is 2 m. [Take $\pi = 22/7$] *[2008]*

Sol. (i) Let the radius of conical tent be r.

$\because$ Area of base floor $= 154\,m^2$ [given]

$\Rightarrow \pi r^2 = 154$

$\Rightarrow \frac{22}{7} \times r^2 = 154 \Rightarrow r^2 = \frac{154 \times 7}{22}$

$\Rightarrow r^2 = 49 \Rightarrow r = 7\,m$ [taking positive square root]

(ii) Let the height of conical tent be h.

$\because$ Volume of conical tent $= 1232\,m^3$

$\Rightarrow \frac{1}{3}\pi r^2 h = 1232 \Rightarrow \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times h = 1232$

$\Rightarrow h = \frac{1232 \times 3}{22 \times 7} \Rightarrow h = 24\,m$

(iii) Curved surface area of cone

= Area of the canvas required to cover conical tent

$\Rightarrow \pi r l = \text{Length} \times \text{Width}$

$\Rightarrow \frac{22}{7} \times 7 \times \sqrt{r^2 + h^2} = \text{Length} \times 2$ [$\because$ width = 2 m (given)]

$\Rightarrow 22 \times \sqrt{(7)^2 + (24)^2} = \text{Length} \times 2$

$\therefore$ Length $= 11 \times \sqrt{625} = 11 \times 25 = 275\,m$

Example 8. A cloth having an area of 165 m^2 is shaped in the form of a conical tent of radius 5 m. How many students can sit in the tent if a student, on an average, occupies $\frac{5}{7}\,m^2$ area on the ground?

Sol. Given, radius of the cone, $r = 5\,m$

$\therefore$ Floor area of the tent $= \pi r^2 = \left(\frac{22}{7} \times 5^2\right) = \frac{550}{7}\,m^2$.

Let n be the number of students.

As each student, on an average, occupies $\frac{5}{7}\,m^2$ area.

$\therefore \frac{5}{7} \times n = \frac{550}{7} \Rightarrow n = 110$

Hence, the number of students are 110.

Example 9. The height of a cone is 40 cm. A small cone is cut off at the top by a plane parallel to its base. If its volume be $\frac{1}{64}$ of the volume of the given cone, at what height above the base is the section cut?

Sol. Let OAB be the given cone of height 40 cm and base radius R cm. Let this cone be cut by the plane CND (parallel to the base plane AMB) to obtain the cone OCD with height h cm and the base radius r cm.

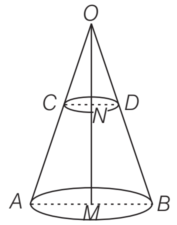
Then, $\Delta OND \sim \Delta OMB$.

$\therefore \frac{ND}{MB} = \frac{ON}{OM} \Rightarrow \frac{r}{R} = \frac{h}{40}$ ...(i)

According to the question,

Volume of cone $OCD = \frac{1}{64} \times$ Volume of cone OAB

$\Rightarrow \frac{1}{3}\pi r^2 h = \frac{1}{64} \times \frac{1}{3}\pi R^2 \times 40$



$\Rightarrow \left(\frac{r}{R}\right)^2 = \frac{5}{8h} \Rightarrow \left(\frac{h}{40}\right)^2 = \frac{5}{8h}$ [using Eq. (i)]

$\Rightarrow h^3 = \frac{40 \times 40 \times 5}{8} = 1000 \Rightarrow h = 10$

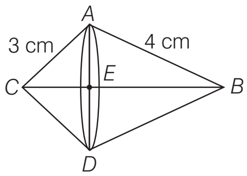
[taking cube root]

Since, the height of the cone OCD is 10 cm.

Here, the section is cut at the height of (40 – 10) cm, i.e. 30 cm from the base.

Example 10. A right angled triangle with sides 3 cm and 4 cm is made to revolve about its hypotenuse. Find the volume of the double cone so formed. [Take $\pi = 22/7$]

Sol. Let ABC be a right angled triangle, right angled at A, having sides $AC = 3\,cm$ and $AB = 4\,cm$.



When we revolve a triangle about hypotenuse BC, then double cone is formed with common base AD.

In right angled ΔBAC,

$BC^2 = AC^2 + AB^2$ [using Pythagoras theorem]

$= 3^2 + 4^2 = 9 + 16 = 25\,cm$

$\Rightarrow BC = 5\,cm$ [taking positive square root]

In ΔACE and ΔBCA,

$\angle CEA = \angle CAB = 90°$

and $\angle ACE = \angle BCA$ [common angles]

$\therefore \Delta ACE \sim \Delta BCA$ [by AA similarity axiom]

$\Rightarrow \frac{CE}{AC} = \frac{AE}{AB} = \frac{AC}{BC} \Rightarrow \frac{CE}{3} = \frac{AE}{4} = \frac{3}{5}$

$\Rightarrow CE = \frac{9}{5}\,cm$ and $AE = \frac{12}{5}\,cm$

Now, $BE = BC - CE = 5 - \frac{9}{5} = \frac{25 - 9}{5} = \frac{16}{5}\,cm$

Now, volume of double cone

= Volume of cone ACD + Volume of cone ABD

$= \frac{1}{3}\pi r^2 h_1 + \frac{1}{3}\pi r^2 h_2 = \frac{1}{3}\pi r^2 (h_1 + h_2)$

$= \frac{1}{3} \times \frac{22}{7} \times \left(\frac{12}{5}\right)^2 \left(\frac{9}{5} + \frac{16}{5}\right)$

$\left[\because r = AE = \frac{12}{5}, h_1 = CE = \frac{9}{5} \text{ and } h_2 = BE = \frac{16}{5}\right]$

$= \frac{1}{3} \times \frac{22}{7} \times \frac{144}{25}\left(\frac{25}{5}\right) = \frac{22 \times 48}{35} = 30.17\,cm^3$

Topic Exercise 2

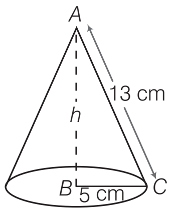
1. If radius and height of a right circular cone be 3 cm and 21 cm, respectively, then find its
 (i) slant height (ii) curved surface area
 (iii) total surface area (iv) volume
2. Find the curved surface area of a right circular cone whose slant height is 10 cm and base radius is 7 cm.
3. Diameter of the base of a cone is 10.5 cm and its slant height is 10 cm. Find
 (i) the curved surface area of a cone.
 (ii) the surface area of a cone.
4. The radius of the base of a cone is 7cm and its vertical height is 15 cm. Find
 (i) the curved surface area of a cone.
 (ii) the volume of a cone.
5. The height of a cone is 15 cm. If its volume is $1570\ \text{cm}^3$, then find the radius of the base.
6. The diameter of the base of a right circular cone is 10 cm and its slant height is 13 cm. Calculate
 (i) height of the cone.
 (ii) curved surface area of the cone.
 (iii) total surface area of the cone.
 (iv) the volume of the cone. [Take $\pi = 3.14$]
7. Find the ratio of the curved surface areas of two cones, if the diameters of their bases are equal and slant heights are in the ratio 3 : 4.
8. The radius and the height of a cone are in the ratio 1 : 3 and its volume is $1078\ \text{cm}^3$. Find its diameter and the lateral surface area.
9. Two right circular cones X and Y are made, X having three times the radius of Y and Y having half the volume of X. Find the ratio of heights of X and Y.
10. A semi-circular thin sheet of metal of radius 7 cm is bent and an open conical cup is made. Find the capacity of the cup.
11. Find the length of canvas 2 m in width is required to make a conical tent 20 m in diameter and 42 m in slant height allowing 10% for folds and the stitching. Also, find the cost of the canvas at the rate of ₹ 80 per metre.
12. A conical military tent is 5 m high and the diameter of the base is 24 m. Find the cost of canvas used in making this tent at the rate of ₹ 14 per sq m.
13. A sector of a circle of radius 8 cm has an angle of 90°. It is rolled up, so that the two bounding radii are joined together to form a cone. Find
 (i) the radius of the cone.
 (ii) the total surface area of the cone.
 (iii) the volume of the cone.
14. A cloth having an area of $165\ \text{m}^2$, is shaped into the form of a conical tent of radius 5 m.
 (i) How many students can sit in the tent, if a student on an average occupies $5/7\ \text{m}^2$ on the ground?
 (ii) Find the volume of the cone. [Take $\pi = 22/7$]
15. A tent in the form of a right circular cone is 3 m high and its base has a diameter of 14 m. If 100 men sleep in it, then find the average number of cubic metres of air space per man.
16. A ΔPQR, where $PQ = 9$ cm, $QR = 12$ cm and $PR = 15$ cm, is rotated about the side QR to trace out a cone. Find the total surface area of the cone.
17. A conical tent is to accommodate 77 persons. Each person must have 16m^3 of air to breathe. Given the radius of the tent as 7 m. Find the height of the tent and also its curved surface area. ***[2017]***

Hints and Answers

1. (i) **Hint** Slant height of a right circular cone,
$$l = \sqrt{h^2 + r^2} \quad \textbf{Ans. } 21.21 \text{ cm}$$
 (ii) **Hint** Curved surface area of a right circular cone $= \pi rl$ **Ans.** $199.98\ \text{cm}^2$
 (iii) **Hint** Total surface area of cone $= \pi r(l + r)$ **Ans.** $228.27\ \text{cm}^2$
 (iv) **Hint** Volume of cone $= \frac{1}{3}\pi r^2 h$. **Ans.** $198\ \text{cm}^3$
2. **Hint** Curved surface area of cone $= \pi rl$. **Ans.** $220\ \text{cm}^2$
3. **Hint** (i) CSA of cone $= \pi rl$ **Ans.** $165\ \text{cm}^2$
 (ii) TSA of cone $= \pi r(l + r)$ **Ans.** $251.625\ \text{cm}^2$
4. Do same as Example 1.
 Ans. (i) $364.1\ \text{cm}^2$ (ii) $770\ \text{cm}^3$
5. **Hint** $\because$ Volume of the cone $= \frac{1}{3}\pi r^2 h$.
$$\Rightarrow \quad 1570 = \frac{1}{3} \times \frac{22}{7}(15) \times r^2$$
$$\Rightarrow \quad 1570 \times 3 \times \frac{7}{22} = 15r^2 \Rightarrow r^2 = 100 \quad \textbf{Ans. } 10 \text{ cm}$$

6. Radius of the cone $= \frac{10}{2} = 5$ cm

(i) **Hint** $h^2 = (13)^2 - (5)^2$ **Ans.** $h = 12$ cm



(ii) **Hint** Curved surface area of the cone $= \pi rl$
Ans. 204.1 cm^2

(iii) **Hint** Total surface area of the cone $= \pi r\,(l + r)$
Ans. 282.6 cm^2

(iv) **Hint** Volume of the cone $= \frac{1}{3}\pi r^2 h$. **Ans.** 314 cm^3

7. **Hint** Let the slant heights of each cone be $3x$ and $4x$ respectively, i.e. $l_1 = 3x$ and $l_2 = 4x$

$\therefore$ Radius of each cone $= \frac{r}{2}$

$\therefore$ Ratio of the curved surface areas $= \dfrac{\pi \times \left(\frac{r}{2}\right) \times 3x}{\pi \times \left(\frac{r}{2}\right) \times 4x}$

Ans. 3 : 4

8. **Hint** Let $r = x$ and $h = 3x$

Volume of cone $= \frac{1}{3}\pi r^2 h$

Now, slant height $= \sqrt{h^2 + r^2}$

Lateral surface area $= \pi rl$

Ans. 14 cm, 48708 cm^2

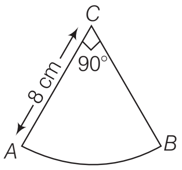
9. **Hint** Let the radius of cone Y be r, then the radius of cone X will be $3r$.
Let the volume of cone X be V, then volume of cone Y will be $\frac{V}{2}$. **Ans.** 2 : 9

10. Do same as Example 4. **Ans.** 77.70 cm^3

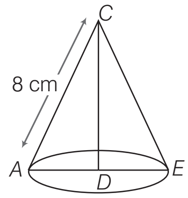
11. Do same as Example 5. **Ans.** 726 m, ₹ 58080

12. Do same as Example 6 (ii). **Ans.** ₹6864

13. A sector of a circle of radius 8 cm having an angle of 90°, is shown below



When we rolled up and joined the bounding radii, a conical shape is formed,



Now, length of an arc of the sector

$$= \frac{1}{4}(2\pi r) = \frac{1}{4}(2\pi \times 8) = 4\pi \text{ cm}$$

(i) **Hint** Let r be the radius of the cone.
Then, circumference of the circular base of the cone = Length of an arc of the sector
$\therefore \quad 2\pi r = 4\pi$ **Ans.** $r = 2$ cm

(ii) **Hint** Total surface area of the cone $= \pi(rl + r^2)$.
Ans. 20π cm^2

(iii) **Hint** In right angled ΔADC,
Volume of a cone $= \frac{1}{3}\pi r^2 h$ **Ans.** $\frac{8\sqrt{15}}{3}\pi$ cm^3

14. (i) **Hint** Area occupied by a student on the ground $= \frac{5}{7}$ m^2

$\therefore$ Area of the base of a conical tent $= \pi r^2$

Now, number of students

$$= \frac{\text{Area of the base of a conical tent}}{\text{Area occupied by a student on the ground}}$$

Ans. 110

(ii) **Hint** Given, area of the cloth in the form of a conical tent = 165 m^2
and radius of the base of a conical tent, $r = 5$ m
Curved surface area of a conical tent
= Area of cloth in the form of a conical tent

$\Rightarrow \quad \pi rl = 165$

Now, height of a conical tent, $h = \sqrt{l^2 - r^2}$

$\therefore$ Volume of a cone (conical tent) $= \frac{1}{3}\pi r^2 h$

Ans. 241.7 m^3

15. Do same as Example 8. **Ans.** 1.54 m^3

16. Do same as Example 10 and also use total surface area of cone $= \pi r\,(r + l)$ sq unit
Ans. 678.82 cm^2

17. **Hint** Volume of tent $= 77 \times 16$ m^3 $= 1232$ m^3 and Curved surface area $= 2\pi rh$
Ans. $h = 24$ m, CSA $= 550$ m^2

Topic 3
Surface Area and Volume of a Sphere

Sphere

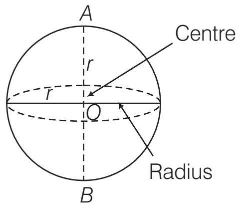
A sphere is three–dimensional figure (solid figure), which is made up of all points in the space, which lie at a constant distance from a fixed point, called the **centre** of the sphere and the constant distance is called its **radius**.

In our daily life, we see many objects having spherical shape; cricket ball, round metallic ball, football etc.

Or

A sphere is a solid generated by the revolution of a circular lamina about any of its diameter is called a solid sphere.

(i) The centre of the circle revolved, is called the **centre** of the sphere.



(ii) The radius of the circle revolved, is called the **radius** of the sphere.

(iii) A line segment through the centre of a sphere and intersecting the end–points on the sphere, is called the **diameter** of the sphere.

Surface Area and Volume of a Sphere

(i) Surface area of a sphere $= 4\pi r^2$ sq units

(ii) Volume of a sphere $= \frac{4}{3}\pi r^3$ cu units

where, r is the radius of the sphere.

Note In sphere, surface area and curved surface area both are same.

Example 1. Find the surface area and volume of a sphere having diameter 12 cm. [Take $\pi = 22/7$]

Sol. Given, diameter of a sphere, $d = 12$ cm

Then, radius of a sphere, $r = \frac{d}{2} = \frac{12}{2} = 6$ cm

Now, surface area of a sphere $= 4\pi r^2 = 4 \times \frac{22}{7} \times (6)^2$

$= \frac{88}{7} \times 36 = \frac{3168}{7} = 452.57 \text{ cm}^2$

and volume of a sphere $= \frac{4}{3}\pi r^3 = \frac{4}{3} \times \frac{22}{7} \times (6)^3$

$= \frac{88}{21} \times 216 = \frac{19008}{21}$

$= 905.14 \text{ cm}^3$

Example 2. The volumes of two spheres are in the ratio 64 : 27. If the sum of their radii is 21 cm, then find their radii.

Sol. Let the radius of one sphere be r_1 and the radius of second sphere be r_2.

Given, ratio of volumes of two spheres $= \frac{64}{27}$

$\therefore \quad \frac{V_1}{V_2} = \frac{64}{27} \Rightarrow \frac{\frac{4}{3}\pi r_1^3}{\frac{4}{3}\pi r_2^3} = \frac{64}{27}$ $\left[\because \text{volume of a sphere} = \frac{4}{3}\pi r^3\right]$

$\Rightarrow \quad \left(\frac{r_1}{r_2}\right)^3 = \frac{64}{27} = \left(\frac{4}{3}\right)^3 \Rightarrow \frac{r_1}{r_2} = \frac{4}{3}$ [taking cube root]

Again, let radius of one sphere, $r_1 = 4x$

and radius of second sphere, $r_2 = 3x$

Given, sum of radii = 21

$\therefore \quad 4x + 3x = 21 \Rightarrow 7x = 21 \Rightarrow x = 3$

Hence, the radius of first sphere $= 4x = 4 \times 3 = 12$ cm

and the radius of second sphere $= 3x = 3 \times 3 = 9$ cm

Example 3. The radius of a sphere is increased by 10%. Prove that the volume will be increased by 33.1%.

Sol. Let r (units) be the radius of the given sphere, then

Volume of sphere $= \frac{4}{3}\pi r^3$

New radius of sphere $= r + 10\%$ of $r = r + \frac{10}{100}r = \frac{11}{10}r$

Volume of new sphere $= \frac{4}{3}\pi\left(\frac{11}{10}r\right)^3 = \frac{4}{3}\pi \times \frac{1331}{1000}r^3$

$\therefore$ Increase in volume $= \frac{4}{3}\pi \times \frac{1331}{1000}r^3 - \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi r^3 (1.331 - 1) = \frac{4}{3}\pi r^3 \times 0.331$

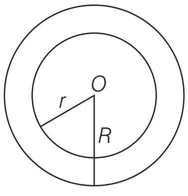
$\therefore$ Percentage increase in volume

$= \left(\frac{\frac{4}{3}\pi r^3 \times 0.331}{\frac{4}{3}\pi r^3} \times 100\right)\% = 33.1\%$

Spherical Shell (Hollow Sphere)

The solid enclosed between two concentric spheres is called a spherical shell.

Let r and R be the radii of inner and outer spheres. Then,



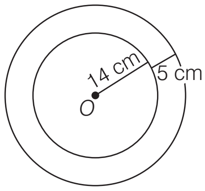
(i) Surface area of a spherical shell

$$= 4\pi (R^2 + r^2) \text{ sq units}$$

(ii) Volume of a spherical shell $= \frac{4}{3}\pi (R^3 - r^3)$ cu units

Example 4. The inner radius and thickness of a hollow spherical shell are 14 cm and 5 cm, respectively. Find the surface area of a spherical shell. [Take $\pi = 3.14$]

Sol. Given, radius of the inner sphere, $r = 14$ cm



Then, radius of the outer sphere,

$R = r + 5 = 14 + 5 = 19$ cm

$\therefore$ Surface area of a spherical shell,

$$S = 4\pi (R^2 + r^2) = 4 \times 3.14 [(19)^2 + (14)^2]$$
$$= 12.56 (361 + 196)$$
$$= 12.56 \times 557 = 6995.92 \text{ cm}^2$$

Example 5. The outer and inner diameters of a spherical shell are 27 cm and 21 cm, respectively. Find the volume of spherical shell. [Take $\pi = 3.14$]

Sol. Given, outer diameter, $D = 27$ cm

and inner diameter, $d = 21$ cm

Then, outer radius, $R = \frac{D}{2} = \frac{27}{2} = 13.5$ cm

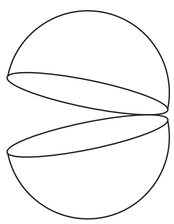
and inner radius, $r = \frac{d}{2} = \frac{21}{2} = 10.5$ cm

$\therefore$ Volume of a spherical shell $= \frac{4}{3}\pi (R^3 - r^3)$

$$= \frac{4}{3}\pi [(13.5)^3 - (10.5)^3]$$
$$= \frac{4}{3} \times 3.14 (2460.38 - 1157.63)$$
$$= \frac{12.56}{3} \times (1302.75) = 4.19 \times 1302.75 = 5458.52 \text{ cm}^3$$

Hemisphere

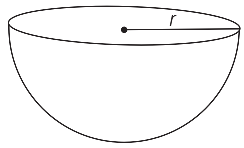
A plane through the centre of a sphere divides the sphere into two equal parts, each part is called a **hemisphere**.



Let r be the radius of hemisphere. Then,

(i) Curved surface area of hemisphere $= 2\pi r^2$ sq units

(ii) Total surface area of hemisphere $= 3\pi r^2$ sq units



(iii) Volume of hemisphere $= \frac{2}{3}\pi r^3$ cu units

Note In hemisphere, surface area and curved surface area both area different.

Example 6. Find the curved and total surface area of a hemisphere of radius 10 cm. [Take $\pi = 3.14$]

Sol. Given, radius of a hemisphere, $r = 10$ cm

Then, curved surface area of a hemisphere

$$= 2\pi r^2 = 2 \times 3.14 \times (10)^2 = 6.28 \times 100 = 628 \text{ cm}^2$$

and total surface area of a hemisphere $= 3\pi r^2$

$$= 3 \times 3.14 \times (10)^2 = 9.42 \times 100 = 942 \text{ cm}^2$$

Example 7. Find the volume of a hemisphere, whose diameter is 14 cm. [Take $\pi = 22/7$]

Sol. Given, diameter of a hemisphere, $d = 14$ cm

Then, radius of a hemisphere, $r = \frac{d}{2} = \frac{14}{2} = 7$ cm

Now, volume of a hemisphere $= \frac{2}{3}\pi r^3$

$$= \frac{2}{3} \times \frac{22}{7} \times (7)^3 = \frac{44}{3} \times 7^2$$
$$= \frac{2156}{3} = 718.67 \text{ cm}^3$$

Example 8. How many litres of milk can a hemispherical bowl of diameter 10.5 cm hold? [Take $\pi = 22/7$]

Sol. Given, diameter of hemispherical bowl, $d = 10.5$ cm

Then, radius, $r = \frac{d}{2} = \frac{10.5}{2} = 5.25$ cm

Now, volume of a hemisphere $= \frac{2}{3}\pi r^3$

$$= \frac{2}{3} \times \frac{22}{7} \times 5.25 \times 5.25 \times 5.25 = \frac{6366.94}{21} = 303.1876 \text{ cm}^3$$

$\therefore$ Quantity of milk that hemispherical bowl can hold

$$= \frac{303.1876}{1000} \text{ L} \qquad \left[\because 1 \text{ cm}^3 = \frac{1}{1000} \text{ L}\right]$$
$$= 0.303 \text{ L (approx.)}$$

Example 9. A hemispherical bowl made of brass has inner diameter 10.5 cm. Find the cost of tin plating it on the inside at the rate of ₹ 16 per 100 cm^2.

Sol. Given, inner diameter, $d = 10.5$ cm

Then, inner radius, $r = \frac{10.5}{2} = 5.25$ cm

Curved surface area of hemispherical bowl of inner side

$= 2\pi r^2 = 2 \times \frac{22}{7} \times (5.25)^2 = 2 \times \frac{22}{7} \times 5.25 \times 5.25$

$= 44 \times 0.75 \times 5.25 = 173.25 \text{ cm}^2$

Since, the cost of tin plating on inside for 100 cm^2 is ₹ 16.

$\therefore$ Cost of tin plating on the inside for 173.25 cm^2

$= \frac{16 \times 173.25}{100} =$ ₹ 27.72

Example 10. A dome of a building is in the form of a hemisphere. From inside, it was white washed at the cost of ₹ 498.96. If the cost of white-washing is ₹ 2 per sq m, then find

(i) inside surface area of the dome.

(ii) volume of the air inside the dome. [Take $\pi = 22/7$]

Sol. Given, cost of white wash = ₹ 498.96

and cost of white-washing per square metre = ₹ 2

(i) Surface area of the dome

$= \frac{\text{Cost of white wash}}{\text{Cost of white-washing per square metre}}$

$= \frac{498.96}{2} = 249.48$ sq m

Let r be the radius of the dome.

Then, surface area $= 249.48$ sq m

$\Rightarrow \quad 2\pi r^2 = 249.48$

$\Rightarrow 2 \times \frac{22}{7} \times r^2 = 249.48 \Rightarrow r^2 = \frac{249.48 \times 7}{44} \Rightarrow r^2 = 39.69$

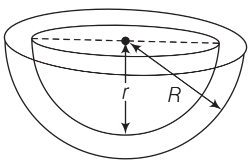
$\therefore \quad r = 6.3$ m [taking positive square root]

(ii) Volume of the air inside the dome $= \frac{2}{3}\pi r^3$

$= \frac{2}{3} \times \frac{22}{7} \times (6.3)^3 = 44 \times 2.1 \times 0.9 \times 6.3 = 523.908 \text{ m}^3$

Hemispherical Shell (Hollow Hemisphere)

The solid enclosed between two concentric hemispheres is called a hemispherical shell.



Let r and R be the radii of the inner and outer hemispheres. Then,

(i) Thickness of shell $= (R - r)$ units

(ii) Area of base $= \pi(R^2 - r^2)$ sq units

(iii) External curved surface area $= 2\pi R^2$ sq units

(iv) Internal curved surface area $= 2\pi r^2$ sq units

(v) Curved surface area of hemispherical shell

$= 2\pi(R^2 + r^2)$ sq units

(vi) Total surface area $= 2\pi R^2 + 2\pi r^2 + \pi(R^2 - r^2)$

$= 3\pi R^2 + \pi r^2 = \pi(3R^2 + r^2)$ sq units

(vii) Volume of a hemispherical shell

$= \frac{2}{3}\pi(R^3 - r^3)$ cu units

Example 11. A hollow spherical shell is made of a metal of density 5.6 g / cm^3. If its internal and external diameters are 10 cm and 12 cm respectively, then find the weight of the shell. [Take $\pi = 3.14$]

Sol. Given, internal diameter, $d = 10$ cm

and external diameter, $D = 12$ cm

Then, internal radius, $r = \frac{d}{2} = \frac{10}{2} = 5$ cm

and external radius, $R = \frac{D}{2} = \frac{12}{2} = 6$ cm

Now, volume of a spherical shell,

$V = \frac{4}{3}\pi(R^3 - r^3) = \frac{4}{3} \times 3.14\,(6^3 - 5^3)$

$= \frac{12.56}{3}(216 - 125) = \frac{12.56 \times 91}{3}$

$= \frac{1142.96}{3} = 380.99 \text{ cm}^3 \approx 381 \text{ cm}^3$

$\therefore$ Weight of the shell $= 5.6 \times V = 5.6 \times 381$

[$\because$ mass = density $\times$ volume]

$= 2133.6$ g $= 2.134$ kg [$\because$ 1 kg = 1000 g]

Topic Exercise 3

1. If radius of a sphere is 21 cm, then find its
(i) surface area (ii) volume

2. Find the volume and surface area of a sphere of radius 4.2 cm. [Take $\pi = 3.14$]

3. Find the radius of a sphere, whose surface area is 154 cm^2. [Take $\pi = 22/7$]

4. The total surface area of a sphere is 616 cm^2. Find the radius and volume of the sphere. [Take $\pi = 22/7$]

5. If volume and surface area of a sphere is numerically equal, then find its radius.

6. Find the amount of water displaced by a solid spherical bowl of diameter 28 cm. [Take $\pi = 22/7$]

7. The difference of the surface areas of two spheres is 3080 cm^2. If the radius of the larger sphere is 21 cm, then find the difference of their volumes.

8. The diameter of a sphere is decreased by 25%. By what per cent does its curved surface area decrease?

9. Find the surface area and volume of a spherical shell, whose external and internal radii are 7 cm and 5 cm, respectively. [Take $\pi = 3.14$]

10. If radius of a hemisphere be 42 cm, then find its
(i) curved surface area (ii) total surface area
(iii) volume

11. The radius of a hemispherical balloon increases from 6 cm to 12 cm as air is being pumped into it. Find ratio of the surface areas of the balloon in the two cases.

12. A hemispherical bowl is made of steel, 0.25 cm thick.The inner radius of the bowl is 5 cm. Find the outer curved surface area of the bowl. [Take $\pi = 3.14$]

13. Find the volume, curved surface area and total surface area of a hemisphere of diameter 12.6 cm. [Take $\pi = 22/7$]

14. A solid sphere and a solid hemisphere have the same total surface areas. Find the ratio of their volumes.

15. The internal and external diameters of a hollow hemispherical vessel are 14 cm and 21 cm, respectively. The cost of silver plating of 1cm^2 of the surface is ₹ 0.40. Find the total cost of silver plating the vessel all over. [Take $\pi = 3.14$]

Hints and Answers

1. (i) **Hint** Surface area of sphere $= 4\pi r^2$ **Ans.** 5544 cm^2

(ii) **Hint** Volume of sphere $= \frac{4}{3}\pi r^3$ **Ans.** 38808 cm^3

2. Do same as Example 1.
Ans. 310.18 cm^3, 221.56 cm^2

3. **Hint** Surface area of sphere $= 4\pi r^2$

$\Rightarrow \quad 154 = 4 \times \frac{22}{7} r^2 \Rightarrow r^2 = \frac{49}{4}$ **Ans.** 3.5 cm

4. **Hint** TSA of sphere $= 4\pi r^2$
and volume of sphere $= \frac{4}{3}\pi r^3$
Ans. 7 cm, 1437.33 cm^3

5. **Hint** Volume of a sphere = Surface area of a sphere

$\Rightarrow \quad \frac{4}{3}\pi r^3 = 4\pi r^2$ **Ans.** 3 cm

6. **Hint** Volume of spherical bowl $= \frac{4}{3}\pi r^3$
Ans. 11498.67 cm^3

7. **Hint** Difference of two spheres $= 4\pi R^2 - 4\pi r^2$

$\Rightarrow \quad 3080 = 4 \times \frac{22}{7}[(21)^2 - r^2]$

$\Rightarrow \quad \frac{3080 \times 7}{4 \times 22} = 441 - r^2 \Rightarrow r = 14 \text{ cm}$

Now, difference of their volumes $= \frac{4}{3}\pi(R^3 - r^3)$
Ans. 27309.33 cm^3

8. **Hint** Let the radius of the sphere be r, then its diameter, $d = 2r$
New diameter, $d_1 = d - 25\%$ of d.
New radius, $r_1 = \frac{d_1}{2}$
Decrease in surface area, S_1 = Original surface area – New surface area

$\because$ Decrease percentage $= \frac{S_1}{S} \times 100\%$. **Ans.** 43.75%

9. **Hint** Surface area of spherical shell $= 4\pi(R^2 + r^2)$
Volume of spherical shell $= \frac{4}{3}\pi(R^3 - r^3)$
Ans. 301.44 cm^2, 912.69 cm^3

10. (i) **Hint** Curved surface area of hemisphere $= 2\pi r^2$
Ans. 11088 cm^2

(ii) **Hint** Surface area of hemisphere $= 3\pi r^2$
Ans. 16632 cm^2

(iii) **Hint** Volume of hemisphere $= \frac{2}{3}\pi r^3$
Ans. 155232 cm^3

11. **Hint** Ratio of surface areas of the balloon in two cases $= \frac{3\pi r_1^2}{3\pi r_2^2}$ **Ans.** 1 : 4

12. **Hint** Here, radius $= 5 + 0.25 = 5.25$ cm
Curved surface area of hemisphere bowl $= 2\pi r^2$
Ans. 173.25 cm^2

13. **Hint** Volume of hemisphere $= \frac{2}{3}\pi r^3$,
curved surface area of hemisphere $= 2\pi r^2$
and total surface area $= 3\pi r^2$
Ans. 523.9 cm^3, 249.48 cm^2, 374.22 cm^2

14. **Hint** $\because$ Surface area of sphere $= 4\pi r^2$
Total surface area of hemisphere $= 3\pi r^2$
Given, $4\pi R^2 = 3\pi r^2$

$$\therefore \quad \frac{\text{Volume of sphere}}{\text{Volume of hemisphere}} = \frac{\frac{4}{3}\pi R^3}{\frac{2}{3}\pi r^3}$$

Ans. $3\sqrt{3} : 4$

15. **Hint** Total surface area $= \pi(3R^2 + r^2)$. **Ans.** ₹ 476.96

Topic 4
Conversion of Solid and Combination of Two Solids

Conversion of Solid from One Shape to Another

When we come across objects which are converted from one shape to another or when a liquid which originally filled in one container of a particular shape is poured into another container of a different shape or size, the volume remains same, but the surface area may change.

e.g. Solid metallic sphere is melted and recast into more than one spherical balls or recast into a conical shape, the earth taken out by digging a well and spreading it uniformly around the well to form an embankment taking the shape of a cylindrical shell from its original shape of right circular cylinder, etc.

Example 1. Two solid spheres of radii 2 cm and 4 cm are melted and recast into a cone of height 8 cm. Find the radius of the cone so formed. ***[2015]***

Sol. Given, radii of spheres are $r_1 = 2\,\text{cm}$ and $r_2 = 4\,\text{cm}$.

$$\therefore \text{Volume of spheres} = \frac{4}{3}\pi r_1^3 + \frac{4}{3}\pi r_2^3$$
$$= \frac{4}{3}\pi(2^3 + 4^3) = \frac{4}{3}\pi(8+64)$$
$$= \frac{4}{3}\times\frac{22}{7}\times 72 = \frac{6336}{21}\ \text{cm}^3$$

Since, these spheres recasted into a cone.

$\because$ Volume of spheres = Volume of cone

$$\therefore \quad \text{Volume of cone} = \frac{6336}{21}\ \text{cm}^3 \qquad \text{[given]}\ldots\text{(i)}$$

Also, height of a cone, $h = 8\,\text{cm}$

From Eq. (i), we get

$$\frac{1}{3}\pi r^2 h = \frac{6336}{21}$$
$$\Rightarrow \quad \frac{1}{3}\times\frac{22}{7}\times r^2 \times 8 = \frac{6336}{21}$$
$$\Rightarrow \quad r^2 = \frac{6336}{21}\times\frac{7}{22}\times\frac{3}{8} \Rightarrow r^2 = 36$$
$$\therefore \quad r = 6\ \text{cm} \quad \text{[taking positive square root]}$$

Hence, the radius of cone is 6 cm.

Example 2. How many shots each having diameter 3 cm can be made from a cuboidal lead solid of dimensions 9 cm × 11 cm × 12 cm?

Sol. Given, dimensions of cuboidal lead solid

$$= 9\ \text{cm}\times 11\ \text{cm}\times 12\ \text{cm}$$

$\therefore$ Volume of a cuboidal lead solid

$$= 9\times 11\times 12 = 1188\ \text{cm}^3 \qquad [\because \text{volume of cuboid} = l\times b\times h]$$

Also, diameter of a shot = 3 cm

$$\therefore \text{Radius of a shot, } r = \frac{3}{2} = 1.5\ \text{cm}$$

Now, volume of a shot

$$= \frac{4}{3}\pi r^3 = \frac{4}{3}\times\frac{22}{7}\times(1.5)^3 = \frac{297}{21}$$
$$= 14.143\ \text{cm}^3$$

$\therefore$ Required number of shots

$$= \frac{\text{Volume of cuboidal lead solid}}{\text{Volume of a shot}}$$
$$= \frac{1188}{14.143} = 84\ \text{(approx.)}$$

Example 3. A metallic cone of base radius 8 cm and height 6.912 cm is melted down and shaped into a sphere without any loss of metal. Find the radius of the sphere. [Take $\pi = 3.14$]

Sol. Given, radius of a cone, $r = 8\,\text{cm}$

and height of a cone, $h = 6.912\,\text{cm}$

Now, volume of a cone

$$= \frac{1}{3}\pi r^2 h = \frac{1}{3}\times\pi\times(8)^2\times 6.912$$
$$= \frac{442.368}{3}\pi\ \text{cm}^3$$

Let r_1 be the radius of a sphere.
Then, according to the given condition,

Volume of cone = Volume of sphere

$$\Rightarrow \quad \frac{442.368}{3}\pi = \frac{4}{3}\pi r_1^3$$
$$\Rightarrow \quad \frac{442.368}{4} = r_1^3 \Rightarrow r_1^3 = 110.592$$
$$\therefore \quad r_1 = 4.8\ \text{cm}$$

Hence, the radius of the sphere is 4.8 cm.

Example 4. How many solid spheres of diameter 6 cm are required to be melted to form a cylindrical solid of height 45 cm and diameter 4 cm?

Sol. Given, diameter of solid sphere, $d_1 = 6\,\text{cm}$

$$\therefore \text{Radius of sphere, } r_1 = \frac{6}{2} = 3\ \text{cm}$$

Also, given diameter of cylinder, $d_2 = 4\,\text{cm}$

$$\therefore \text{Radius of cylinder, } r_2 = \frac{4}{2} = 2\ \text{cm}$$

$\because$ Height of cylinder, $h = 45\,\text{cm}$ [given]

Let the required number of spheres be N.

$\therefore$ $N\times$ Volume of sphere = Volume of cylinder

$$\Rightarrow \quad N\times\frac{4}{3}\pi r_1^3 = \pi r_2^2 h$$
$$\Rightarrow \quad N\times\frac{4}{3}\pi\times(3)^3 = \pi\times(2)^2\times 45$$
$$\therefore \quad N = \frac{2\times 2\times 45}{4\times 3\times 3} = 5$$

Hence, the required number of solid spheres is 5.

Example 5. The surface area of a solid metallic sphere is 2464 cm^2. It is melted and recast into solid right circular cones of radius 3.5 cm and height 7 cm. Calculate

(i) the radius of the sphere.

(ii) the number of cones recast. [Take $\pi = 22/7$] *[2014]*

Sol. (i) Given, surface area of a sphere = 2464 cm^2

Let the radius of sphere be r cm.

Then, surface area of sphere = $4\pi r^2$

$\therefore \quad 2464 = 4\pi r^2$

$\Rightarrow \quad r^2 = \dfrac{2464 \times 7}{4 \times 22} \Rightarrow r^2 = 196 \Rightarrow r = \pm 14$

$\therefore \quad r = 14 \text{ cm}$

[taking positive square root, because radius cannot be negative]

Hence, the radius of sphere is 14 cm.

(ii) Also, given radius of cone, $r_1 = 3.5$ cm and height of cone, $h = 7$ cm

Volume of cone $= \dfrac{1}{3}\pi r_1^2 h = \dfrac{1}{3}\pi (3.5)^2 \times 7$

Now, volume of sphere $= \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi (14)^3$

$\therefore$ Number of cones recast

$$= \frac{\text{Volume of sphere}}{\text{Volume of cone}} = \frac{\frac{4}{3}\pi (14)^3}{\frac{1}{3}\pi (3.5)^2 \times 7}$$

$$= \frac{4 \times 14 \times 14 \times 14}{3.5 \times 3.5 \times 7} = \frac{3200}{25} = 128$$

Hence, the number of cones recast from solid metallic sphere is 128.

Example 6. A hemispherical bowl of diameter 7.2 cm is filled completely with chocolate sauce. This sauce is poured into an inverted cone of radius 4.8 cm. Find the height of the cone. *[2010]*

Sol. Given, diameter of hemispherical bowl, $d = 7.2$ cm

$\therefore$ Radius of hemispherical bowl,

$$r_1 = \frac{d}{2} = \frac{7.2}{2} = 3.6 \text{ cm}$$

Now, volume of hemispherical bowl $= \dfrac{2}{3} \times \pi r_1^3$

$$= \frac{2}{3} \times \pi \times (3.6)^3 \text{ cm}^3$$

Also, radius of cone, $r_2 = 4.8$ cm [given]

Let the height of cone be h cm.

$\therefore$ Volume of cone $= \dfrac{1}{3}\pi r_2^2 h = \dfrac{1}{3}\pi (4.8)^2 \times h \text{ cm}^2$

Since, the chocolate sauce from hemispherical bowl is completely poured into an inverted cone.

$\therefore$ Volume of hemispherical bowl = Volume of cone

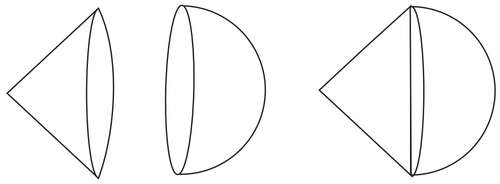
$\Rightarrow \quad \dfrac{2}{3}\pi \times (3.6)^3 = \dfrac{1}{3}\pi (4.8)^2 \times h \Rightarrow 93.312 = 23.04\, h$

$\therefore \quad h = \dfrac{93.312}{23.04} = 4.05 \text{ cm}$

Hence, the height of the cone is 4.05 cm.

Combination of Two (or More) Solids

Sometimes, we have to find the curved surface area and volume of a solid, which is a combination of two (or more) solids. Then, for finding the surface area, we add the curved surface areas of individual solids and for finding the volume of this solid, we add the volumes of individual solids. e.g. A combined solid is formed by joining hemisphere and right circular cone.



(i) Surface area of combined solid figure

= CSA of cone + CSA of hemisphere

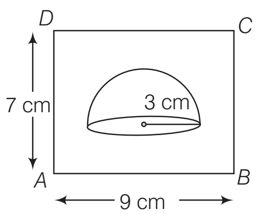
(ii) Volume of combined solid figure

= Volume of cone + Volume of hemisphere

While calculating the surface area, we have not added the surface areas of the two individual solids, rather we have added curved surface area because some part of the surface area disappeared in the process of joining them. But this will not be in the case, when we calculate the volume.

Example 7. A circular side of hemisphere of radius 3 cm is kept on a rectangular sheet of dimensions 7 cm × 9 cm. Find the surface area of the combined figure.

Sol. Given, dimensions of rectangular sheet are $l = 7$ cm, $b = 9$ cm and radius of hemisphere, $r = 3$ cm.



Now, area of rectangle $= l \times b = 9 \times 7 = 63 \text{ cm}^2$

and area of circular base of hemisphere $= \pi r^2$

$= 3.14 \times 3^2 = 3.14 \times 9 = 28.26 \text{ cm}^2$

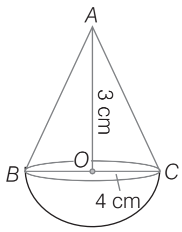
$\therefore$ Surface area of the combined figure

= Area of rectangle – Area of circular base of hemisphere

$= 63 - 28.26 = 34.74 \text{ cm}^2$

Example 8. A toy is in the form of a cone mounted on a hemisphere with the same radius. The radius of the conical portion is 4 cm and its height is 3 cm. Calculate the surface area and volume of the toy. [Take $\pi = 3.14$]

Sol. Given, toy is formed with combination of a cone and a hemisphere.



Here, radius of cone = radius of sphere = 4 cm
and height of cone, $h = 3$ cm
Now, in right angled ΔAOC,

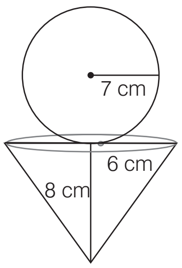
$$AC = \sqrt{AO^2 + OC^2} \quad \text{[by Pythagoras theorem]}$$
$$= \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \text{ cm}$$

$\therefore$ Slant height of a cone, $l = AC = 5$ cm
Now, curved surface area of conical portion = πrl
$$= 3.14 \times 4 \times 5 = 62.8 \text{ cm}^2$$
and curved surface area of hemispherical portion
$$= 2\pi r^2 = 2 \times 3.14 \times (4)^2$$
$$= 6.28 \times 16 = 100.48 \text{ cm}^2$$
$\therefore$ Required surface area of toy
= Curved surface area of conical portion
+ Curved surface area of the hemispherical portion
$= 62.8 + 100.48 = 163.28 \text{ cm}^2$

Now, volume of conical portion $= \dfrac{1}{3}\pi r^2 h$
$$= \frac{1}{3} \times 3.14 \times (4)^2 \times 3 = \frac{3.14 \times 16 \times 3}{3}$$
$$= \frac{150.72}{3} = 50.24 \text{ cm}^3$$
and volume of hemispherical portion $= \dfrac{2}{3}\pi r^3$
$$= \frac{2}{3} \times 3.14 \times (4)^3 = \frac{6.28 \times 64}{3} = \frac{401.92}{3} = 133.97 \text{ cm}^3$$
$\therefore$ Required volume of toy = Volume of conical portion + Volume of hemispherical portion
$$= 50.24 + 133.97 = 184.21 \text{ cm}^3$$

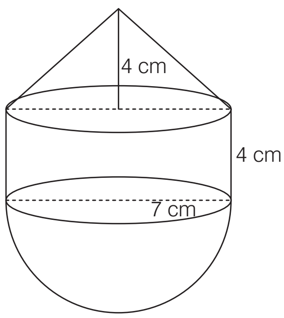
Example 9. A sphere of radius 7 cm is mounted on the solid cone of radius 6 cm and height 8 cm. Find the volume of the combined solid. [Take $\pi = 3.14$]

Sol. Given, radius of sphere, $r_1 = 7$ cm
Radius of cone, $r_2 = 6$ cm and height of cone, $h = 8$ cm



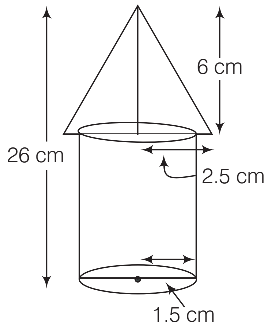
Now, volume of sphere
$$= \frac{4}{3}\pi r_1^3 = \frac{4}{3} \times 3.14 \times (7)^3$$
$$= \frac{12.56}{3} \times 343 = \frac{4308.08}{3} = 1436.03 \text{ cm}^3$$
and volume of cone $= \dfrac{1}{3}\pi r_2^2 h = \dfrac{1}{3} \times 3.14 \times (6)^2 \times 8$
$$= \frac{3.14 \times 36 \times 8}{3}$$
$$= \frac{904.32}{3} = 301.44 \text{ cm}^3$$
$\therefore$ Volume of combined two solids
= Volume of sphere + Volume of cone
$= 1436.03 + 301.44 = 1737.47 \text{ cm}^3$

Example 10. The following figure represents a solid consisting of a right circular cylinder with a hemisphere at one end and a cone at the other. Their common radius is 7 cm. The height of the cylinder and cone are each of 4 cm. Find the volume of the solid. ***[2018]***



Sol. Given, common radius, $r = 7$ cm
and height of cone = height of cylinder, $h = 4$ cm
$\therefore$ Volume of solid = Volume of cone + Volume of cylinder + Volume of hemisphere
$$= \frac{1}{3}\pi r^2 h + \pi r^2 h + \frac{2}{3}\pi r^3$$
$$= \frac{1}{3}\pi r^2(h + 3h + 2r)$$
$$= \frac{1}{3}\pi r^2(4h + 2r)$$
$$= \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times (16 + 14)$$
$$= \frac{1}{3} \times 22 \times 7 \times 30 = \frac{1}{3} \times 22 \times 7 \times 30$$
$$= 154 \times 10 = 1540 \text{ cm}^3$$

Example 11. A wooden toy rocket is in the shape of a cone mounted on a cylinder as shown in the following figure. The height of the entire rocket is 26 cm, while the height of the conical part is 6 cm. The base of the conical portion has a diameter of 5 cm, while the base diameter of the cylindrical portion is 3 cm. Find the volume of the solid figure. [Take $\pi = 3.14$]



Sol. Given, diameter and height of the conical portion are $d_1 = 5\,\text{cm}$ and $h_1 = 6\,\text{cm}$.

$\therefore$ Radius, $r_1 = \dfrac{5}{2} = 2.5\,\text{cm}$

Also, given diameter and height of a cylindrical portion are $d_2 = 3$ cm, $h_2 = 26 - 6 = 20\,\text{cm}$.

$\therefore$ Radius, $r_2 = \dfrac{3}{2} = 1.5\,\text{cm}$

Now, volume of the solid figure

$= \text{Volume of a cone} + \text{Volume of a cylinder}$

$= \dfrac{1}{3}\pi r_1^2 h_1 + \pi r_2^2 h_2$

$= \dfrac{3.14 \times (2.5)^2 \times 6}{3} + 3.14 \times (1.5)^2 \times 20$

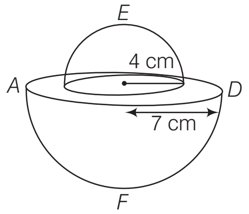
$= \dfrac{117.75}{3} + 141.3 = 39.25 + 141.3 = 180.55\ \text{cm}^3$

Example 12. A small hemisphere of radius 4 cm is mounted on a larger hemisphere of radius 7 cm with common base. Find the surface area and volume of the combined figure. [Take $\pi = 3.14$]

Sol. Given, radii of smaller and larger hemispheres are $r_1 = 4$ cm and $r_2 = 7$ cm, respectively.

Now, surface area of combined figure

= Curved surface area of a smaller hemisphere
+ Curved surface area of larger hemisphere
+ (Area of circular base of larger hemisphere
− Area of circular base of smaller hemisphere)



$= 2\pi r_1^2 + 2\pi r_2^2 + [\pi r_2^2 - \pi r_1^2] = 3\pi r_2^2 + \pi r_1^2$

$= 3 \times 3.14 \times (7)^2 + 3.14 \times (4)^2$

$= 461.58 + 50.24 = 511.82\ \text{cm}^2$

Now, volume of combined figure

= Volume of smaller hemisphere
+ Volume of larger hemisphere

$= \dfrac{2}{3}\pi r_1^3 + \dfrac{2}{3}\pi r_2^3 = \dfrac{2}{3}\pi (r_1^3 + r_2^3)$

$= \dfrac{2}{3} \times 3.14\,[(4)^3 + (7)^3] = \dfrac{6.28}{3} \times (64 + 343)$

$= \dfrac{6.28 \times 407}{3} = 851.99\ \text{cm}^3$

Problems Related to Inner and Outer Solid Figures

In such types of problems, a cone or sphere or hemisphere etc., is given either inside or outside the another figure and we have to find inner and outer surface area or volume by using the given information according to the questions.

Example 13. In a solid hemisphere of radius 10 cm, a maximum volume of sphere is cut out. Find the surface area and volume of the remaining solid. [Take $\pi = 3.14$]

Sol. Given, radius of hemisphere, $r_1 = 10$ cm

Since, the maximum sphere is cut out of diameter 10 cm.

Therefore, radius of the sphere, $r_2 = \dfrac{10}{2} = 5\,\text{cm}$

Now, surface area of the hemisphere $= 3\pi r_1^2$

$= 3 \times 3.14 \times (10)^2 = 942\ \text{cm}^2$

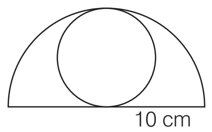
and surface area of the sphere $= 4\pi r_2^2 = 4 \times 3.14 \times (5)^2$

$= 314\ \text{cm}^2$

$\therefore$ Surface area of the remaining solid

= Surface area of the hemisphere
+ Surface area of the sphere

$= 942 + 314 = 1256\ \text{cm}^2$



Now, volume of the hemisphere

$= \dfrac{2}{3}\pi r_1^3 = \dfrac{2}{3} \times 3.14 \times (10)^3 = \dfrac{6280}{3} = 2093.33\ \text{cm}^3$

and volume of the sphere

$= \dfrac{4}{3}\pi r_2^3 = \dfrac{4}{3} \times 3.14 \times (5)^3 = \dfrac{1570}{3} = 523.33\ \text{cm}^3$

$\therefore$ Volume of the remaining solid

= Volume of hemisphere − Volume of sphere

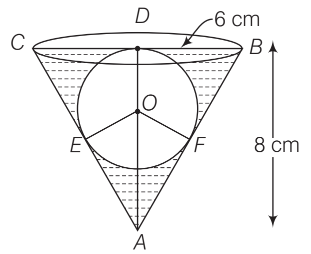
$= 2093.33 - 523.33 = 1570\ \text{cm}^3$

Example 14. A conical vessel of radius 6 cm and height 8 cm is completely filled with water. A sphere is lowered into the water such that when it touches the sides, it is just immersed. What fraction of the water overflow?

Sol. Given, radius of conical vessel, $r_1 = 6\,\text{cm}$

and height, $h = 8\,\text{cm}$

In a conical vessel, a sphere is immersed as shown in the figure.



In right angled ΔADB,

$AB^2 = AD^2 + DB^2$ [using Pythagoras theorem]

$= 8^2 + 6^2$ [$\because AD = 8$ cm and $DB = 6$ cm, given]

$= 64 + 36 = 100$

$\Rightarrow$ $AB = 10$ cm [taking positive square root] ... (i)

In the figure, BC is a tangent to the circle at point D and AB is a tangent to the circle at point F

$\therefore$ $BD = BF = 6$ cm

[$\because$ tangents drawn from an external point to the circle are equal in lengths]

Now, $AF = AB - BF = 10 - 6 = 4$ cm [from Eq. (i)]

Let r cm be the radius of the sphere. Then,

$OE = OF = r$ cm

Now, $AO = AD - OD = (8 - r)$ cm

In right angled ΔAFO,

$AO^2 = AF^2 + OF^2$ [using Pythagoras theorem]

$\Rightarrow$ $(8 - r)^2 = 4^2 + r^2$

$\Rightarrow$ $64 + r^2 - 16r = 16 + r^2$

$\Rightarrow$ $48 = 16r \Rightarrow r = 3$ cm

Now, volume of sphere $= \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi \times (3)^3 = 36\pi$ cm^3

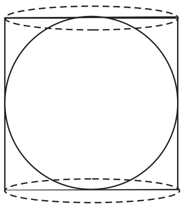
$\therefore$ The volume of water which overflows by immersing the sphere = Volume of sphere = 36π cm^3

Now, total volume of water in the cone before immersing the sphere = Volume of the cone $= \dfrac{1}{3}\pi r_1^2 h$

$= \dfrac{1}{3}\pi \times (6)^2 \times 8 = \pi \times 12 \times 8 = 96\pi$ cm^3

$\therefore$ Fraction of water which overflows by immersing the sphere $= \dfrac{\text{Volume of water overflows}}{\text{Total volume of water}} = \dfrac{36\pi}{96\pi} = \dfrac{3}{8}$

Example 15. In the given figure, a sphere is inscribed in a cylinder. Prove that the surface area of the sphere is equal to the curved surface area of the cylinder.



Sol. Let r be the radius of a sphere. Then, diameter of the sphere is $2r$. It is clear from the figure that radius and diameter of a sphere are equal to the radius and height of a cylinder, respectively.

Therefore, height of cylinder, $h = 2r$

Now, surface area of sphere $= 4\pi r^2$ ...(i)

and curved surface area of cylinder $= 2\pi rh$

$= 2\pi r \times 2r = 4\pi r^2$ ...(ii)

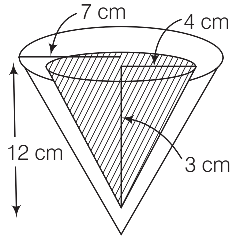
$\therefore$ From Eqs. (i) and (ii),

Surface area of a sphere

= Curved surface area of a cylinder **Hence proved.**

Example 16. A solid metallic cone, with radius 7 cm and height 12 cm, is made of some heavy metal A. In order to reduce its weights, a conical hole is made in the cone as shown and it is completely filled with a lighter metal B. The conical hole has a radius 4 cm and depth 3 cm. Calculate the ratio of the volume of metal A to the volume of the metal B in the solid. [Take $\pi = 3.14$]

Sol. Given, radius and height of the cone of heavy metal A are $r_1 = 7$ cm and $h_1 = 12$ cm, respectively.



Also, radius and height (depth) of the cone of lighter metal B are $r_2 = 4$ cm and $h_2 = 3$ cm.

Now, volume of a cone having metal A,

$V_1 = \dfrac{1}{3}\pi r_1^2 h_1 = \dfrac{1}{3} \times 3.14 \times (7)^2 \times 12$

$= \dfrac{3.14 \times 49 \times 12}{3} = \dfrac{1846.32}{3} = 615.44$ cm^3

and volume of a cone having metal B,

$V_2 = \dfrac{1}{3}\pi r_2^2 h_2 = \dfrac{1}{3} \times 3.14 \times (4)^2 \times 3$

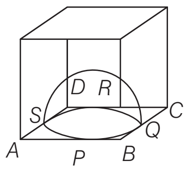
$= 3.14 \times 16 = 50.24$ cm^3

$\therefore$ Ratio of the volume of metal A to the volume of the metal $B = \dfrac{V_1}{V_2} = \dfrac{615.44}{50.24} = \dfrac{61544}{5024} = \dfrac{7693}{628} = 7693 : 628$

Example 17. A hemispherical depression is cut out from one face of a cubical wooden block, such that the diameter l of the hemisphere is equal to the edge of the cube. Determine the surface area of the remaining solid.

Sol. Given, side of the cube = diameter of the hemisphere = l

$\therefore$ Radius of the hemisphere $= \dfrac{l}{2}$



Now, required surface area of the remaining solid

= Area of one face $ABCD$ of cube – Area of circle $PQRS$

+ Area of remaining five faces of cube

+ curved surface area of hemisphere

$= l^2 - \pi\dfrac{l^2}{4} + 5l^2 + 2\pi\dfrac{l^2}{4}$ $\left[\because \text{radius of hemisphere} = \dfrac{l}{2}\right]$

$= 6l^2 + \dfrac{\pi l^2}{4} = \dfrac{l^2}{4}(\pi + 24)$ sq units

Topic Exercise 4

1. A hollow sphere of internal and external radii 6 cm and 8 cm respectively, is melted and recast into small cones of base radius 2 cm and height 8 cm. Find the number of cones. *[2012]*

2. A hollow sphere of internal and external diameters 4 cm and 8 cm respectively, is melted into a cone of base diameter 8 cm. Find the height of the cone.

3. A solid rectangular block of metal 49 cm $\times$ 44 cm $\times$ 18 cm is melted and formed into a solid sphere. Calculate the radius of the sphere.

4. A solid cone of radius 5 cm and height 8 cm is melted and recast into small spheres of radius 0.5 cm. Find the number of spheres formed. *[2011]*

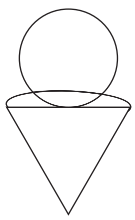
5. The surface area of a solid metallic sphere is 616 cm^2. It is melted and recast into smaller spheres of diameter 3.5 cm. How many such spheres can be obtained? [Take $\pi = 22/7$] *[2007]*

6. A metallic disc in the shape of a right circular cylinder is of height 2.5 mm and base radius 12 cm. Metallic disc is melted and made into a sphere. Calculate the radius of the sphere.

7. A solid sphere of radius 15 cm is melted and recast into solid right circular cones of radius 2.5 cm and height 8 cm. Calculate the number of cones recast.

8. A certain number of metallic cones, each of radius 2 cm and height 3 cm are melted and recast into a solid sphere of radius 6 cm. Find the number of cones. *[2016]*

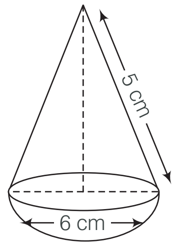
9. The given block is made of two solids; a cone and a hemisphere. If the height and the base radius of the cone are 9 cm and 7 cm, respectively and the radius of the sphere is 5 cm, then find the volume of the block. [Take $\pi = 3.14$]



10. A small solid cone of radius 4 cm and having slant height 6 cm is mounted on a bigger solid cone of radius 6 cm and having slant height 10 cm with common base. Find the surface area of combined figure. [Take $\pi = 3.14$]

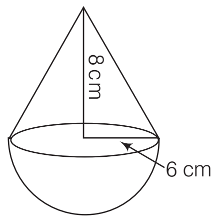
11. In a solid hemisphere of radius 10 cm, a right cone of same radius is removed out. Find the volume and surface area of the remaining solid. [Take $\pi = 3.14$ and $\sqrt{2} = 1.41$]

12. The given figure represents a hemisphere surmounted by a conical block of wood. The diameter of their bases is 6 cm each and the slant height of the cone is 5 cm.



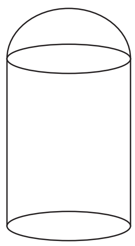
Calculate
(i) the height of the cone.
(ii) the volume of the solid. *[2009]*

13. In the given figure, a hemisphere of radius 6 cm surmounted by a right circular cone of base radius 6 cm. The height of the cone is 8 cm.



Calculate
(i) the total surface area of the solid.
(ii) the volume of the solid. [Take $\pi = 3.14$]

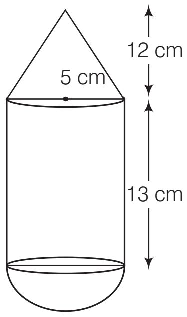
14. In the given figure, it shows a metal container in the form of a cylinder surmounted by a hemisphere of the same radius. The internal height of the cylinder is 7 m and the internal radius is 3.5 m.



Calculate
(i) the total area of the internal surface, excluding the base (in m^2).
(ii) the internal volume of the container (in m^3).

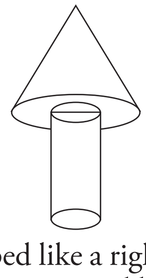
15. An iron sphere of diameter 12 cm is dropped into a cylindrical cone of diameter 24 cm containing water. Find the rise in the level of water when the sphere is completely immersed.

16. In the given figure, toy is in the shape of a right circular cylinder with a hemisphere on one end and a cone on the other. The height and radius of the cylindrical part are 13 cm and 5 cm, respectively. The radii of the hemispherical and conical parts are the same as that of the cylindrical part. Calculate the surface area of the toy if height of the conical part is 12 cm.



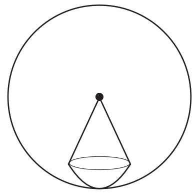
17. The interior of a building is in the form of a cylinder of diameter 4.3 m and 3.8 m height, surmounted by a cone whose vertical angle is a right angle. Find the height and slant height of the cone. Also, find the surface area of building. [Take $\pi = 3.14$]

18. A wooden toy is in the shape of a cone mounted on a cylinder as shown in the following the figure. If the height of the cone is 24 cm, the total height of the toy is 60 cm and the radius of the base of the cone is equal to twice the radius of the base of the cylinder is 10 cm; then find the total surface area of the toy. [Take $\pi = 3.14$]



19. A container shaped like a right circular cylinder having diameter 12 cm and height 15 cm is full of ice-cream. The ice-cream is to be filled into cones of height 12 cm and diameter 6 cm, having a hemispherical shape on the top. Find the number of such cones which can be filled with ice-cream. [Take $\pi = 22/7$]

20. In the given figure, from a solid sphere of radius 9 cm, a cone mounted on a hemisphere shape is carved out. The radius and height of a cone are 4 cm and 5 cm, respectively. **Find the surface area of the remaining figure.**



21. A small solid cone of radius 5 cm and slant height 7 cm is mounted on a bigger solid cone of radius 8 cm and slant height 12 cm with common base. Find the surface area and volume of the combined figure. [Take $\pi = 3.14$]

22. A vessel in the form of an inverted cone is filled with water to the brim. Its height is 20 cm and diameter is 16.8 cm. Two equal solid cones are dropped in it, so that they are fully submerged. As a result, one-third of the water in the original cone overflows. What is the volume of each of the solid cones submerged? ***[2006]***

Hints and Answers

1. Hint Volume of hollow sphere

$$= \frac{4}{3}\pi(R^3 - r^3) \quad [\because R = 8 \text{ cm}, r = 6 \text{ cm}]$$

$$\text{Volume of cone} = \frac{1}{3}\pi r_1^2 h \quad [\because r_1 = 2 \text{ cm}]$$

$$\therefore \text{ Number of cones} = \frac{\text{Volume of hollow spheres}}{\text{Volume of cone}}$$

Ans. 37

2. Hint Volume of hollow sphere = Volume of cone

$$\Rightarrow \quad \frac{4}{3}\pi(R^3 - r^3) = \frac{1}{3}\pi r_1^2 h$$ **Ans.** 56 cm

3. Hint Volume of rectangular block

= Volume of solid sphere

$$\Rightarrow \quad l \times b \times h = \frac{4}{3}\pi r^3$$ **Ans.** 21 cm

4. Hint $\text{Number of spheres} = \dfrac{\text{Volume of cone}}{\text{Volume of each sphere}}$

Ans. 400

5. Hint $\text{Number of spheres} = \dfrac{\text{Volume of a sphere}}{\text{Volume of a smaller sphere}}$

Ans. 64

6. Hint Volume of right circular cylinder

= Volume of sphere **Ans.** 3 cm

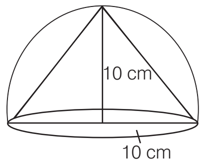
7. **Hint** Number of cones
$$= \frac{\text{Volume of a sphere}}{\text{Volume of each cone}} \quad \textbf{Ans. } 270$$

8. **Hint** Number of cones
$$= \frac{\text{Volume of a sphere}}{\text{Volume of a small cone}} \quad \textbf{Ans. } 72$$

9. Do same as Example 9. **Ans.** 984.91 cm^3

10. Do same as Example 12. **Ans.** 326.56 cm^2

11. **Hint** Volume of remaining solid
= Volume of hemisphere − Volume of cone



Surface area of remaining solid = Curved surface area of hemisphere + Curved surface area of cone
Ans. 1046.67 cm^3; 1070.74 cm^2

12. (i) **Hint** $h^2 = l^2 - r^2 = (5)^2 - (3)^2$ **Ans.** $h = 4 \text{ cm}$
(ii) **Hint** Volume of solid = Volume of the hemisphere + Volume of cone
Ans. 94.28 cm^3

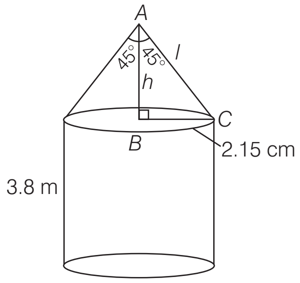
13. (i) **Hint** Total surface area of solid = Curved surface area of cone + Curved surface area of hemisphere.
Ans. 414.74 cm^2.
(ii) Do same as Q. 12 (ii). **Ans.** 754.08 cm^3

14. (i) **Hint** Total area of the internal surface = curved surface area of hemisphere + curved surface area of cylinder + Area of circular base of cylinder
Ans. 66 m^2
(ii) **Hint** Volume of container = Volume of hemisphere + Volume of cylinder.

15. Do same as Example 14. **Ans.** 150

16. **Hint** Surface area of the toy = Curved surface area of cone + Curved surface area of cylinder + Curved surface area of hemisphere. **Ans.** 770 cm

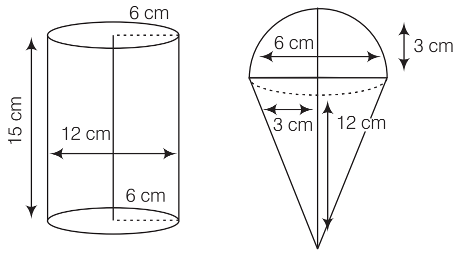
17. **Hint** In right angled ΔABC,
$$\sin 45° = \frac{BC}{AC} = \frac{2.15}{l} \Rightarrow l = 2.15 \times \sqrt{2}$$



Also, $\angle BCA = 45°$
Here, ΔABC is an isosceles triangle.
$\therefore \quad AB = BC = 2.15 \text{ m}$
$\therefore$ Surface area of building = Curved surface area of cone + Curved surface area of cylinder.
Ans. Height = 2.15 m, slant height = 3.04 m and surface area = 71.83 m^2

18. **Hint** Total surface area of toy = Curved surface area of cone + Curved surface area of cylinder + [Area of circular base of cone − Area of circular base of cylinder]
Ans. 2260.8 cm^2

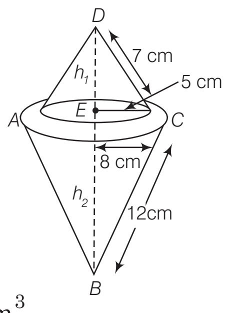
19. **Hint** Volume of full ice-cream cone, V_1
= Volume of cone + Volume of hemisphere



Number of cones
$$= \frac{\text{Volume of cylinderical ice - cream container}}{V_1}$$
Ans. 10.

20. **Hint** Surface area of the remaining figure = Surface area of the sphere + Curved surface area of cone + Curved surface area of hemisphere.
Ans. 1198.22 cm^2

21. **Hint** Now, surface area of combined figure
= Curved surface area of small cone
+ Curved surface area of larger cone
+ (Area of circular base of larger cone − Area of circular base of smaller cone)



Ans. 727.31 cm^3

22. **Hint** Volume of water in conical vessel, i.e. original volume $= \frac{1}{3}\pi r^2 h$.
$\therefore$ Volume of water overflow $= \frac{1}{3} \times$ Original volume
Ans. 246.4 cm^3

CHAPTER EXERCISE

a 3 Marks Questions

1. The height of a right circular cylinder with lateral surface area 792 cm^2 is 21 cm. Find the diameter of its base.

2. The curved surface area of a right circular cylinder is 4400 cm^2. If the circumference of the base is 110 cm, then find the height of the cylinder.

3. The diameter of a garden roller is 1.4 m and it is 2 m long. How much area will it level in 5 revolutions?

4. A school provides milk to the students daily in a cylindrical glasses of diameter 7 cm. If the glass is filled with milk upto an height of 12 cm, find how many litres of milk is needed to serve 1600 students?

5. The diameter of the roller 120 cm long is 84 cm. If it takes 500 complete revolutions to level a playground, then find the cost of levelling at the rate of 30 paise per m^2.

6. The pillars of a temple are cylindrically shaped. If each pillar has a circular base of radius 20 cm and height 10 m, how much concrete mixture would be required to build 14 such pillars?

7. The radius and the slant height of a cone are in the ratio of 4 : 7. If its curved surface area is 792 cm^2. Then, find the radius.

8. A cone and a cylinder are having equal base radius. Find the ratio of the heights of cone and cylinder, if their volumes are equal.

9. The height and base diameter of a conical tent is 16 m and 24 m, respectively. Find the cost of canvas required to make it at the rate of ₹ 210 per m^2?

10. How many metres of 5 m wide cloth will be required to make a conical tent, the radius of whose base is 3.5 m and height is 12 m?

11. Find the volume of a sphere which is equal to two-third of the volume of a cylinder whose height and diameter are equal to the diameter of the sphere.

12. The radius of a spherical balloon is inflated from 1.5 cm to 2.5 cm by pumping more air in it. Find the ratio of surface area of resulting balloon to the original balloon.

13. The radius of the base of a cone and the radius of a sphere are the same, each being 8 cm. Given that, the volume of these two solids are also the same. Calculate the slant height of the cone.

14. A hemispherical bowl of internal diameter 36 cm contains a liquid. This liquid is filled in cylindrical bottles of radius 3 cm and height 6 cm. How many bottles are required to empty the bowl?

15. A cone is filled of water poured into the hemisphere, which is also filled. The radius and height of a cone are 10 cm and 20 cm, respectively. Find the radius of the hemisphere.

16. How many spherical bullets can be made out of a solid cube of lead whose edge measures 44 cm, each bullet being 4 cm in diameter? [Take $\pi = 22/7$]

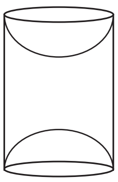
17. Find the maximum volume of a cone, that can be carved out of a solid hemisphere of radius r.

18. The rainwater from a roof 22 m $\times$ 20 m drain into a conical vessel having diameter of base as 2 m and height 3.5 m. If the vessel is just full, then find the rainfall.

19. If a sphere is inscribed in a cube, then find the ratio of the volume of the cube to the volume of the sphere.

20. The water for a factory is stored in a hemispherical tank, whose internal diameter is 14 m. The tank contains 50 kL of water. Water is pumped into the tank to fill to its capacity. Calculate the volume of water pumped into the tank. [Take $\pi = 22/7$]

21. A wooden article was made by scooping out a hemisphere from each end of a solid cylinder, as shown in figure. If the height of the cylinder is 10 cm and its base is of radius 3.5 cm,then find the total surface area of the article. [Take $\pi = 22/7$]



22. A storage oil tanker consists of a cylindrical portion 7m in diameter with two hemispherical ends of the same diameter. The oil tanker lying horizontally. If the total length of the tanker is 20 m, then find the capacity of the container.

23. A cylindrical beaker, whose base has a radius of 15 cm, is filled with water upto a height of 20 cm. A heavy iron spherical ball of radius 10 cm is dropped to submerge completely in water in the beaker. Find the increase in the level of water. [Take $\pi = 3.14$]

24. The radii of the internal and external surfaces of a metallic spherical shell are 3 cm and 5 cm, respectively. It is melted and recast into a solid right circular cylinder of height 102/3 cm. Find the diameter of the base of the cylinder.

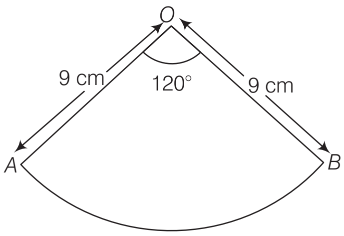
b 4 Marks Questions

25. Circumference of the base of a cylinder, open at the top, is 132 cm. The sum of radius and height is 41 cm. Find the cost of polishing the outer surface area of cylinder at the rate ₹ 10 per sq dm (decimetre). [Take $\pi = 22/7$]

26. A cloth having an area of 165 m^2 is shaped into the form of a conical tent of radius 5 m.

(i) How many students can sit in the tent, if a student on an average occupies $5/7 m^2$ on the ground?
(ii) Find the volume of the cone.

27. In the given figue, A sector of a circle of radius 9 cm and central angle of $120°$. It is rolled up so that the two bounding radii are joined together to form a cone. Find



(i) the slant height of the cone.
(ii) the radius of the base of the cone.
(iii) the volume of the cone.
(iv) the total surface area of the cone.

28. The internal and external diameters of a hollow hemispherical vessel are 7cm and 14 cm, respectively. The cost of silver plating of 1 sq cm surface is ₹ 0.60. Find the total cost of silver plating the vessel all over.

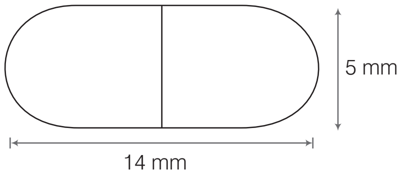
29. A cylindrical bucket 32 cm high and radius of base 18 cm, is filled with sand. This bucket is emptied on the ground and a conical heap of sand is formed. If the height of the conical heap is 24 cm, then find the radius and slant height of the heap.

30. A cylindrical water tank of diameter 1.4 m and height 2.1 m is being filled by a pipe of diameter 3.5 cm through which water flows at the rate of 2 m/s. Find the time taken in min by it to fill the tank.

31. A circular hall (i.e. cylindrical) has a hemispherical roof. The greatest height is equal to the inner diameter. Find the area of the floor, given that the capacity of the hall is 48510 m^3.

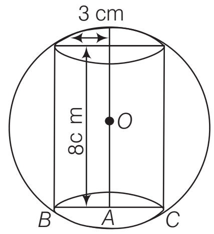
32. An ice-cream cone consisting of cone surmounted by a hemisphere. The radius of the hemisphere is 3.5 cm and height of the ice-cream cone is 12.5 cm. Calculate the volume of the ice-cream in the cone. [Take $\pi = 22/7$]

33. A medicine capsule is in the shape of a cylinder with two hemispheres stuck to each of its ends (see figure). The length of the entire capsule is 14 mm and the diameter of the capsule is 5 mm. Find the surface area of the figure. [Take $\pi = 22/7$]



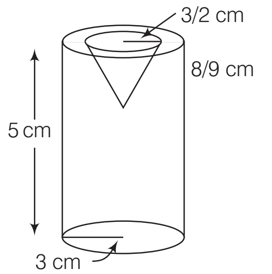
34. A rocket is in the form of a right circular cylinder closed at the lower end and surmounted by a cone with the same radius as that of the cylinder. The diameter and height of the cylinder are 6 cm and 12 cm, respectively. If the slant height of the conical portion is 5 cm, then find the total surface area and volume of the rocket. [take $\pi = 3.14$]

35. In the given figure, a sphere circumscribes a right cylinder whose height is 8 cm and radius of the base is 3 cm. Find the ratio of the volumes of the sphere and the cylinder.

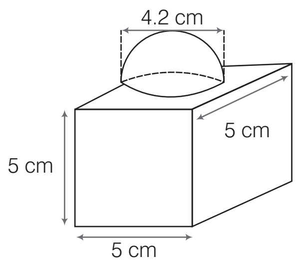


36. A metallic cylinder has radius 3 cm and height 5 cm. It is made of metal A. To reduce its weight, a conical hole is drilled in the cylinder as shown in the figure and it is completely filled with a lighter metal B. The conical hole has a radius of 3/2 cm and its depth is

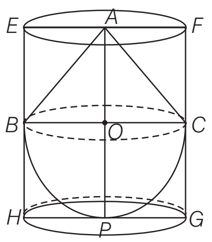
8/9 cm. Calculate the ratio of the volume of the metal A to the volume of the metal B in the solid.



37. The decorative block as shown in the figure is made of two solids, a cube and a hemisphere. The base of the block is a cube with edge 5 cm and the hemisphere fixed on the top has a diameter of 4.2 cm, then find the total surface area of the block and find the total area to be painted.



38. In the given figure, a solid toy is in the form of a hemisphere surmounted by a right circular cone. Height of the cone is 2 cm and the diameter of the base is 4 cm. If a right circular cylinder circumscribes the solid, then find how much more space it will cover?



Hints and Answers

1. Hint Lateral surface area of a right circular cylinder $= 2\pi rh$
Ans. 12 cm

2. Hint Curved surface area of right circular cylinder $= 2\pi rh = 4400 \text{ cm}^2$. Also, given $2\pi r = 110$ cm
Ans. 40 cm

3. Do same as Example 3 of Topic 1. **Ans.** 44 m^2

4. Hint Volume of cylinderical glass $= \pi r^2 h$
Volume of milk needed to serve 1600 students
$= 1600 \times \pi r^2 h$
Also, use $1 \text{ cm}^3 = \dfrac{1}{1000}$ litres **Ans.** 739.2 L

5. Hint Lateral surface area of the roller, S_1
$=$ Lateral surface area of a cylinder $= 2\pi rh$
$\because$ Area covered in 1 revolution
$=$ Lateral surface area of the roller $= S_1$
$\therefore$ Area covered in 500 revolutions $= 500 \times S_1$
Ans. ₹ 475.20

6. Hint Volume of each cylindrical shaped pillar $= \pi r^2 h$
Now, required concrete mixture to make 14 such pillars $=$ Volume of one cylindrical pillar $\times 14$
Ans. 17.6 m^3

7. Hint Let radius of a cone $= 4x$
and slant height of a cone $= 7x$
Now, curved surface area of a cone $= 792 \text{ cm}^2$
$\Rightarrow \dfrac{22}{7} \times 4x \times 7x = 792.$ **Ans.** $x = 3$

8. Hint Volume of cone, $V_1 = \dfrac{1}{3}\pi r^2 h_1$
and volume of cylinder, $V_2 = \pi r^2 h_2$
$\because \quad V_1 = V_2$
Ans. $h_1 : h_2 = 3 : 1$

9. Hint Do same as Example 6 (i) and (ii) of Topic 2.
Ans. ₹ 158400

10. Hint Slant height $(l) = \sqrt{h^2 + r^2}$
Now, curved surface area of the conical tent
$=$ Curved surface area of a cone $= \pi rl$
$\because$ Length of cloth required to make conical tent
$= \dfrac{\text{Curved surface area of conical tent}}{\text{Width of the cloth}}$
Ans. 27.5 m

11. Hint Volume of sphere $= \dfrac{2}{3} \times$ Volume of cylinder
Ans. $\dfrac{4}{3}\pi r^3$

12. Hint Here $r_1 = 1.5$ cm, $r_2 = 2.5$ cm
$\because$ Ratio $= \dfrac{4\pi r_2^2}{4\pi r_1^2}$ **Ans.** 25 : 9

13. Hint Volume of sphere $=$ Volume of cone
Ans. 33 cm

14. Hint Volume of the hemispherical bowl $= \dfrac{2}{3}\pi r^3$
and volume of the cylindrical bottle of radius 3 cm and height 6 cm $=$ Volume of a cylinder $= \pi \times (3)^2 \times 6$
$\therefore$ Required number of bottle to empty the bowl

$$= \frac{\text{Volume of the bowl}}{\text{Volume of one bottle}}$$

Ans. 72

15. **Hint** Volume of cone = Volume of the hemisphere

Ans. 10 cm

16. **Hint** Number of bullets

$$= \frac{\text{Volume of cube}}{\text{Volume of one spherical bullet}}$$

Ans. 2541

17. **Hint** For the maximum volume of cone curved out from the hemisphere, radius of the cone should be equal radius of hemisphere.

Volume of cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (r)$

Ans. $\frac{\pi r^3}{3}$

18. **Hint** Let the rainfall be x m.

Then, volume of water on the roof

= Volume of conical vessel

$\Rightarrow \quad 22 \times 20 \times x = \frac{1}{3}\pi r^2 h$ **Ans.** 0.83 cm

19. **Hint** Let the side of a cube be x.

Then, radius of the sphere, $r = \frac{x}{2}$

$\therefore \quad \text{Ratio} = \frac{\text{Volume of the cube}}{\text{Volume of the sphere}}$

Ans. $6 : \pi$

20. **Hint** Water in the hemispherical tank = 50 kL

$= 50 \times 1000\ \text{L} = 50\ \text{m}^3$

Volume of water in the hemispherical tank $V_1 = \frac{2}{3}\pi r^3$

$\therefore$ Volume of water pumped in the hemispherical tank

$= V_1 - 50$

Ans. 668.67 m^3

21. **Hint** Total surface area of the wooden article

= 2 × Curved surface area of the hemisphere

+ Curved surface area of cylinder.

Ans. 374 cm^2

22. **Hint** Capacity of container = Volume of cylinder

+ 2(Volume of hemisphere)

Ans. 680.17 m^3

23. **Hint** Volume of cylinder, $V_1 = \pi r^2 h$

and Volume of spherical ball, $V_2 = \frac{4}{3}\pi r^3$

When spherical ball is dropped in the cylinderical beaker, then total volume $= V_1 + V_2$

$\Rightarrow \quad \frac{1}{3}\pi r^2 h_1 = V_1 + V_2$

Increase in the level of water $= h_1 - h$

Ans. 4186.66 cm.

24. **Hint** Volume of spherical shell

= Volume of right circular cylinder

Ans. 7 cm

25. **Hint** Circumference of base of cylinder $= 2\pi r$

Surface area of cylinder whose top is open,

$S_1 = 2\pi rh + \pi r^2$

$\therefore$ Total cost of polishing at the rate of ₹ 10 per dm^2

$= 10 \times S_1$

Ans. ₹ 402. 60

26. (i) **Hint** Area of the base of a conical tent $= \pi r^2$

Now, number of students

$$= \frac{\text{Area of the base of a conical tent}}{\text{Area occupied by a student on the ground}}$$

Ans. 110

(ii) **Hint** Curved surface area of conical tent

= Area of cloth to form conical tent

$\Rightarrow \quad \pi rl = 165$

Now, height of conical tent,

$h = \sqrt{l^2 - r^2}$

$\therefore$ Volume of the conical tent = Volume of a cone

Ans. 241.7 m^3

27. (i) **Hint** The slant height of the cone = Radius of the given sector of a circle.

Ans. 9 cm

(ii) **Hint** Circumference of base of the cone = Length of arc of given sector whose central angle is 120°

$\Rightarrow \quad 2\pi r = \frac{\theta}{360°} \times 2\pi r_1$

Ans. $r = 3$

(iii) **Hint** Let h be the height of the cone, then

$h = \sqrt{l^2 - r^2}$

$\therefore$ Volume of the cone $= \frac{1}{3}\pi r^2 h$

Ans. $18\pi\sqrt{2}\ \text{cm}^3$

(iv) **Hint** Total surface area of the cone

= Area of the given sector + Area of base of the cone

$= \frac{\theta}{360°} \times \pi (r_1)^2 + \pi r^2$

Ans. $36\pi\ \text{cm}^2$

28. **Hint** Total surface area of hemisphere,

$S = \pi(3R^2 + r^2)$

$\therefore$ Cost of silver plating $= S \times 0.60$ **Ans.** ₹ 27694.8

29. Hint Volume of cylinderical bucket = Volume of cone

$$\Rightarrow \quad \pi r^2 h = \frac{1}{3}\pi r_1^2 h_1$$

Also, $\quad l = \sqrt{h^2 + r_1^2}$

Ans. Radius = 36 cm and slant height $= 12\sqrt{13}$ cm

30. Hint Volume of the cylindrical water tank, $V_1 = \pi r^2 h$

Length of water which flows through the pipe in 1s

= 2 m = 200 cm

$\therefore$ Volume of water delivered in 1s through pipe, V_2

= Volume of a cylinder

Hence, time taken to fill the cylindrical tank by the pipe

$$= \frac{V_1}{V_2}$$

Ans. 28 min

31. Hint Volume of hemisphere $= \frac{2}{3}\pi r^3$

and area of floor $= \pi r^2$ **Ans.** 1386 m^2

32. Hint Volume of ice-cream

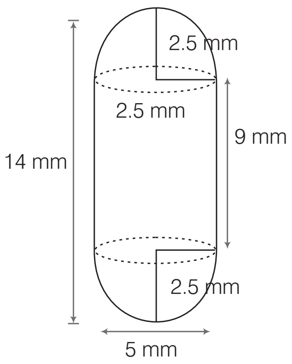
= Volume of cone + Volume of hemisphere

Ans. 205.33 cm^3

33. Hint Surface area of figure

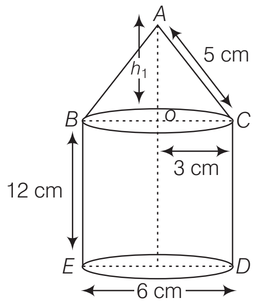
$= 2 \times$ Curved surface area of hemisphere

+ Curved surface area of cylinder



Ans. 220 mm^2

34. Hint In ΔAOC, $h_1 = \sqrt{5^2 - 3^2}$



Total surface area of rocket

= Curved surface area of cone

+ Curved surface area of cylinder

$\therefore$ Volume of rocket = Volume of cone

+ Volume of cylinder

Ans. 301.44 cm^2, 376.8 cm^3

35. Hint Ratio $= \dfrac{\text{Volume of sphere}}{\text{Volume of cylinder}}$

$$= \frac{\frac{4}{3}\pi r^3}{\pi r_1^2 h}$$

Ans. 125 : 54

36. Hint Ratio $= \dfrac{\text{Volume of metal } A}{\text{Volume of metal } B}$

$$= \frac{\frac{1}{3}\pi(3)^2 \times 5}{\frac{1}{3}\pi\left(\frac{3}{2}\right)^2 \times \frac{8}{9}}$$

Ans. 135 : 2

37. Hint Total surface area of decorative block

= Total surface area of cube

− Area of base of the hemisphere

+ Curve surface area of the hemisphere.

Ans. 163.86 cm^2

38. Hint The more space it will cover = Volume of cylinder

− Volume of cone − Volume of hemisphere.

Ans. 8π cm^3

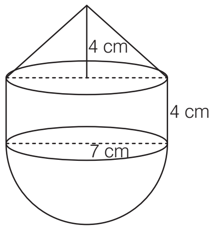
ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1. The circumference of the base of a cylindrical vessel is 132 cm and its height is 25 cm. Find the
(i) radius of the cylinder. (ii) volume of cylinder. [use $\pi = 22/7$]

2. The given figure represents a solid consisting of a right circular cylinder with a hemisphere at one end and a cone at the other. Their common radius is 7 cm. The height of the cylinder and cone are each of 4 cm. Find the volume of the solid.



2017

3. A conical tent is to accommodate 77 persons. Each person must have 16 m^3 of air to breathe. Given the radius of the tent as 7m. Find the height of the tent and also its curved surface area.

2016

4. A certain number of metallic cones, each of radius 2 cm and height 3 cm are melted and recast into a solid sphere of radius 6 cm. Find the number of cones.

2015

5. Two solid spheres of radii 2 cm and 4 cm are melted and recast into a cone of height 8 cm. Find the radius of the cone so formed.

2014

6. The surface area of a solid metallic sphere is 2464 cm^2. It is melted and recast into solid right circular cones of radius 3.5 cm and height 7 cm. Calculate
(i) the radius of the sphere. (ii) the number of cones recast. [Take $\pi = 22/7$]

2013

7. A solid sphere of radius 15 cm is melted and recast into solid right circular cones of radius 2.5 cm and height 8 cm. Calculate the number of cones recast.

2012

8. A hollow sphere of internal and external radii 6 cm and 8 cm respectively, is melted and recast into small cones of base radius 2 cm and height 8 cm. Find the number of cones.

2011

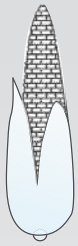
9. A solid cone of radius 5 cm and height 8 cm is melted and recast into small spheres of radius 0.5 cm. Find the number of spheres formed.

* All these questions are completely covered in chapter either as solved examples or in topic/chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1 A corn cob (see figure), shaped somewhat like a cone, has the radius of its broadest end as 2.1 cm and length as 20 cm. If each 1 cm^2 of the surface of the cob carries an average of four grains, then how many grains you would find on the entire cob? [Take $\pi = 22/7$]



(a) 132 (b) 530
(c) 531 (d) 513

2 A sphere of radius a units is immersed completely in water contained in a right circular cone of semi-vertical angle 30° and water is drained off from the cone till its surface touches the sphere. Then, the volume of water remaining in the cone will be

(a) $\frac{5\pi}{3}a^3$ (b) $5\pi a^3$ (c) $\frac{\pi a^3}{3}$ (d) $\frac{5}{3}\pi a^2$

Directions (Q. Nos. 3-4) *A semi-circle is drawn, which is passing through the points* $A(3, 0)$, $B(0, 3)$ *and* $C(-3, 0)$. [Take $\pi = 3.14$]

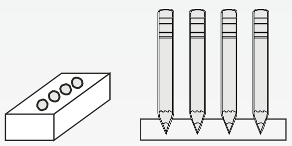
3 When we rotate this figure along the side AC, then write the name of the figure formed and then the space occupied by this will be

(a) 113.40 cm^3 (b) 113.04 cm^3
(c) 114.03 cm^3 (d) 11.304 cm^3

4 When we rotate this figure along the side BO, then write the name of the figure formed and then the space occupied by this will be

(a) 56.52 cm^3 (b) 52.56 cm^3
(c) 50.52 cm^3 (d) 55.50 cm^3

5 A pen-stand made of wood is in the shape of a cuboid with four conical depressions to hold pens. The dimensions of the cuboid are 15 cm × 10 cm × 3.5 cm. The radius of each of the depressions is 0.5 cm and the depth is 1.4 cm. Then, the volume of wood in the entire stand (see the figure). [Take $\pi = 22/7$]

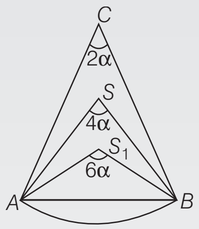


(a) 523.54 cm^3 (b) 523.45 cm^3
(c) 523 cm^3 (d) 532.45 cm^3

6 A hollow cone of radius 6 cm and height 8 cm is vertical standing at the origin, such that the vertex of the cone is at the origin. Some pipes are hanging around the circular base of the cone, such that they touch the surface of the graph paper. Then, the total surface area of the formed by the figure will be

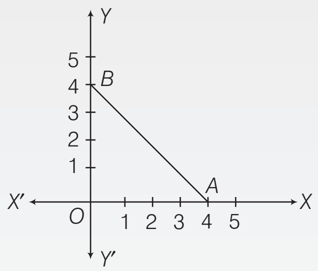
(a) 484.98 cm^2 (b) 489.84 cm^2
(c) 948.84 cm^2 (d) None of these

7 A cone made of paper has height $3h$ and vertical angle 2α. It contains two other cones of height $2h$ and h and vertical angles 4α and 6α, respectively. Then, the ratio of the volumes of regions S and S_1 will be



(a) $(8\tan^2\alpha - 27\tan^2 2\alpha) : (5\tan^2\alpha - \tan 3\alpha)$
(b) $(27\tan^2\alpha - 8\tan^2 2\alpha) : (8\tan^2 2\alpha - \tan^2 3\alpha)$
(c) $(25\tan^2\alpha - 7\tan^2 2\alpha) : (8\tan^2\alpha - \tan 3\alpha)$
(d) None of the above

8 A right isosceles triangle is shown below



When we rotate a right angled triangle about the vertex O, then write the name of the formed figure and then the curved surface area of the formed figure will be

(a) 284.15 sq units (b) 254.15 sq units
(c) 215.84 sq units (d) 215.54 sq units

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 413.

CHAPTER

18

Trigonometric Identities

In earlier class, we have already learnt about the various trigonometric ratios and their relations as well as trigonometric ratio of complementary angles. Now, we will establish some identities involving trigonometric ratios as well as trigonometric ratios of complementary angles.

Chapter Objectives

- Trigonometric Ratios
- Trigonometric Identities

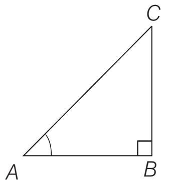
Trigonometric Ratios

The ratios of the sides of a right angled triangle with respect to its acute angles, are called **trigonometric ratios**. Trigonometric ratios are also called **T-ratios**.

To understand the trigonometric ratios, first of all we will understand the concept of perpendicular, base and hypotenuse in a right angled triangle.

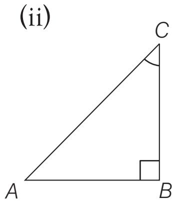
For any acute angle (which is also known as the **angle of reference**) in a right angled triangle, the side opposite to the acute angle is called the **perpendicular** (P), the side adjacent to this acute angle is called the **base** (B) and side opposite to the right angle is called the **hypotenuse** (H).

(i)



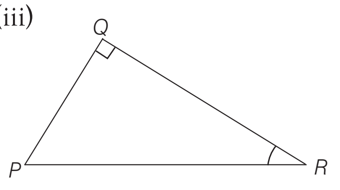
Angle of reference = $\angle A$
Perpendicular = BC
Base = AB
Hypotenuse = AC

(ii)



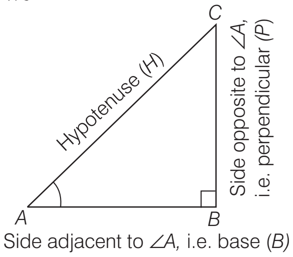
Angle of reference = $\angle C$
Perpendicular = AB
Base = BC
Hypotenuse = AC

(iii)



Angle of reference = $\angle R$
perpendicular = PQ
Base = QR
Hypotenuse = PR

Trigonometric ratios of $\angle A$ in right angled ΔABC are defined as follows



(i) sine of $\angle A$ or $\sin A = \dfrac{\text{Side opposite to } \angle A}{\text{Hypotenuse}} = \dfrac{P}{H} = \dfrac{BC}{AC}$

(ii) cosine of $\angle A$ or $\cos A = \dfrac{\text{Side adjacent to } \angle A}{\text{Hypotenuse}} = \dfrac{B}{H} = \dfrac{AB}{AC}$

(iii) tangent of $\angle A$ or $\tan A = \dfrac{\text{Side opposite to } \angle A}{\text{Side adjacent to } \angle A} = \dfrac{P}{B} = \dfrac{BC}{AB}$

(iv) cosecant of $\angle A$ or cosec A

$$= \frac{\text{Hypotenuse}}{\text{Side opposite to } \angle A} = \frac{H}{P} = \frac{AC}{BC}$$

(v) secant of $\angle A$ or $\sec A = \dfrac{\text{Hypotenuse}}{\text{Side adjacent to } \angle A} = \dfrac{H}{B} = \dfrac{AC}{AB}$

(vi) cotangent of $\angle A$ or $\cot A$

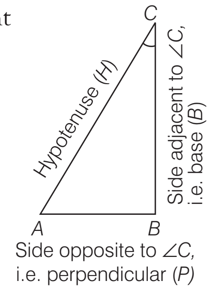
$$= \frac{\text{Side adjacent to } \angle A}{\text{Side opposite to } \angle A} = \frac{B}{P} = \frac{AB}{BC}$$

Similarly, trigonometric ratios of $\angle C$ in right angled ΔABC are as follows:

(i) $\sin C = \dfrac{AB}{AC}$ (ii) $\cos C = \dfrac{BC}{AC}$

(iii) $\tan C = \dfrac{AB}{BC}$ (iv) $\text{cosec } C = \dfrac{AC}{AB}$

(v) $\sec C = \dfrac{AC}{BC}$ (vi) $\cot C = \dfrac{BC}{AB}$



Note The symbol sin A is used as an abbreviation for 'the sine of $\angle A$.' sin A is not the product of 'sin' and 'A'. 'sin' separated from 'A' has no meaning. This interpretation follows for other trigonometric ratios also.

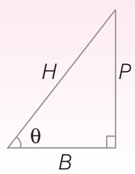
A Popular Technique to Remember T-Ratios

$\sin\theta\downarrow$	$\cos\theta\downarrow$	$\tan\theta\downarrow$
Pandit (P)	Badari (B)	Prasad (P)
Har (H)	Har (H)	Bholay (B)
$\text{cosec}\,\theta\uparrow$	$\sec\theta\uparrow$	$\cot\theta\uparrow$

Then, $\sin\theta = \dfrac{P}{H}$, $\cos\theta = \dfrac{B}{H}$, $\tan\theta = \dfrac{P}{B}$,

$\text{cosec}\,\theta = \dfrac{H}{P}$, $\sec\theta = \dfrac{H}{B}$, $\cot\theta = \dfrac{B}{P}$

where, P is perpendicular, B is base and H is hypotenuse.



Relations between Trigonometric Ratios

Reciprocal Relation

(i) $\sin\theta = \dfrac{1}{\text{cosec}\,\theta}$ or $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ or $\sin\theta \cdot \text{cosec}\,\theta = 1$

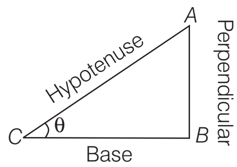
(ii) $\cos\theta = \dfrac{1}{\sec\theta}$ or $\sec\theta = \dfrac{1}{\cos\theta}$ or $\cos\theta \cdot \sec\theta = 1$

(iii) $\tan\theta = \dfrac{1}{\cot\theta}$ or $\cot\theta = \dfrac{1}{\tan\theta}$ or $\tan\theta \cdot \cot\theta = 1$

Quotient Relation

(i) $\tan\theta = \dfrac{AB}{BC} = \dfrac{AB/AC}{BC/AC} = \dfrac{\sin\theta}{\cos\theta}$

(ii) $\cot\theta = \dfrac{BC}{AB} = \dfrac{BC/AC}{AB/AC} = \dfrac{\cos\theta}{\sin\theta}$



Example 1. Prove that $\dfrac{\sec A - 1}{\sec A + 1} = \dfrac{1 - \cos A}{1 + \cos A}$.

Sol. LHS $= \dfrac{\sec A - 1}{\sec A + 1} = \dfrac{(1/\cos A) - 1}{(1/\cos A) + 1}$ $\left[\because \sec A = \dfrac{1}{\cos A}\right]$

$= \dfrac{1 - \cos A}{\cos A} \times \dfrac{\cos A}{1 + \cos A}$

$= \dfrac{1 - \cos A}{1 + \cos A} = \text{RHS}$ **Hence proved.**

Trigonometric Ratios of Complementary Angles

$\sin(90° - \theta) = \cos\theta$, $\cos(90° - \theta) = \sin\theta$

$\tan(90° - \theta) = \cot\theta$, $\text{cosec}(90° - \theta) = \sec\theta$

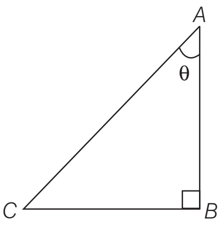
$\cot(90° - \theta) = \tan\theta$, $\sec(90° - \theta) = \text{cosec}\,\theta$

Values of Trigonometric Ratios for Some Specific Angles

Angles	0° or 0	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$	90° or $\frac{\pi}{2}$
$\sin\theta$	0	1/2	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞
$\text{cosec}\,\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞
$\cot\theta$	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0

Here, ∞ = Undefined [not defined]

Trigonometric Identities



We know that an equation is called **identity** when it is true for all values of the variables involved. Similarly, an equation involving trigonometric ratios of an angle is called a **trigonometric identity**, if it is true for all values of the angle(s) involved. For any acute angle θ, we have three identities

(i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $1 + \tan^2\theta = \sec^2\theta$ (iii) $\cot^2\theta + 1 = \text{cosec}^2\theta$

Note $\sin^2\theta = (\sin\theta)^2$ but $\sin\theta^2 \neq (\sin\theta)^2$. The same is true for all other trigonometric ratios.

Conversion of Trigonometric Ratios in Terms of Other Trigonometric Ratios

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\text{cosec}\,\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\text{cosec}\,\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\text{cosec}^2\theta-1}}{\text{cosec}\,\theta}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\text{cosec}^2\theta-1}}$
$\cot\theta$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\text{cosec}^2\theta-1}$
$\sec\theta$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\text{cosec}\,\theta}{\sqrt{\text{cosec}^2\theta-1}}$
$\text{cosec}\,\theta$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\text{cosec}\,\theta$

Example 2. Write all the other trigonometric ratios of $\angle A$ in terms of $\sec A$.

Sol.

(i) We know that, $\sin^2 A + \cos^2 A = 1$

$$\Rightarrow \sin^2 A = 1 - \cos^2 A$$

$$\Rightarrow \sin A = \sqrt{1-\cos^2 A} \quad \text{[taking positive square root]}$$

$$= \sqrt{1 - \frac{1}{\sec^2 A}} \quad \left[\because \cos A = \frac{1}{\sec A}\right]$$

$$= \sqrt{\frac{\sec^2 A - 1}{\sec^2 A}} = \frac{\sqrt{\sec^2 A - 1}}{\sec A}$$

(ii) $\cos A = \dfrac{1}{\sec A}$

(iii) We know that, $\sec^2 A - \tan^2 A = 1 \Rightarrow \tan^2 A = \sec^2 A - 1$

$$\therefore \quad \tan A = \sqrt{\sec^2 A - 1}$$

[taking positive square root]

(iv) $\cot A = \dfrac{1}{\tan A} = \dfrac{1}{\sqrt{\sec^2 A - 1}}$ [using part (iii)]

(v) $\text{cosec}\,A = \dfrac{1}{\sin A} = \dfrac{\sec A}{\sqrt{\sec^2 A - 1}}$ [using part (i)]

Example 3. If A is an acute angle and $\tan A = 12/5$, then, find all other trigonometric ratios of $\angle A$ using trigonometric identities.

Sol. Given, $\tan A = \dfrac{12}{5} \Rightarrow \cot A = \dfrac{1}{\tan A} = \dfrac{5}{12}$

Now, $\sec^2 A = 1 + \tan^2 A = 1 + \left(\dfrac{12}{5}\right)^2 = \dfrac{25+144}{25} = \dfrac{169}{25}$

$$\Rightarrow \quad \sec A = \frac{13}{5} \quad \text{[taking positive square root]}$$

$$\therefore \quad \cos A = \frac{1}{\sec A} = \frac{5}{13},$$

$$\sin A = \frac{\sin A}{\cos A}\cdot\cos A = \tan A\cos A = \frac{12}{5}\times\frac{5}{13} = \frac{12}{13},$$

$$\text{cosec}\,A = \frac{1}{\sin A} = \frac{13}{12}$$

Hence, $\sin A = \dfrac{12}{13}$, $\cos A = \dfrac{5}{13}$, $\cot A = \dfrac{5}{12}$,

$\sec A = \dfrac{13}{5}$, $\text{cosec}\,A = \dfrac{13}{12}$.

Example 4. Evaluate the following.

(i) $9\sec^2 A - 9\tan^2 A$

(ii) $(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \text{cosec}\,\theta)$

(iii) $(\sec A + \tan A)(1 - \sin A)$ ***[2018]***

Sol.

(i) We have, $9\sec^2 A - 9\tan^2 A = 9(\sec^2 A - \tan^2 A)$

$$= 9\times 1 = 9 \quad [\because \sec^2 A - \tan^2 A = 1]$$

(ii) We have, $(1+\tan\theta+\sec\theta)(1+\cot\theta-\text{cosec}\,\theta)$

$$= \left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)\left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)$$

$$\left[\because \tan\theta = \frac{\sin\theta}{\cos\theta}, \sec\theta = \frac{1}{\cos\theta}, \cot\theta = \frac{\cos\theta}{\sin\theta} \text{ and } \text{cosec}\,\theta = \frac{1}{\sin\theta}\right]$$

$$= \left[\frac{(\cos\theta + \sin\theta) + 1}{\cos\theta}\right] \times \left[\frac{(\sin\theta + \cos\theta) - 1}{\sin\theta}\right]$$

$$= \frac{(\cos\theta + \sin\theta)^2 - 1^2}{\cos\theta \sin\theta} \quad [\because (a+b)(a-b) = a^2 - b^2]$$

$$= \frac{\cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta - 1}{\cos\theta\sin\theta} \quad [\because (a+b)^2 = a^2 + b^2 + 2ab]$$

$$= \frac{1 + 2\cos\theta\sin\theta - 1}{\cos\theta\sin\theta} \quad [\because \cos^2 A + \sin^2 A = 1]$$

$$= \frac{2\cos\theta\sin\theta}{\cos\theta\sin\theta} = 2$$

(iii) We have, $(\sec A + \tan A)(1 - \sin A)$

$$= \left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)(1 - \sin A) \quad \left[\because \sec A = \frac{1}{\cos A} \text{ and } \tan A = \frac{\sin A}{\cos A}\right]$$

$$= \frac{(1 + \sin A)(1 - \sin A)}{\cos A} = \frac{1 - \sin^2 A}{\cos A} \quad [\because (a+b)(a-b) = a^2 - b^2]$$

$$= \frac{\cos^2 A}{\cos A} = \cos A \quad [\because \sin^2 A + \cos^2 A = 1 \Rightarrow \cos^2 A = 1 - \sin^2 A]$$

Example 5. Prove that

$$\sqrt{\frac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1}} + \sqrt{\frac{\operatorname{cosec} A + 1}{\operatorname{cosec} A - 1}} = 2\sec A.$$

Sol. LHS $= \sqrt{\dfrac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1}} + \sqrt{\dfrac{\operatorname{cosec} A + 1}{\operatorname{cosec} A - 1}}$

$$= \frac{(\sqrt{\operatorname{cosec} A - 1})(\sqrt{\operatorname{cosec} A - 1}) + (\sqrt{\operatorname{cosec} A + 1})(\sqrt{\operatorname{cosec} A + 1})}{(\sqrt{\operatorname{cosec} A + 1})(\sqrt{\operatorname{cosec} A - 1})}$$

$$= \frac{(\sqrt{\operatorname{cosec} A - 1})^2 + (\sqrt{\operatorname{cosec} A + 1})^2}{\sqrt{(\operatorname{cosec} A + 1)(\operatorname{cosec} A - 1)}}$$

$$= \frac{(\operatorname{cosec} A - 1) + (\operatorname{cosec} A + 1)}{\sqrt{\operatorname{cosec}^2 A - 1}} \quad [\because (a+b)(a-b) = a^2 - b^2]$$

$$= \frac{2\operatorname{cosec} A}{\sqrt{\cot^2 A}} \quad [\because \operatorname{cosec}^2 A = 1 + \cot^2 A \Rightarrow \operatorname{cosec}^2 A - 1 = \cot^2 A]$$

$$= \frac{2\operatorname{cosec} A}{\cot A} = \frac{2}{\sin A} \times \frac{\sin A}{\cos A} \quad \left[\because \operatorname{cosec} A = \frac{1}{\sin A} \text{ and } \cot A = \frac{\cos A}{\sin A}\right]$$

$$= \frac{2}{\cos A} = 2\sec A \quad \left[\because \frac{1}{\cos A} = \sec A\right]$$

= RHS **Hence proved.**

Example 6. Prove that

$$\frac{\sin A + \cos A}{\sin A - \cos A} + \frac{\sin A - \cos A}{\sin A + \cos A} = \frac{2}{\sin^2 A - \cos^2 A}.$$

Sol. LHS $= \dfrac{\sin A + \cos A}{\sin A - \cos A} + \dfrac{\sin A - \cos A}{\sin A + \cos A}$

$$= \frac{(\sin A + \cos A)^2 + (\sin A - \cos A)^2}{(\sin A - \cos A)(\sin A + \cos A)}$$

$$= \frac{\left[\begin{array}{c}\sin^2 A + 2\sin A\cos A + \cos^2 A + \sin^2 A \\ - 2\sin A\cos A + \cos^2 A\end{array}\right]}{\sin^2 A - \cos^2 A}$$

$[\because (a \pm b)^2 = a^2 \pm 2ab + b^2$ and $(a-b)(a+b) = a^2 - b^2]$

$$= \frac{2\sin^2 A + 2\cos^2 A}{\sin^2 A - \cos^2 A} = \frac{2(\sin^2 A + \cos^2 A)}{\sin^2 A - \cos^2 A}$$

$$= \frac{2}{\sin^2 A - \cos^2 A} \quad [\because \sin^2 A + \cos^2 A = 1]$$

= RHS **Hence proved.**

Example 7. Prove that $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{1}{\sec\theta - \tan\theta}$, using the identity $\sec^2\theta = 1 + \tan^2\theta$.

Sol. LHS $= \dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{\dfrac{\sin\theta}{\cos\theta} - \dfrac{\cos\theta}{\cos\theta} + \dfrac{1}{\cos\theta}}{\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\cos\theta} - \dfrac{1}{\cos\theta}}$

[dividing numerator and denominator by $\cos\theta$]

$$= \frac{\tan\theta - 1 + \sec\theta}{\tan\theta + 1 - \sec\theta} \quad \left[\because \tan A = \frac{\sin A}{\cos A} \text{ and } \frac{1}{\cos A} = \sec A\right]$$

$$= \frac{(\tan\theta + \sec\theta) - 1}{(\tan\theta - \sec\theta) + 1} = \frac{[(\tan\theta + \sec\theta) - 1][\tan\theta - \sec\theta]}{[(\tan\theta - \sec\theta) + 1][\tan\theta - \sec\theta]}$$

[multiplying and dividing by $(\tan\theta - \sec\theta)$]

$$= \frac{(\tan\theta + \sec\theta)(\tan\theta - \sec\theta) - (\tan\theta - \sec\theta)}{(\tan\theta - \sec\theta - 1)(\tan\theta - \sec\theta)}$$

$$= \frac{(\tan^2\theta - \sec^2\theta) - (\tan\theta - \sec\theta)}{(\tan\theta - \sec\theta + 1)(\tan\theta - \sec\theta)} \quad [\because (a+b)(a-b) = a^2 - b^2]$$

$$= \frac{-1 - \tan\theta + \sec\theta}{(\tan\theta - \sec\theta + 1)(\tan\theta - \sec\theta)} \quad [\because \tan^2\theta - \sec^2\theta = -1]$$

$$= \frac{-(\tan\theta - \sec\theta + 1)}{(\tan\theta - \sec\theta + 1)(\tan\theta - \sec\theta)}$$

$$= \frac{-1}{\tan\theta - \sec\theta}$$

$$= \frac{1}{\sec\theta - \tan\theta} = \text{RHS}$$ **Hence proved.**

Example 8. Show that $\sqrt{\dfrac{1 - \cos A}{1 + \cos A}} = \dfrac{\sin A}{1 + \cos A}$.

[2013, 2000]

Sol. LHS $= \sqrt{\dfrac{1 - \cos A}{1 + \cos A}} \times \sqrt{\dfrac{1 + \cos A}{1 + \cos A}}$

[multiplying numerator and denominator by $\sqrt{1 + \cos A}$]

$$= \sqrt{\frac{(1 - \cos A)(1 + \cos A)}{(1 + \cos A)^2}} = \frac{\sqrt{1^2 - \cos^2 A}}{1 + \cos A} \quad [\because (a-b)(a+b) = a^2 - b^2]$$

$$= \frac{\sqrt{\sin^2 A}}{1 + \cos A} \quad [\because \sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2\theta = 1 - \cos^2\theta]$$

$$= \frac{\sin A}{1 + \cos A} = \text{RHS}$$

Example 9. Prove the following identities.

(i) $\sqrt{\sec^2\theta + \text{cosec}^2\theta} = \tan\theta + \cot\theta$ ***[2018]***

(ii) $\sec A(1-\sin A)(\sec A + \tan A) = 1$

Sol.

(i) $\text{LHS} = \sqrt{\sec^2\theta + \text{cosec}^2\theta}$

$= \sqrt{1+\tan^2\theta + 1 + \cot^2\theta}$

$[\because 1+\tan^2\theta = \sec^2\theta \text{ and } 1+\cot^2\theta = \text{cosec}^2\theta]$

$= \sqrt{\tan^2\theta + \cot^2\theta + 2}$

$= \sqrt{\tan^2\theta + \cot^2\theta + 2\tan\theta\cot\theta}$ $\quad [\because \tan\theta\cdot\cot\theta = 1]$

$= \sqrt{(\tan\theta + \cot\theta)^2}$ $\quad [\because a^2 + b^2 + 2ab = (a+b)^2]$

$= \tan\theta + \cot\theta$

$=$ RHS **Hence proved.**

(ii) $\text{LHS} = \sec A(1-\sin A)(\sec A + \tan A)$

$= \left(\dfrac{1}{\cos A}\right)(1-\sin A)\left(\dfrac{1}{\cos A} + \dfrac{\sin A}{\cos A}\right)$

$= \dfrac{(1-\sin A)(1+\sin A)}{\cos^2 A} = \dfrac{1-\sin^2 A}{\cos^2 A} = \dfrac{\cos^2 A}{\cos^2 A}$

$[\because 1-\sin^2 A = \cos^2 A]$

$= 1 =$ RHS **Hence proved.**

Example 10. Prove that $\dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta.$ ***[2017]***

Sol. $\text{LHS} = \dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \dfrac{\sin\theta(1-2\sin^2\theta)}{\cos\theta\,(2\cos^2\theta - 1)}$ $\quad [\because \sin^2\theta = 1-\cos^2\theta]$

$= \dfrac{\sin\theta[1-2(1-\cos^2\theta)]}{\cos\theta\,(2\cos^2\theta - 1)}$ $\quad [\because 1-\sin^2\theta = \cos^2\theta]$

$= \dfrac{\sin\theta\,(1-2+2\cos^2\theta)}{\cos\theta\,(2\cos^2\theta-1)} = \dfrac{\sin\theta(2\cos^2\theta-1)}{\cos\theta\,(2\cos^2\theta-1)} = \dfrac{\sin\theta}{\cos\theta}$

$= \tan\theta =$ RHS **Hence proved.**

Example 11. Prove the following identities, where the angles involved are acute angles for which the expressions are defined.

(i) $(\text{cosec}\,\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

(ii) $\dfrac{\cos A}{1+\sin A} + \dfrac{1+\sin A}{\cos A} = 2\sec A$

Sol.

(i) $\text{LHS} = (\text{cosec}\,\theta - \cot\theta)^2$

$= \left(\dfrac{1}{\sin\theta} - \dfrac{\cos\theta}{\sin\theta}\right)^2$ $\quad \left[\because \text{cosec}\,\theta = \dfrac{1}{\sin\theta},\ \cot\theta = \dfrac{\cos\theta}{\sin\theta}\right]$

$= \left(\dfrac{1-\cos\theta}{\sin\theta}\right)^2 = \dfrac{(1-\cos\theta)^2}{\sin^2\theta} = \dfrac{(1-\cos\theta)^2}{1-\cos^2\theta}$

$[\because \sin^2\theta = 1-\cos^2\theta]$

$= \dfrac{(1-\cos\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$ $\quad [\because a^2 - b^2 = (a+b)(a-b)]$

$= \dfrac{1-\cos\theta}{1+\cos\theta} =$ RHS **Hence proved.**

(ii) $\text{LHS} = \dfrac{\cos A}{1+\sin A} + \dfrac{1+\sin A}{\cos A} = \dfrac{\cos^2 A + (1+\sin A)^2}{(1+\sin A)\cos A}$

$= \dfrac{\cos^2 A + \sin^2 A + 1 + 2\sin A}{(1+\sin A)\cos A}$

$[\because (a+b)^2 = a^2 + b^2 + 2ab]$

$= \dfrac{1+1+2\sin A}{(1+\sin A)\cos A}$ $\quad [\because \sin^2 A + \cos^2 A = 1]$

$= \dfrac{2+2\sin A}{(1+\sin A)\cos A} = \dfrac{2(1+\sin A)}{(1+\sin A)\cos A}$

$= \dfrac{2}{\cos A} = 2\sec A$ $\quad \left[\because \dfrac{1}{\cos A} = \sec A\right]$

$=$ RHS **Hence proved.**

Example 12. Show that

$\dfrac{\cos A - \sin A + 1}{\cos A + \sin A - 1} = \text{cosec}\,A + \cot A$, using the identity $\text{cosec}^2 A = 1 + \cot^2 A$.

Sol. $\text{LHS} = \dfrac{\cos A - \sin A + 1}{\cos A + \sin A - 1} = \dfrac{\dfrac{\cos A}{\sin A} - \dfrac{\sin A}{\sin A} + \dfrac{1}{\sin A}}{\dfrac{\cos A}{\sin A} + \dfrac{\sin A}{\sin A} - \dfrac{1}{\sin A}}$

[dividing of numerator and denominator by $\sin A$]

$= \dfrac{\cot A - 1 + \text{cosec}\,A}{\cot A + 1 - \text{cosec}\,A}$ $\quad \left[\because \dfrac{\cos A}{\sin A} = \cot A \text{ and } \dfrac{1}{\sin A} = \text{cosec}\,A\right]$

$= \dfrac{\cot A + \text{cosec}\,A - 1}{\cot A + 1 - \text{cosec}\,A} = \dfrac{(\cot A + \text{cosec}\,A) - (\text{cosec}^2 A - \cot^2 A)}{\cot A + 1 - \text{cosec}\,A}$

$[\because 1 = \text{cosec}^2 A - \cot^2 A]$

$= \dfrac{(\cot A + \text{cosec}\,A) - [(\text{cosec}\,A + \cot A)(\text{cosec}\,A - \cot A)]}{\cot A + 1 - \text{cosec}\,A}$

$[\because a^2 - b^2 = (a+b)(a-b)]$

$= \dfrac{(\cot A + \text{cosec}\,A)[1 - (\text{cosec}\,A - \cot A)]}{\cot A + 1 - \text{cosec}\,A}$

$= \dfrac{(\cot A + \text{cosec}\,A)(1 - \text{cosec}\,A + \cot A)}{(\cot A + 1 - \text{cosec}\,A)}$

$= \cot A + \text{cosec}\,A =$ RHS

Example 13. Evaluate the following in terms of $\tan A$ and $\cot A$.

(i) $(\sin A + \text{cosec}\,A)^2 + (\cos A + \sec A)^2$

(ii) $(\text{cosec}\,A - \sin A)(\sec A - \cos A)$

Sol.

(i) We have, $(\sin A + \text{cosec}\,A)^2 + (\cos A + \sec A)^2$

$= \sin^2 A + \text{cosec}^2 A + 2\sin A\,\text{cosec}\,A + \cos^2 A + \sec^2 A + 2\cos A\sec A$

$[\because (a+b)^2 = a^2 + b^2 + 2ab]$

$= (\sin^2 A + \cos^2 A) + (1 + \cot^2 A) + 2\sin A\dfrac{1}{\sin A} + (1+\tan^2 A) + 2\cos A\dfrac{1}{\cos A}$

$\left[\because \text{cosec}^2 A = 1 + \cot^2 A,\ \sec^2 A = 1 + \tan^2 A,\ \text{cosec}\,A = \dfrac{1}{\sin A} \text{ and } \sec A = \dfrac{1}{\cos A}\right]$

$= 1 + 1 + \cot^2 A + 2 + 1 + \tan^2 A + 2$ $\quad [\because \sin^2 A + \cos^2 A = 1]$

$= 7 + \tan^2 A + \cot^2 A$

(ii) We have, $(\operatorname{cosec} A - \sin A)(\sec A - \cos A)$

$$= \left(\frac{1}{\sin A} - \frac{\sin A}{1}\right)\left(\frac{1}{\cos A} - \frac{\cos A}{1}\right) \quad \left[\because \operatorname{cosec} A = \frac{1}{\sin A}, \sec A = \frac{1}{\cos A}\right]$$

$$= \left(\frac{1-\sin^2 A}{\sin A}\right)\left(\frac{1-\cos^2 A}{\cos A}\right) = \frac{\cos^2 A}{\sin A} \times \frac{\sin^2 A}{\cos A}$$

$$= \cos A \sin A \qquad \text{...(i)}$$

$$= \frac{1}{\dfrac{1}{\cos A \sin A}} = \frac{1}{\left(\dfrac{\sin^2 A + \cos^2 A}{\cos A \sin A}\right)} = \frac{1}{\left(\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}\right)}$$

$$\left[\because \tan A = \frac{\sin A}{\cos A}, \cot A = \frac{\cos A}{\sin A}\right]$$

$$= \frac{1}{\tan A + \cot A} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

Example 14. Evaluate the following in terms of $\sin A$.

(i) $(1 + \tan^2 A) + \left(1 + \dfrac{1}{\tan^2 A}\right)$

(ii) $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1}$

Sol.

(i) We have, $(1+\tan^2 A) + \left(1 + \dfrac{1}{\tan^2 A}\right)$

$$= \sec^2 A + (1 + \cot^2 A) \quad \left[\because 1 + \tan^2 A = \sec^2 A, \frac{1}{\tan A} = \cot A\right]$$

$$= \sec^2 A + \operatorname{cosec}^2 A \qquad [\because 1 + \cot^2 A = \operatorname{cosec}^2 A]$$

$$= \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} = \frac{\sin^2 A + \cos^2 A}{\cos^2 A \sin^2 A} \quad \left[\because \sec A = \frac{1}{\cos A}, \ \operatorname{cosec} A = \frac{1}{\sin A}\right]$$

$$= \frac{1}{\cos^2 A \sin^2 A} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= \frac{1}{(1-\sin^2 A)\sin^2 A}$$

$$= \frac{1}{\sin^2 A - \sin^4 A}$$

(ii) We have, $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{\tan A + \sec A - (\sec^2 A - \tan^2 A)}{\tan A - \sec A + 1}$ $\quad [\because \sec^2 A - \tan^2 A = 1]$

$$= \frac{(\tan A + \sec A) - (\sec A + \tan A)(\sec A - \tan A)}{\tan A - \sec A + 1} \quad [\because a^2 - b^2 = (a+b)(a-b)]$$

$$= \frac{(\tan A + \sec A)(1 - \sec A + \tan A)}{\tan A - \sec A + 1}$$

$$= \tan A + \sec A$$

$$= \frac{\sin A}{\cos A} + \frac{1}{\cos A} \quad \left[\because \tan A = \frac{\sin A}{\cos A}, \ \sec A = \frac{1}{\cos A}\right]$$

$$= \frac{\sin A + 1}{\cos A} = \frac{\sin A + 1}{\sqrt{1 - \sin^2 A}} \qquad [\because \cos^2\theta = 1 - \sin^2\theta]$$

Alternate Method

We have, $\dfrac{\dfrac{\sin A}{\cos A} + \dfrac{1}{\cos A} - 1}{\dfrac{\sin A}{\cos A} - \dfrac{1}{\cos A} + 1} = \dfrac{\dfrac{\sin A + 1 - \cos A}{\cos A}}{\dfrac{\sin A - 1 + \cos A}{\cos A}}$

$$= \frac{\sin A + 1 - \cos A}{\sin A - 1 + \cos A} = \frac{\sin A + 1 - \cos A}{\sin A - 1 + \cos A} \times \frac{1 + \sin A}{1 + \sin A}$$

[multiplying and dividing by $(1 + \sin A)$]

$$= \frac{(\sin A + 1 - \cos A)(1 + \sin A)}{\sin A - 1 + \cos A + \sin^2 A - \sin A + \sin A \cos A}$$

$$= \frac{(\sin A + 1 - \cos A)(1 + \sin A)}{-1 + \cos A + (1 - \cos^2 A) + \sin A \cos A} \qquad [\because \sin^2 A = 1 - \cos^2 A]$$

$$= \frac{(\sin A + 1 - \cos A)(1 + \sin A)}{\cos A - \cos^2 A + \sin A \cos A}$$

$$= \frac{(\sin A + 1 - \cos A)(1 + \sin A)}{\cos A(1 - \cos A + \sin A)}$$

$$= \frac{1 + \sin A}{\cos A} = \frac{\sin A + 1}{\sqrt{1 - \sin^2 A}}$$

Example 15. Convert the following in terms of $\sec\theta$ and $\operatorname{cosec}\theta$.

(i) $(\sin\theta + \cos\theta)(\tan\theta + \cot\theta)$ ***[2014]***

(ii) $\dfrac{\sin\theta \tan\theta}{1 - \cos\theta}$ ***[2006]***

Sol.

(i) We have, $(\sin\theta + \cos\theta)(\tan\theta + \cot\theta)$

$$= (\sin\theta + \cos\theta)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= (\sin\theta + \cos\theta)\frac{\sin^2\theta + \cos^2\theta}{\cos\theta \sin\theta}$$

$$= (\sin\theta + \cos\theta) \times \frac{1}{\cos\theta\sin\theta} = \frac{\cos\theta + \sin\theta}{\cos\theta\sin\theta}$$

$$[\because \sin^2 A + \cos^2 A = 1]$$

$$= \frac{\cos\theta}{\cos\theta\sin\theta} + \frac{\sin\theta}{\cos\theta\sin\theta} = \frac{1}{\sin\theta} + \frac{1}{\cos\theta}$$

$$= \operatorname{cosec}\theta + \sec\theta$$

(ii) We have, $\dfrac{\sin\theta\tan\theta}{1 - \cos\theta} = \dfrac{\sin\theta\tan\theta}{1 - \cos\theta} \times \dfrac{1 + \cos\theta}{1 + \cos\theta}$

$$= \frac{\sin\theta\tan\theta(1 + \cos\theta)}{1 - \cos^2\theta}$$

$$= \frac{\sin\theta\tan\theta(1 + \cos\theta)}{\sin^2\theta} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= \frac{\tan\theta(1 + \cos\theta)}{\sin\theta} = \frac{\dfrac{\sin\theta}{\cos\theta} \times (1 + \cos\theta)}{\sin\theta}$$

$$= \frac{1 + \cos\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\cos\theta}{\cos\theta} = \sec\theta + 1$$

Example 16. Prove the following.

(i) $(\sec A - \cos A)(\tan A + \cot A) = \tan A \sec A$

(ii) $(\sin\theta + \cos\theta + 1)(\sin\theta - 1 + \cos\theta)\sec\theta\,\mathrm{cosec}\,\theta = 2$

Sol.

(i) LHS $= (\sec A - \cos A)(\tan A + \cot A)$

$$= \left(\frac{1}{\cos A} - \cos A\right)\left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)$$

$$= \left(\frac{1-\cos^2 A}{\cos A}\right)\cdot\left(\frac{\sin^2 A + \cos^2 A}{\cos A \sin A}\right) = \frac{\sin^2 A}{\cos A}\cdot\frac{1}{\cos A \sin A}$$

$[\because \sin^2 A + \cos^2 A = 1]$

$$= \frac{\sin A}{\cos^2 A} = \frac{\sin A}{\cos A}\cdot\frac{1}{\cos A} = \tan A \sec A = \text{RHS}$$

Hence proved.

(ii) LHS $= (\sin\theta + \cos\theta + 1)(\sin\theta - 1 + \cos\theta)\sec\theta\,\mathrm{cosec}\,\theta$

$= (\sin\theta + \cos\theta + 1)(-1)\sec\theta\,\mathrm{cosec}\,\theta$

$= [(\sin\theta + \cos\theta)^2 - 1^2]\sec\theta\,\mathrm{cosec}\,\theta$

$[\because (a+b)(a-b) = a^2 - b^2]$

$= (\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1)\sec\theta\,\mathrm{cosec}\,\theta$

$[\because (a+b)^2 = a^2 + b^2 + 2ab]$

$= (1 + 2\sin\theta\cos\theta - 1)\sec\theta\,\mathrm{cosec}\,\theta$

$[\because \sin^2 A + \cos^2 A = 1]$

$$= 2\sin\theta\cos\theta \times \frac{1}{\cos\theta} \times \frac{1}{\sin\theta} = 2 = \text{RHS}$$

Hence proved.

Example 17. Convert the following trigonometric identities into unity.

(i) $[\sin\theta(1+\tan\theta) + \cos\theta(1+\cot\theta)] \div (\sec\theta + \mathrm{cosec}\,\theta)$

(ii) $(1 + \cot\theta - \mathrm{cosec}\,\theta)(1 + \tan\theta + \sec\theta) \div 2$

Sol.

(i) We have, $[\sin\theta(1+\tan\theta) + \cos\theta(1+\cot\theta)] \div (\sec\theta + \mathrm{cosec}\,\theta)$

$$= \left[\sin\theta\left(1 + \frac{\sin\theta}{\cos\theta}\right) + \cos\theta\left(1 + \frac{\cos\theta}{\sin\theta}\right)\right] \times \left[\frac{1}{\sec\theta + \mathrm{cosec}\,\theta}\right]$$

$$= \left[\sin\theta\left(\frac{\cos\theta + \sin\theta}{\cos\theta}\right) + \cos\theta\left(\frac{\sin\theta + \cos\theta}{\sin\theta}\right)\right] \times \left[\frac{1}{\sec\theta + \mathrm{cosec}\,\theta}\right]$$

$$= (\cos\theta + \sin\theta)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) \times \frac{1}{(\sec\theta + \mathrm{cosec}\,\theta)}$$

$$= \frac{(\cos\theta + \sin\theta)}{(\sec\theta + \mathrm{cosec}\,\theta)}\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right)$$

$$= \frac{(\cos\theta + \sin\theta)}{(\sec\theta + \mathrm{cosec}\,\theta)}\left(\frac{1}{\cos\theta\sin\theta}\right) \quad [\because \sin^2 A + \cos^2 A = 1]$$

$$= \left[\frac{\cos\theta}{\cos\theta\sin\theta} + \frac{\sin\theta}{\cos\theta\sin\theta}\right] \times \left[\frac{1}{\sec\theta + \mathrm{cosec}\,\theta}\right]$$

$$= \frac{(\mathrm{cosec}\,\theta + \sec\theta)}{\mathrm{cosec}\,\theta + \sec\theta} = 1$$

(ii) We have, $\dfrac{(1 + \cot\theta - \mathrm{cosec}\,\theta)(1 + \tan\theta + \sec\theta)}{2}$

$$= \frac{1}{2}\left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)\left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)$$

$$= \frac{1}{2}\left(\frac{\sin\theta + \cos\theta - 1}{\sin\theta}\right)\left(\frac{\cos\theta + \sin\theta + 1}{\cos\theta}\right)$$

$$= \frac{(\sin\theta + \cos\theta)^2 - 1}{2\sin\theta\cos\theta} \quad [\because (a-b)(a+b) = a^2 - b^2]$$

$$= \frac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1}{2\sin\theta\cos\theta}$$

$[\because (a+b)^2 = a^2 + b^2 + 2ab]$

$$= \frac{1 + 2\sin\theta\cos\theta - 1}{2\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta}{2\sin\theta\cos\theta} = 1$$

Example 18. Prove the following identities.

(i) $\dfrac{\sec A + \tan A}{\mathrm{cosec}\,A + \cot A} = \dfrac{\mathrm{cosec}\,A - \cot A}{\sec A - \tan A}$

(ii) $\dfrac{\tan A}{1 - \cot A} + \dfrac{\cot A}{1 - \tan A} = 1 + \tan A + \cot A$

Sol.

(i) $\dfrac{\sec A + \tan A}{\mathrm{cosec}\,A + \cot A} = \dfrac{\mathrm{cosec}\,A - \cot A}{\sec A - \tan A}$ is true,

if $(\sec A + \tan A)(\sec A - \tan A) = (\mathrm{cosec}\,A + \cot A)(\mathrm{cosec}\,A - \cot A)$ is true

i.e. if $\sec^2 A - \tan^2 A = \mathrm{cosec}^2 A - \cot^2 A$ is true

i.e. if $(1 + \tan^2 A) - \tan^2 A = (1 + \cot^2 A) - \cot^2 A$ is true

i.e. if $1 = 1$ is true, which is true.

(ii) LHS $= \dfrac{\tan A}{1 - \cot A} + \dfrac{\cot A}{1 - \tan A} = \dfrac{\frac{\sin A}{\cos A}}{1 - \frac{\cos A}{\sin A}} + \dfrac{\frac{\cos A}{\sin A}}{1 - \frac{\sin A}{\cos A}}$

$$= \frac{\sin^2 A}{\cos A(\sin A - \cos A)} + \frac{\cos^2 A}{\sin A(\cos A - \sin A)}$$

$$= \frac{1}{\sin A - \cos A}\left(\frac{\sin^2 A}{\cos A} - \frac{\cos^2 A}{\sin A}\right)$$

$$= \frac{1}{\sin A - \cos A}\cdot\frac{\sin^3 A - \cos^3 A}{\cos A \sin A}$$

$$= \frac{(\sin A - \cos A)(\sin^2 A + \sin A\cos A + \cos^2 A)}{(\sin A - \cos A)\sin A\cos A}$$

$[\because a^3 - b^3 = (a-b)(a^2 + b^2 + ab)]$

$$= \frac{\sin^2 A + \sin A\cos A + \cos^2 A}{\sin A\cos A}$$

$$= \frac{\sin^2 A}{\sin A\cos A} + \frac{\sin A\cos A}{\sin A\cos A} + \frac{\cos^2 A}{\sin A\cos A}$$

$$= \frac{\sin A}{\cos A} + 1 + \frac{\cos A}{\sin A}$$

$= \tan A + 1 + \cot A$

$= 1 + \tan A + \cot A = \text{RHS}$ **Hence proved.**

Example 19. Prove the following identities.

(i) $(\sec A - \sin A)(\text{cosec } A + \cos A) = \sin^2 A \tan A + \cot A$

(ii) $(1 + \cot A + \tan A)(\sin A - \cos A) = \dfrac{\sec A}{\text{cosec}^2 A} - \dfrac{\text{cosec } A}{\sec^2 A}$

Sol.

(i) $\text{LHS} = (\sec A - \sin A)(\text{cosec } A + \cos A)$

$$= \left(\frac{1}{\cos A} - \sin A\right)\left(\frac{1}{\sin A} + \cos A\right)$$

$$= \left(\frac{1 - \sin A \cos A}{\cos A}\right)\left(\frac{1 + \sin A \cos A}{\sin A}\right)$$

$$= \frac{(1 - \sin^2 A \cos^2 A)}{\sin A \cos A} \qquad [\because (a-b)(a+b) = a^2 - b^2]$$

$$= \frac{\sin^2 A + \cos^2 A - \sin^2 A \cos^2 A}{\sin A \cos A} \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{\sin^2 A - \sin^2 A \cos^2 A + \cos^2 A}{\sin A \cos A}$$

$$= \frac{\sin^2 A(1 - \cos^2 A) + \cos^2 A}{\sin A \cos A}$$

$$= \frac{\sin^2 A \sin^2 A + \cos^2 A}{\sin A \cos A} = \frac{\sin^4 A + \cos^2 A}{\sin A \cos A}$$

$$= \frac{\sin^4 A}{\sin A \cos A} + \frac{\cos^2 A}{\sin A \cos A} = \frac{\sin^3 A}{\cos A} + \frac{\cos A}{\sin A}$$

$$= \sin^2 A \cdot \frac{\sin A}{\cos A} + \cot A$$

$= \sin^2 A \cdot \tan A + \cot A = \text{RHS}$ **Hence proved.**

(ii) $\text{LHS} = (1 + \cot A + \tan A)(\sin A - \cos A)$

$$= \left(1 + \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}\right)(\sin A - \cos A)$$

$$= \left(\frac{\sin A \cos A + \cos^2 A + \sin^2 A}{\sin A \cos A}\right)(\sin A - \cos A)$$

$$= \frac{(\sin A - \cos A)(\sin^2 A + \cos^2 A + \sin A \cos A)}{\sin A \cos A}$$

$$= \frac{\sin^3 A - \cos^3 A}{\sin A \cos A} \qquad [\because (a-b)(a^2 + b^2 + ab) = a^3 - b^3] \quad \text{...(i)}$$

and $$\text{RHS} = \frac{\sec A}{\text{cosec}^2 A} - \frac{\text{cosec } A}{\sec^2 A} = \frac{\frac{1}{\cos A}}{\frac{1}{\sin^2 A}} - \frac{\frac{1}{\sin A}}{\frac{1}{\cos^2 A}}$$

$$= \frac{\sin^2 A}{\cos A} - \frac{\cos^2 A}{\sin A} = \frac{\sin^3 A - \cos^3 A}{\cos A \sin A} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

LHS = RHS **Hence proved.**

Example 20. Prove the following identities.

(i) $\tan^2\theta + \cot^2\theta + 2 = \sec^2\theta \text{ cosec}^2\theta$

(ii) $\sqrt{\sec^2\theta + \text{cosec}^2\theta} = \tan\theta + \cot\theta$

Sol.

(i) $\text{LHS} = \tan^2\theta + \cot^2\theta + 2 = (1 + \tan^2\theta) + (1 + \cot^2\theta)$

$= \sec^2\theta + \text{cosec}^2\theta$

$[\because 1 + \tan^2 A = \sec^2 A \text{ and } 1 + \cot^2 A = \text{cosec}^2 A]$

$$= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta \sin^2\theta}$$

$$= \frac{1}{\cos^2\theta \sin^2\theta} = \frac{1}{\cos^2\theta} \cdot \frac{1}{\sin^2\theta} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$= \sec^2\theta \text{ cosec}^2\theta = \text{RHS}$ **Hence proved.**

(ii) $\text{LHS} = \sqrt{\sec^2\theta + \text{cosec}^2\theta} = \sqrt{(1 + \tan^2\theta) + (1 + \cot^2\theta)}$

$= \sqrt{\tan^2\theta + \cot^2\theta + 2}$

$= \sqrt{\tan^2\theta + \cot^2\theta + 2\tan\theta\cot\theta} \qquad [\because \tan\theta \cot\theta = 1]$

$= \sqrt{(\tan\theta + \cot\theta)^2}$

$= \tan\theta + \cot\theta = \text{RHS}$ **Hence proved.**

Example 21. Prove the following identities.

(i) $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta \cos^2\theta$

(ii) $\dfrac{1}{\text{cosec }\theta - \cot\theta} - \dfrac{1}{\sin\theta} = \dfrac{1}{\sin\theta} - \dfrac{1}{\text{cosec }\theta + \cot\theta}$

Sol.

(i) $\text{LHS} = \sin^4\theta + \cos^4\theta = (\sin^2\theta)^2 + (\cos^2\theta)^2$

$= (\sin^2\theta + \cos^2\theta)^2 - 2 \cdot \sin^2\theta \cos^2\theta$

$[\because a^2 + b^2 = (a+b)^2 - 2ab]$

$= 1^2 - 2\sin^2\theta\cos^2\theta = 1 - 2\sin^2\theta\cos^2\theta$

$[\because \sin^2 A + \cos^2 A = 1]$

$= \text{RHS}$ **Hence proved.**

(ii) $\dfrac{1}{\text{cosec }\theta - \cot\theta} - \dfrac{1}{\sin\theta} = \dfrac{1}{\sin\theta} - \dfrac{1}{\text{cosec }\theta + \cot\theta}$ is true

if $\dfrac{1}{\text{cosec }\theta - \cot\theta} + \dfrac{1}{\text{cosec }\theta + \cot\theta} = \dfrac{1}{\sin\theta} + \dfrac{1}{\sin\theta}$ is true

i.e. if $\dfrac{(\text{cosec }\theta + \cot\theta) + (\text{cosec }\theta - \cot\theta)}{(\text{cosec }\theta - \cot\theta)(\text{cosec }\theta + \cot\theta)} = \dfrac{2}{\sin\theta}$ is true

i.e. if $\dfrac{2\text{ cosec }\theta}{\text{cosec}^2\theta - \cot^2\theta} = 2\text{ cosec }\theta$ is true

i.e. if $\dfrac{2\text{ cosec }\theta}{1} = 2\text{ cosec }\theta$ is true. $[\because \text{cosec}^2\theta - \cot^2\theta = 1]$

which is true. **Hence proved.**

Example 22. Prove the following identities.

(i) $\dfrac{1 + \tan^2\theta}{1 + \cot^2\theta} = \left(\dfrac{1 - \tan\theta}{1 - \cot\theta}\right)^2 = \tan^2\theta$

(ii) $\sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta = 1$

Sol.

(i) We have, $\dfrac{1 + \tan^2\theta}{1 + \cot^2\theta} = \dfrac{\sec^2\theta}{\text{cosec}^2\theta} = \dfrac{1/\cos^2\theta}{1/\sin^2\theta}$

$$= \frac{1}{\cos^2\theta} \times \frac{\sin^2\theta}{1} = \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \tan^2\theta$$

Also, $$\left(\frac{1 - \tan\theta}{1 - \cot\theta}\right)^2 = \left(\frac{1 - \frac{\sin\theta}{\cos\theta}}{1 - \frac{\cos\theta}{\sin\theta}}\right)^2 = \left(\frac{\frac{\cos\theta - \sin\theta}{\cos\theta}}{\frac{\sin\theta - \cos\theta}{\sin\theta}}\right)$$

$= \left(-\frac{\sin\theta - \cos\theta}{\cos\theta} \times \frac{\sin\theta}{\sin\theta - \cos\theta}\right)^2 = \left(-\frac{\sin\theta}{\cos\theta}\right)^2$

$= (-\tan\theta)^2 = \tan^2\theta$

$\therefore \quad \frac{1+\tan^2\theta}{1+\cot^2\theta} = \left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2 = \tan^2\theta$ **Hence proved.**

(ii) LHS $= \sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta$

$= (\sin^2\theta)^3 + (\cos^2\theta)^3 + 3\sin^2\theta\cos^2\theta$

$= (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta) + 3\sin^2\theta\cos^2\theta$

$[\because a^3 + b^3 = (a+b)^3 - 3ab(a+b)]$

$= (1)^3 - 3\sin^2\theta\cos^2\theta(1) + 3\sin^2\theta\cos^2\theta$ $[\because \sin^2 A + \cos^2 A = 1]$

$= 1 - 3\sin^2\theta\cos^2\theta + 3\sin^2\theta\cos^2\theta = 1 =$ RHS

Hence proved.

Example 23. If $1 + \sin^2\theta = 3\sin\theta\cos\theta$, then prove that $\tan\theta = 1$ or $1/2$.

Sol. Given, $1 + \sin^2\theta = 3\sin\theta\cos\theta$

On dividing both sides by $\cos^2\theta$, we get

$\frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = 3\frac{\sin\theta\cos\theta}{\cos^2\theta}$

$\Rightarrow \sec^2\theta + \tan^2\theta = 3\tan\theta$

$\Rightarrow (1+\tan^2\theta) + \tan^2\theta = 3\tan\theta$ $[\because \sec^2 A = 1 + \tan^2 A]$

$\Rightarrow 2\tan^2\theta - 3\tan\theta + 1 = 0$

$\Rightarrow 2\tan^2\theta - 2\tan\theta - \tan\theta + 1 = 0$

[by factorisation method]

$\Rightarrow 2\tan\theta(\tan\theta - 1) - 1(\tan\theta - 1) = 0$

$\Rightarrow (\tan\theta - 1)(2\tan\theta - 1) = 0$

$\Rightarrow \tan\theta - 1 = 0$ or $2\tan\theta - 1 = 0$

$\Rightarrow \tan\theta = 1$ or $\tan\theta = \frac{1}{2}$

Example 24. If $\sin\theta + \cos\theta = \sqrt{3}$, then prove that $\tan\theta + \cot\theta = 1$.

Sol. Given, $\sin\theta + \cos\theta = \sqrt{3}$

$\Rightarrow (\sin\theta + \cos\theta)^2 = (\sqrt{3})^2$ [squaring on both sides]

$\Rightarrow \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 3$

$[\because (a+b)^2 = a^2 + b^2 + 2ab]$

$\Rightarrow 1 + 2\sin\theta\cos\theta = 3$ $[\because \sin^2 A + \cos^2 A = 1]$

$\Rightarrow 2\sin\theta\cos\theta = 2 \Rightarrow \sin\theta\cos\theta = 1$

$\Rightarrow \sin\theta\cos\theta = \sin^2\theta + \cos^2\theta$ $[\because \sin^2 A + \cos^2 A = 1]$

$\Rightarrow 1 = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$ [dividing both sides by $\sin\theta\cos\theta$]

$\Rightarrow 1 = \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta}$

$\Rightarrow 1 = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$

$\Rightarrow 1 = \tan\theta + \cot\theta$ **Hence proved.**

Example 25. If $\operatorname{cosec}\theta - \sin\theta = \sqrt{5}$ and $\operatorname{cosec}\theta$, $\sin\theta$ are positive, then show that $\operatorname{cosec}\theta + \sin\theta = 3$.

Sol. Given, $\operatorname{cosec}\theta - \sin\theta = \sqrt{5}$

$\Rightarrow (\operatorname{cosec}\theta - \sin\theta)^2 = (\sqrt{5})^2$ [squaring on both sides]

$\Rightarrow \operatorname{cosec}^2\theta + \sin^2\theta - 2\operatorname{cosec}\theta\sin\theta = 5$

$[\because (a-b)^2 = a^2 + b^2 - 2ab]$

$\Rightarrow \operatorname{cosec}^2\theta + \sin^2\theta + 2\operatorname{cosec}\theta\sin\theta = 5 + 4\operatorname{cosec}\theta\sin\theta$

[adding $4\operatorname{cosec}\theta\sin\theta$ on both sides]

$\Rightarrow (\operatorname{cosec}\theta + \sin\theta)^2 = 5 + 4\times 1$ $[\because \operatorname{cosec}\theta\sin\theta = 1]$

$\Rightarrow (\operatorname{cosec}\theta + \sin\theta)^2 = 9$

$\Rightarrow \operatorname{cosec}\theta + \sin\theta = \pm 3$ [taking square root]

But $\operatorname{cosec}\theta > 0$ and $\sin\theta > 0$, so $\operatorname{cosec}\theta + \sin\theta > 0$.

$\therefore \operatorname{cosec}\theta + \sin\theta = 3$ **Hence proved.**

Example 26. If $a\sin\theta + b\cos\theta = c$, then prove that $a\cos\theta - b\sin\theta = \pm\sqrt{a^2 + b^2 - c^2}$.

Sol. Given, $a\sin\theta + b\cos\theta = c$

$\Rightarrow (a\sin\theta + b\cos\theta)^2 = c^2$ [squaring on both sides]

$\Rightarrow a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta = c^2$

$\Rightarrow a^2(1-\cos^2\theta) + b^2(1-\sin^2\theta) + 2ab\sin\theta\cos\theta = c^2$

$[\because \sin^2 A + \cos^2 A = 1]$

$\Rightarrow a^2 + b^2 - (a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta) = c^2$

$\Rightarrow a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta = a^2 + b^2 - c^2$

$\Rightarrow (a\cos\theta - b\sin\theta)^2 = a^2 + b^2 - c^2$

$\Rightarrow a\cos\theta - b\sin\theta = \pm\sqrt{a^2 + b^2 - c^2}$ **Hence proved.**

Example 27. If $a\cos\theta - b\sin\theta = x$ and $a\sin\theta + b\cos\theta = y$, then prove that $a^2 + b^2 = x^2 + y^2$.

Sol. Given, $a\cos\theta - b\sin\theta = x$ ...(i)

and $a\sin\theta + b\cos\theta = y$ ...(ii)

On adding the squares of Eqs. (i) and (ii), we get

$x^2 + y^2 = (a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2$

$\Rightarrow x^2 + y^2 = a^2\cos^2\theta + b^2\sin^2\theta - 2ab\cos\theta\sin\theta + a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta$

$[\because (a \pm b)^2 = a^2 + b^2 \pm 2ab]$

$\Rightarrow x^2 + y^2 = a^2\cos^2\theta + b^2\sin^2\theta + a^2\sin^2\theta + b^2\cos^2\theta$

$\Rightarrow x^2 + y^2 = a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$

$\Rightarrow x^2 + y^2 = a^2 + b^2$ $[\because \sin^2 A + \cos^2 A = 1]$

Hence proved.

Example 28.

(i) If $\tan\theta + \sec\theta = l$, then prove that $\sec\theta = \frac{l^2+1}{2l}$.

(ii) If $\operatorname{cosec}\theta + \cot\theta = p$, then prove that $\cos\theta = \frac{p^2-1}{p^2+1}$.

Sol.

(i) Given, $\tan\theta + \sec\theta = l$ ...(i)

We know that, $\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow (\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

$[\because a^2 - b^2 = (a+b)(a-b)]$

$\Rightarrow l(\sec\theta - \tan\theta) = 1$ [from Eq. (i)]

$\Rightarrow \sec\theta - \tan\theta = \frac{1}{l}$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2\sec\theta = l + \frac{1}{l}$$

$$\Rightarrow \quad \sec\theta = \frac{l^2+1}{2l}$$ **Hence proved.**

(ii) Given, $\operatorname{cosec}\theta + \cot\theta = p$ ...(i)

We know that, $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

$\Rightarrow$ $(\operatorname{cosec}\theta + \cot\theta)(\operatorname{cosec}\theta - \cot\theta) = 1$ $[\because a^2 - b^2 = (a+b)(a-b)]$

$\Rightarrow$ $p(\operatorname{cosec}\theta - \cot\theta) = 1$ [from Eq. (i)]

$\Rightarrow$ $\operatorname{cosec}\theta - \cot\theta = \frac{1}{p}$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2\operatorname{cosec}\theta = p + \frac{1}{p} \Rightarrow \operatorname{cosec}\theta = \frac{p^2+1}{2p} \quad ...(iii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$2\cot\theta = p - \frac{1}{p} \Rightarrow \cot\theta = \frac{p^2-1}{2p} \quad ...(iv)$$

On dividing Eq. (iv) by Eq. (iii), we get

$$\frac{\cot\theta}{\operatorname{cosec}\theta} = \frac{p^2-1}{p^2+1}$$

$$\Rightarrow \quad \cos\theta = \frac{p^2-1}{p^2+1} \quad \left[\because \frac{\cot\theta}{\operatorname{cosec}\theta} = \frac{\cos\theta \times \sin\theta}{\sin\theta} = \cos\theta\right]$$

Hence proved.

Example 29. If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, then show that $(m^2 - n^2)^2 = 16\,mn$ or $(m^2 - n^2) = 4\sqrt{mn}$.

Sol. Given, $\tan\theta + \sin\theta = m$ ...(i)

and $\tan\theta - \sin\theta = n$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2\tan\theta = m + n$$

$$\Rightarrow \quad \tan\theta = \frac{m+n}{2}$$

$$\therefore \quad \cot\theta = \frac{1}{\tan\theta} = \frac{2}{m+n} \quad ...(iii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$2\sin\theta = m - n$$

$$\Rightarrow \quad \sin\theta = \frac{m-n}{2} \quad ...(iv)$$

$$\therefore \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{2}{m-n}$$

We know that, $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

$$\Rightarrow \quad \left(\frac{2}{m-n}\right)^2 - \left(\frac{2}{m+n}\right)^2 = 1 \quad \text{[from Eqs. (iii) and (iv)]}$$

$$\Rightarrow \quad \frac{4}{(m-n)^2} - \frac{4}{(m+n)^2} = 1$$

$$\Rightarrow \quad 4\left[\frac{1}{(m-n)^2} - \frac{1}{(m+n)^2}\right] = 1$$

$$\Rightarrow \quad 4\left[\frac{(m+n)^2 - (m-n)^2}{(m-n)^2(m+n)^2}\right] = 1$$

$$\Rightarrow \quad 4\left[\frac{(m^2+n^2+2mn) - (m^2+n^2-2mn)}{(m-n)^2(m+n)^2}\right] = 1$$

$[\because (a \pm b)^2 = a^2 + b^2 \pm 2ab]$

$$\Rightarrow \quad 4\left[\frac{2mn + 2mn}{(m-n)^2(m+n)^2}\right] = 1$$

$$\Rightarrow \quad \frac{16mn}{[(m-n)(m+n)]^2} = 1$$

$$\Rightarrow \quad \frac{16mn}{(m^2-n^2)^2} = 1$$

$\Rightarrow$ $(m^2-n^2)^2 = 16mn$ $[\because (a-b)(a+b) = a^2 - b^2]$

$\therefore$ $(m^2-n^2) = 4\sqrt{mn}$ [taking positive square root]

Hence proved.

Example 30. If $\operatorname{cosec}\theta = x + \frac{1}{4x}$, then prove that $\operatorname{cosec}\theta + \cot\theta = 2x$ or $\frac{1}{2x}$.

Sol. Given, $\operatorname{cosec}\theta = x + \frac{1}{4x}$ ...(i)

We know that, $\cot^2\theta = \operatorname{cosec}^2\theta - 1$

$$\Rightarrow \quad \cot^2\theta = \left(x + \frac{1}{4x}\right)^2 - 1 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \cot^2\theta = x^2 + \frac{1}{16x^2} + 2x\cdot\frac{1}{4x} - 1$$

$[\because (a+b)^2 = a^2 + b^2 + 2ab]$

$$= x^2 + \frac{1}{16x^2} + \frac{1}{2} - 1 = x^2 + \frac{1}{16x^2} - \frac{1}{2}$$

$$= x^2 + \frac{1}{16x^2} - 2x\cdot\frac{1}{4x} = \left(x - \frac{1}{4x}\right)^2$$

$[\because a^2 + b^2 - 2ab = (a-b)^2]$

$$\Rightarrow \quad \cot\theta = x - \frac{1}{4x} \quad ...(ii)$$

$$\text{or} \quad \cot\theta = -\left(x - \frac{1}{4x}\right) \quad ...(iii)$$

On adding Eqs. (i) and (ii), we get

$$\operatorname{cosec}\theta + \cot\theta = 2x$$

Now, adding Eqs. (i) and (iii), we get

$$\operatorname{cosec}\theta + \cot\theta = \frac{1}{2x}$$

Hence, $\operatorname{cosec}\theta + \cot\theta = 2x$ or $\frac{1}{2x}$.

Example 31. Without using trigonometric tables, evaluate $\operatorname{cosec}^2 57° - \tan^2 33° + \cos 44° \operatorname{cosec} 46° - \sqrt{2}\cos 45° - \tan^2 60°$. ***[2016]***

Sol. We have, $\operatorname{cosec}^2 57° - \tan^2 33° + \cos 44° \operatorname{cosec} 46° - \sqrt{2}\cos 45° - \tan^2 60°$

$= \operatorname{cosec}^2(90° - 33°) - \tan^2 33° + \cos(90° - 46°)\operatorname{cosec} 46° - \sqrt{2}\cos 45° - \tan^2 60°$

$= (\sec^2 33° - \tan^2 33°) + (\sin 46° \operatorname{cosec} 46°) - \sqrt{2}\cos 45° - \tan^2 60°$

$[\because \operatorname{cosec}(90° - \theta) = \sec\theta$ and $\cos(90° - \theta) = \sin\theta]$

$= 1 + 1 - \sqrt{2} \times \frac{1}{\sqrt{2}} - (\sqrt{3})^2$

$[\because \sec^2\theta - \tan^2\theta = 1, \sin\theta \cdot \mathrm{cosec}\,\theta = 1, \cos 45° = \frac{1}{\sqrt{2}}$ and $\tan 60° = \sqrt{3}]$

$= 2 - 1 - 3 = 2 - 4 = -2$

Example 32. Without using trigonometric tables evaluate $\sin^2 28° + \sin^2 62° + \tan^2 38° - \cot^2 52° + \frac{1}{4}\sec^2 30°$. ***[2017]***

Sol. We have, $\sin^2 28° + \sin^2 62° + \tan^2 38° - \cot^2 52° + \frac{1}{4}\sec^2 30°$

$= \sin^2 28° + \sin^2(90° - 28°) + \tan^2(90° - 52°) - \cot^2 52° + \frac{1}{4}\sec^2 30°$

$= \sin^2 28° + \cos^2 28° + \cot^2 52° - \cot^2 52° + \frac{1}{4}\sec^2 30°$

$[\because \sin(90° - \theta) = \cos\theta$ and $\tan(90° - \theta) = \cot\theta]$

$= 1 + 0 + \frac{1}{4}\sec^2 30°$ $\quad [\because \sin^2\theta + \cos^2\theta = 1]$

$= 1 + \frac{1}{4}\left(\frac{2}{\sqrt{3}}\right)^2$

$= 1 + \frac{1}{4} \times \frac{4}{3}$

$= 1 + \frac{1}{3} = \frac{4}{3}$ $\quad \left[\because \sec 30° = \frac{2}{\sqrt{3}}\right]$

Example 33. Without using trigonometric tables, evaluate the following.

(i) $\dfrac{\frac{2}{3}\mathrm{cosec}^2 58° - \frac{2}{3}\cot 58° \tan 32° - \frac{5}{3}\tan 13° \tan 37°}{\tan 45° \tan 53° \tan 77°}$

(ii) $\left(\frac{\tan 20°}{\mathrm{cosec}\,70°}\right)^2 + \left(\frac{\cot 20°}{\sec 70°}\right)^2 + 2\tan 15° \tan 45° \tan 75°$

Sol.

(i) We have, $\dfrac{\frac{2}{3}\mathrm{cosec}^2 58° - \frac{2}{3}\cot 58° \tan 32° - \frac{5}{3}\tan 13° \tan 37°}{\tan 45° \tan 53° \tan 77°}$

$= \frac{2}{3}\mathrm{cosec}^2 58° - \frac{2}{3}\cot 58° \tan(90° - 58°)$

$- \frac{5}{3}\tan 13° \tan 37° \cdot 1 \cdot \tan(90° - 37°)\tan(90° - 13°)$

$[\because \tan 45° = 1]$

$= \dfrac{\frac{2}{3}\mathrm{cosec}^2 58° - \frac{2}{3}\cot 58° \cot 58° - \frac{5}{3}\tan 13°}{\tan 37° \cot 37° \cot 13°}$ $\quad [\because \tan(90° - \theta) = \cot\theta]$

$= \frac{2}{3}(\mathrm{cosec}^2 58° - \cot^2 58°) - \frac{5}{3}\tan 13° \cdot 1 \cdot \cot 13°$

$[\because \tan\theta \cot\theta = 1]$

$= \frac{2}{3} \cdot 1 - \frac{5}{3} \cdot 1 = \frac{2}{3} - \frac{5}{3} = -1$ $\quad [\because \mathrm{cosec}^2 A - \cot^2 A = 1]$

(ii) We have,

$\left(\frac{\tan 20°}{\mathrm{cosec}\,70°}\right)^2 + \left(\frac{\cot 20°}{\sec 70°}\right)^2 + 2\tan 15° \tan 45° \tan 75°$

$= \left(\frac{\tan 20°}{\mathrm{cosec}\,(90° - 20°)}\right)^2 + \left(\frac{\cot 20°}{\sec(90° - 20°)}\right)^2$

$+ 2\tan 15° \cdot 1 \cdot \tan(90° - 15°)$ $[\because \tan 45° = 1]$

$= \left(\frac{\tan 20°}{\sec 20°}\right)^2 + \left(\frac{\cot 20°}{\mathrm{cosec}\,20°}\right)^2 + 2\tan 15° \cot 15°$

$[\because \mathrm{cosec}\,(90° - \theta) = \sec\theta, \sec(90° - \theta) = \mathrm{cosec}\,\theta$ and $\tan(90° - \theta) = \cot\theta]$

$= \left(\frac{\sin 20°}{\cos 20°} \cdot \cos 20°\right)^2 + \left(\frac{\cos 20°}{\sin 20°} \cdot \sin 20°\right)^2 + 2 \cdot 1$

$[\because \tan\theta \cdot \cot\theta = 1]$

$= \sin^2 20° + \cos^2 20° + 2 = 1 + 2 = 3$ $\quad [\because \sin^2 A + \cos^2 A = 1]$

CHAPTER EXERCISE

3 Marks Questions

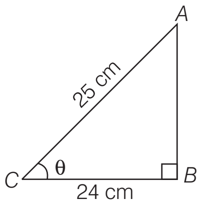
1. If $4\cos^2\theta = 3$, then find the value of θ in 1st quadrant.

2. If $2\sin^2\theta - \cos^2\theta = 2$, then find the value of θ, when θ lies in first quadrant.

3. If $\tan\theta = \frac{20}{21}$, then prove that $\frac{1-\sin\theta+\cos\theta}{1+\sin\theta+\cos\theta} = \frac{3}{7}$.

4. If $7\tan\theta = 4$, then find the value of $\frac{7\sin\theta - 3\cos\theta}{7\sin\theta + 3\cos\theta}$.

5. With the help of following figure, find the value of
(i) $\sec^2\theta + \tan^2\theta$. (ii) $\operatorname{cosec}^2\theta - \cot^2\theta$.



6. If $2\sin\theta - 1 = 0$, then prove that $\sec\theta + \tan\theta = \sqrt{3}$.

7. If $\sin\theta = \frac{1}{2}$, then prove that $3\cos\theta - 4\cos^3\theta = 0$.

8. In right angled ΔACB, $\angle C = 90°$, $AB = 29$ units and $BC = 21$ units. If $\angle ABC = \theta$, then find $\cos^2\theta - \sin^2\theta$ and $\sin^2\theta + \cos^2\theta$.

9. If $A = 30°$ and $B = 60°$, then verify that
$\sin(A+B) = \sin A\cos B + \cos A\sin B$.

10. Find an acute angle θ, when $\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \frac{1-\sqrt{3}}{1+\sqrt{3}}$.

11. If $\tan\theta = \cot(30° + \theta)$, then find the value of θ.

12. Prove that
$(\tan\theta + 2)(2\tan\theta + 1) = 5\tan\theta + 2\sec^2\theta$.

13. Prove that

(i) $\sec A(1-\sin A)(\sec A + \tan A) = 1$

(ii) $\frac{\cos A}{1+\sin A} + \tan A = \sec A$ **[2016]**

(iii) $\frac{\sin\theta}{1-\cot\theta} + \frac{\cos\theta}{1-\tan\theta} = \cos\theta + \sin\theta$ **[2015]**

(iv) $\frac{\tan^2\theta}{(\sec\theta - 1)^2} = \frac{1+\cos\theta}{1-\cos\theta}$ **[2012]**

14. If $\cot\theta = \frac{15}{8}$, then find the value of $\frac{(2+2\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(2-2\cos\theta)}$.

15. If $4x = \operatorname{cosec}\theta$ and $\frac{4}{x} = \cot\theta$, then find the value of $4\left(x^2 - \frac{1}{x^2}\right)$.

16. If $\cos A + \cos^2 A = 1$, then find the value of $\sin^2 A + \sin^4 A$.

17. If $\sin A + \sin^2 A = 1$, then find the value of $\cos^2 A + \cos^4 A$.

18. If $\sqrt{3}\tan\theta = 3\sin\theta$, then find the value of $\sin^2\theta - \cos^2\theta$.

19. If $\tan A = \sqrt{2} - 1$, then prove that $\frac{\tan A}{1+\tan^2 A} = \frac{\sqrt{2}}{4}$.

20. If $\tan\theta = \frac{1}{\sqrt{3}}$, then evaluate $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}$.

21. If $\tan\theta = 1$ and $\sin\phi = \frac{1}{\sqrt{2}}$, then find the value of $\cos(\theta + \phi)$, where θ and ϕ are both acute angles.

22. If $A + B = 90°$, then prove that
$$\sqrt{\frac{\tan A\tan B + \tan A\cot B}{\sin A\sec B} - \frac{\sin^2 B}{\cos^2 A}} = \tan A.$$

23. Prove that

(i) $\frac{\sin\theta}{1-\cos\theta} = \operatorname{cosec}\theta + \cot\theta$.

(ii) $(\operatorname{cosec} A - \sin A)(\sec A - \cos A)\sec^2 A = \tan A$. **[2011]**

24. Show that $\frac{\sin\theta}{\operatorname{cosec}\theta - 1} + \frac{\cos\theta}{1+\sec\theta} = \frac{\sin\theta\cos\theta}{(\sin\theta - \cos\theta)}$.

25. Prove that $\frac{\sin^4\theta + \cos^4\theta}{1 - 2\sin^2\theta\cdot\cos^2\theta} = 1$.

26. Simplify $\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} + \sin\theta\cos\theta$.

27. Prove that $\frac{\operatorname{cosec}\theta}{1+\sec\theta} + \frac{1+\sec\theta}{\operatorname{cosec}\theta} = 2\operatorname{cosec}^3\theta\,(\sec\theta - 1)$.

28. Prove that $\dfrac{1}{(\sec x - \tan x)} - \dfrac{1}{\cos x}$
$= \dfrac{1}{\cos x} - \dfrac{1}{\sec x + \tan x}$.

29. Prove that $\sqrt{\dfrac{\sec A + \tan A}{\sec A - \tan A}} \cdot \sqrt{\dfrac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1}} = 1$.

30. Prove that $\dfrac{\operatorname{cosec}\theta + \cot\theta}{\operatorname{cosec}\theta - \cot\theta}$
$= 1 + 2\cot^2\theta + 2\operatorname{cosec}^2\theta\cos\theta$.

31. Prove that $\dfrac{\sin\theta}{\cot\theta + \operatorname{cosec}\theta} = 2 + \dfrac{\sin\theta}{\cot\theta - \operatorname{cosec}\theta}$.

32. If $7\sin^2 A + 3\cos^2 A = 4$, then prove that $\tan A = \dfrac{1}{\sqrt{3}}$.

33. If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, then prove that
$\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

34. If $\tan\theta + \dfrac{1}{\tan\theta} = 2$, then find the value of
$\cot^2\theta + \dfrac{1}{\cot^2\theta}$.

35. In ΔABC, show that $\sin^2\dfrac{A}{2} + \sin^2\dfrac{B+C}{2} = 1$.

36. If $\cot\theta = 3x - \dfrac{1}{12x}$, then show that
$\cot\theta + \operatorname{cosec}\theta = 6x$ or $-\dfrac{1}{6x}$.

37. Simplify $\dfrac{\tan 28^\circ}{\cot 62^\circ} \div \dfrac{1}{\sqrt{3}}(\tan 20^\circ \cdot \tan 60^\circ \cdot \tan 70^\circ)$.

38. If A, B and C are interior angles of ΔABC, then show that $\tan^2\left(\dfrac{B+C}{2}\right) = \operatorname{cosec}^2\dfrac{A}{2} - 1$.

39. If $\sin^6 A + \cos^6 A + 3\sin^2\theta \cdot \cos^2\theta + 4 = k$, then find the value of k.

40. If $\sec\theta = x + \dfrac{1}{4x}$, then find the value of $\sec\theta + \tan\theta$.

41. Prove that $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$.

42. Eliminate θ from the following equations.
(i) $x = a\sec\theta$, $y = b\tan\theta$
(ii) $x = k + a\cos\theta$, $y = h + b\sin\theta$

43. If $p = m\operatorname{cosec}\theta + n\cot\theta$ and $q = m\cot\theta + n\operatorname{cosec}\theta$, then prove that $p^2 - q^2 = m^2 - n^2$.

44. If $\dfrac{\cos\alpha}{\cos\beta} = m$ and $\dfrac{\cos\alpha}{\sin\beta} = n$, then show that $(m^2 + n^2)\cos^2\alpha = m^2 n^2$.

45. Show that $2(\cos^4 60^\circ + \sin^4 30^\circ)$
$-(\tan^2 60^\circ + \cot^2 45^\circ) + 3\sec^2 30^\circ = \dfrac{1}{4}$.

46. If $\alpha + \beta = 90^\circ$, then prove that
$\sqrt{\cos\alpha\operatorname{cosec}\beta - \cos\alpha\sin\beta} = \sin\alpha$.

47. Find the value of
$\dfrac{\sin^2 22^\circ + \sin^2 68^\circ}{\cos^2 22^\circ + \cos^2 68^\circ} + \sin^2 63^\circ + \cos 63^\circ \sin 27^\circ$.

48. If $\dfrac{1}{x}\left(\dfrac{\sin^2 5^\circ + \sin^2 85^\circ}{\cos^2 5^\circ + \cos^2 85^\circ}\right) - \dfrac{3}{4} = 1$, then find the value of x.

49. Evaluate $\dfrac{\cot(90^\circ - \theta)\sin(90^\circ - \theta)}{\sin\theta} + \dfrac{\cot 40^\circ}{\tan 50^\circ}$
$-(\cos^2 20^\circ + \cos^2 70^\circ)$.

50. Without using trigonometric tables, evaluate the following.

(i) $\dfrac{\sec 39^\circ}{\operatorname{cosec} 51^\circ} + \dfrac{2}{\sqrt{3}}\tan 17^\circ \tan 38^\circ$
$\tan 60^\circ \tan 52^\circ \tan 73^\circ - 3(\sin^2 31^\circ + \sin^2 59^\circ)$

(ii) $\cos^2 26^\circ + \cos 64^\circ \sin 26^\circ + \dfrac{\tan 36^\circ}{\cot 54^\circ}$. **[2012]**

(iii) $2\left(\dfrac{\tan 35^\circ}{\cot 55^\circ}\right)^2 + \left(\dfrac{\cot 55^\circ}{\tan 35^\circ}\right)^2 - 3\left(\dfrac{\sec 40^\circ}{\operatorname{cosec} 50^\circ}\right)$ **[2011]**

(iv) $3\cos 80^\circ \operatorname{cosec} 10^\circ + 2\sin 59^\circ \sec 31^\circ$ **[2013]**

(v) $\sin^2 34^\circ + \sin^2 56^\circ + 2\tan 18^\circ \tan 72^\circ - \cot^2 30^\circ$ **[2014]**

51. Evaluate $\dfrac{\sec^2(90^\circ - \theta) - \cot^2\theta}{2(\sin^2 25^\circ + \sin^2 65^\circ)}$
$+ \dfrac{2\cos^2 60^\circ \tan^2 28^\circ \tan^2 62^\circ}{\sin 30^\circ \cos 60^\circ}$.

52. Prove that $\cos\theta\sin\theta - \dfrac{\sin\theta\cos(90^\circ - \theta)\cos\theta}{\sec(90^\circ - \theta)}$
$- \dfrac{\cos\theta\sin(90^\circ - \theta)\sin\theta}{\operatorname{cosec}(90^\circ - \theta)} + \operatorname{cosec}(90^\circ - \theta) = \dfrac{1}{\cos\theta}$.

b 4 Marks Questions

53. If $\sec A = \frac{17}{8}$, then show that
$$\frac{3-4\sin^2 A}{4\cos^2 A-3} = \frac{3-\tan^2 A}{1-3\tan^2 A}.$$

54. Prove that $(1+\tan A\tan B)^2 + (\tan A - \tan B)^2 = \sec^2 A\sec^2 B$.

55. If $\sqrt{3}\cot^2\theta - 4\cot\theta + \sqrt{3} = 0$, then find the value of $\tan^2\theta + \cot^2\theta$.

56. Prove that $\frac{\sin A - \cos A + 1}{\sin A + \cos A - 1} = \frac{1}{\sec A - \tan A}$.

57. If $\sin\theta + 2\cos\theta = 1$, then prove that $2\sin\theta - \cos\theta = 2$.

58. If $1+\cos^2\theta = 3\sin\theta\cos\theta$, then prove that $\tan\theta = 1$ or 2.

59. Prove that $\left(\tan\theta + \frac{1}{\cos\theta}\right)^2 + \left(\tan\theta - \frac{1}{\cos\theta}\right)^2 = 2\left(\frac{1+\sin^2\theta}{1-\sin^2\theta}\right)$.

60. Prove that $\sec^2 A - \left(\frac{\sin^2 A - 2\sin^4 A}{2\cos^4 A - \cos^2 A}\right) = 1$.

61. Prove that
$$\cot^2 A\left(\frac{\sec A - 1}{1+\sin A}\right) + \sec^2 A\left(\frac{\sin A - 1}{1+\sec A}\right) = 0.$$

62. Prove that
$$\left(\frac{1}{\sec^2\theta - \cos^2\theta} + \frac{1}{\csc^2\theta - \sin^2\theta}\right)\sin^2\theta\cos^2\theta = \frac{1-\sin^2\theta\cos^2\theta}{2+\sin^2\theta\cos^2\theta}.$$

63. Prove that $\frac{(1+\cot\theta+\tan\theta)(\sin\theta-\cos\theta)}{\sec^3\theta - \csc^3\theta} = \sin^2\theta\cos^2\theta$.

64. Show that $(\sin^8 A - \cos^8 A) = (2\sin^2 A - 1)(1-2\sin^2 A\cos^2 A)$.

65. If $\sin\theta + \cos\theta = p$ and $\sec\theta + \csc\theta = q$, then prove that $q(p^2-1) = 2p$.

66. If $\csc A - \cot A = q$, show that $\frac{q^2-1}{q^2+1} + \cos A = 0$.

67. Prove that
$$2\sec^2\theta - \sec^4\theta - 2\csc^2\theta + \csc^4\theta = \cot^4\theta - \tan^4\theta.$$

68. If $x = r\sin A\cos B$, $y = r\sin A\sin B$ and $z = r\cos A$, then show that $x^2+y^2+z^2 = r^2$.

69. Prove that $l^2m^2(l^2+m^2+3) = 1$, if $\csc\theta - \sin\theta = l$ and $\sec\theta - \cos\theta = m$.

70. Show that $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1+\sec\theta\csc\theta$.

71. If $\tan A = a\tan B$ and $\sin A = b\sin B$, then prove that $\cos^2 A = \frac{b^2-1}{a^2-1}$.

72. If $\csc\theta - \sin\theta = m$ and $\sec\theta - \cos\theta = n$, then prove that $(m^2n)^{2/3} + (mn^2)^{2/3} = 1$.

73. If $a\cos\theta - b\sin\theta = c$, then prove that
$$a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}.$$

74. Find the value of
$$\frac{\sec\theta\csc(90^\circ-\theta) - \tan\theta\cot(90^\circ-\theta) + \sin^2 55^\circ + \sin^2 35^\circ}{\tan 10^\circ\tan 20^\circ\tan 60^\circ\tan 70^\circ\tan 80^\circ}.$$

75. Evaluate $\frac{\cos^2 35^\circ + \cos^2 55^\circ}{\csc^2 15^\circ - \tan^2 75^\circ} + \sqrt{3}(\tan 13^\circ\tan 23^\circ\tan 30^\circ\tan 67^\circ\tan 77^\circ)$.

76. Prove that
$$\frac{\sin^2 40^\circ + \sin^2 50^\circ}{\cos^2 20^\circ + \cos^2 70^\circ} + \tan 10^\circ\tan 20^\circ\tan 60^\circ\tan 70^\circ\tan 80^\circ = 1+\sqrt{3}.$$

77. Evaluate $\frac{\sec^2(90^\circ-\theta) - \cot^2\theta}{2(\sin^2 25^\circ + \sin^2 65^\circ)} - \frac{2\sin^2 30^\circ\tan^2 32^\circ\cdot\tan^2 58^\circ}{2(\sec^2 33^\circ - \cot^2 57^\circ)}$.

Hints and Answers

1. Hint $\cos\theta = \pm\frac{\sqrt{3}}{2}$

In first quadrant, we have $\cos\theta = \frac{\sqrt{3}}{2}$

Ans. 30°

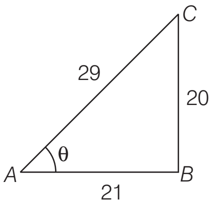
2. Hint $2\sin^2\theta - (1-\sin^2\theta) = 2 \Rightarrow 3\sin^2\theta = 3$

$\Rightarrow \quad \sin\theta = \pm 1$

But θ lies in 1st quadrant.

$\therefore \quad \sin\theta = 1$ **Ans.** $\theta = \frac{\pi}{2}$

3. Hint $\sin\theta = \frac{20}{29}$ and $\cos\theta = \frac{21}{29}$



4. Hint Divide each term of numerator and denominator of the given expression by $\cos\theta$ and then put $\tan\theta = \frac{4}{7}$.

Ans. $\frac{1}{7}$

5. Hint $AB = \sqrt{(25)^2 - (24)^2} = \sqrt{49} = 7$

(i) $\sec^2\theta + \tan^2\theta = \left(\frac{25}{24}\right)^2 + \left(\frac{7}{24}\right)^2$

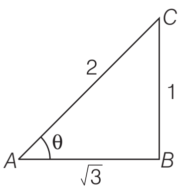
(ii) $\text{cosec}^2\theta - \cot^2\theta = \left(\frac{25}{7}\right)^2 - \left(\frac{24}{7}\right)^2$

Ans. (i) $\frac{337}{288}$ (ii) 1

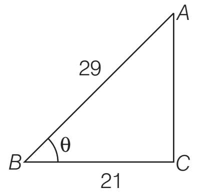
6. Hint $\sin\theta = \frac{1}{2}$ and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$

$\therefore \sec\theta = \frac{2}{\sqrt{3}}$ and $\tan\theta = \frac{1}{\sqrt{3}}$

7. Hint $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$



8. Hint In ΔACB, $\cos\theta = \frac{21}{29}$



$\therefore \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{21}{29}\right)^2} = \frac{20}{29}$

Ans. $\frac{41}{841}$, 1

9. Hint $\sin(A+B) = \sin(30° + 60°) = \sin 90° = 1$

and $\sin A\cos B + \cos A \sin B = \sin 30° \cos 60° + \cos 30° \sin 60°$

$= \frac{1}{2}\times\frac{1}{2} + \frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}$

10. Hint On dividing numerator and denominator of LHS by $\cos\theta$, we get

$$\frac{1-\tan\theta}{1+\tan\theta} = \frac{1-\sqrt{3}}{1+\sqrt{3}}$$

On comparing both sides, we get

$$\tan\theta = \sqrt{3}$$

Ans. 60°

11. Hint We have, $\tan\theta = \cot(30° + \theta)$

$\Rightarrow \quad \cot(90° - \theta) = \cot(30° + \theta)$

$\Rightarrow \quad 90° - \theta = 30° + \theta$ **Ans.** $\theta = 30°$

12. Hint LHS $= 2\tan^2\theta + 5\tan\theta + 2$

Put $\tan^2\theta = \sec^2\theta - 1$

13. (i) **Hint** Put $\sec A = \frac{1}{\cos A}$, $\tan A = \frac{\sin A}{\cos A}$

and $\quad 1 - \sin^2 A = \cos^2 A$

(ii) **Hint** Put $\tan A = \frac{\sin A}{\cos A}$ and $\sin^2 A + \cos^2 A = 1$

(iii) **Hint** Put $\cot\theta = \frac{\cos\theta}{\sin\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$

(iv) **Hint** Put $\tan^2\theta = \sec^2\theta - 1$ in LHS of the given expression.

14. Hint $\frac{(2+2\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(2-2\cos\theta)} = \frac{2(1-\sin^2\theta)}{2(1-\cos^2\theta)} = \frac{\cos^2\theta}{\sin^2\theta}$

Ans. 225 / 64

15. Hint $4x = \text{cosec}\,\theta \Rightarrow x = \dfrac{\text{cosec}\,\theta}{4}$

and $\dfrac{4}{x} = \cot\theta \Rightarrow \dfrac{1}{x} = \dfrac{\cot\theta}{4}$

Now, $4\left(x^2 - \dfrac{1}{x^2}\right) = 4\left[\left(\dfrac{\text{cosec}\,\theta}{4}\right)^2 - \left(\dfrac{\cot\theta}{4}\right)^2\right]$

Ans. 1/4

16. Hint $\cos A = 1 - \cos^2 A \Rightarrow \cos A = \sin^2 A.$

Put $\sin^4 A = \cos^2 A$ **Ans.** 1

17. Hint $\sin A = 1 - \sin^2 A = \cos^2 A$. **Ans.** 1

18. Hint $\sqrt{3}\tan\theta = 3\sin\theta$

$\Rightarrow \cos\theta = \dfrac{1}{\sqrt{3}}$ and $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\dfrac{1}{3}} = \sqrt{\dfrac{2}{3}}$

Ans. $\dfrac{1}{3}$

19. Hint Put $\tan\theta = \sqrt{2} - 1$ in LHS and simplify it.

20. Hint $\dfrac{\text{cosec}^2\theta - \sec^2\theta}{\text{cosec}^2\theta + \sec^2\theta} = \dfrac{\dfrac{1}{\sin^2\theta} - \dfrac{1}{\cos^2\theta}}{\dfrac{1}{\sin^2\theta} + \dfrac{1}{\cos^2\theta}}$

$= \dfrac{\cos^2\theta - \sin^2\theta}{\sin^2\theta + \cos^2\theta} = \cos^2\theta - \sin^2\theta$

Also, $\tan\theta = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = 30°$ **Ans.** $\dfrac{1}{2}$

21. Hint $\tan\theta = 1 \Rightarrow \theta = 45°$ and $\sin\phi = \dfrac{1}{\sqrt{2}} \Rightarrow \phi = 45°$

$\therefore \quad \cos(\theta + \phi) = \cos(45° + 45°)$

Ans. 0

22. Given, $A + B = 90° \Rightarrow B = 90° - A$

Now, LHS $= \sqrt{\dfrac{\tan A \tan B + \tan A \cot B}{\sin A \sec B} - \dfrac{\sin^2 B}{\cos^2 A}}$

$= \sqrt{\dfrac{\tan A \tan(90° - A) + \tan A \cot(90° - A)}{\sin A \sec(90° - A)} - \dfrac{\sin^2(90° - A)}{\cos^2 A}}$

23. (i) **Hint** LHS $= \dfrac{\sin\theta}{1-\cos\theta} = \dfrac{\sin\theta\,(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$

(ii) Do same as Example 12 (ii).

24. Hint LHS $= \dfrac{\sin^2\theta}{1-\sin\theta} + \dfrac{\cos^2\theta}{1+\cos\theta}$

25. Hint LHS $= \dfrac{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta}{1 - 2\sin^2\theta\cos^2\theta}$

26. Hint LHS $= \dfrac{(\sin\theta + \cos\theta)(\sin^2\theta + \cos^2\theta - \sin\theta\cos\theta)}{\sin\theta + \cos\theta} + \sin\theta\cos\theta$

$= 1 - \sin\theta\cos\theta + \sin\theta\cos\theta$

Ans. 1

27. Hint Put $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$ in LHS and RHS.

28. Hint Put $\sec x = \dfrac{1}{\cos x}$ and $\tan x = \dfrac{\sin x}{\cos x}$

and then simplify both sides separately.

29. Hint Put $\sec A = \dfrac{1}{\cos A}$, $\tan A = \dfrac{\sin A}{\cos A}$

and $\text{cosec}\,A = \dfrac{1}{\sin A}$ in LHS.

30. Hint Use $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

31. Hint Put $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ and $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ in both sides and simplify both sides separately.

32. Hint Put $\sin A = \dfrac{\tan A}{\sqrt{1+\tan^2 A}}$ and

$\cos A = \dfrac{1}{\sqrt{1+\tan^2 A}}$ in given equation and simplify it.

33. Hint $(\cos\theta + \sin\theta)^2 = (\sqrt{2}\cos\theta)^2$

$\Rightarrow \cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta = 2\cos^2\theta$

$\Rightarrow 2\cos^2\theta - \cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta = \sin^2\theta + \sin^2\theta$

$\Rightarrow \cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta = 2\sin^2\theta$

$\Rightarrow (\cos\theta - \sin\theta)^2 = (\sqrt{2}\sin\theta)^2$

34. Hint $\left(\tan\theta + \dfrac{1}{\tan\theta}\right)^2 = (2)^2$

$\Rightarrow \tan^2\theta + \dfrac{1}{\tan^2\theta} + 2 = 4$

Now, put $\tan\theta = \dfrac{1}{\cot\theta}$, we get the required result.

Ans. 2

35. Hint Consider $A + B + C = \pi \Rightarrow B + C = \pi - A$

LHS $= \sin^2\dfrac{A}{2} + \sin^2\left(\dfrac{\pi - A}{2}\right)$

$= \sin^2\dfrac{A}{2} + \sin^2\left(\dfrac{\pi}{2} - \dfrac{A}{2}\right) = \sin^2\dfrac{A}{2} + \cos^2\dfrac{A}{2}$

36. Hint Do same as Example 30.

37. Hint $\dfrac{\tan 28°}{\cot 62°} \times \dfrac{\sqrt{3}}{\tan 20° \tan 60° \tan 70°}$

$$= \frac{\cot 62^\circ}{\cot 62^\circ} \times \frac{\sqrt{3}}{\tan 20^\circ \times \sqrt{3} \times \cot 20^\circ}$$ **Ans.** 1

38. Hint $\because \; A + B + C = \pi$

$$\text{LHS} = \tan^2\left(\frac{\pi}{2} - \frac{A}{2}\right) = \cot^2 \frac{A}{2}$$

39. Hint $\text{LHS} = (\sin^2 A + \cos^2 A)^3 - 3\sin^2 A \cos^2 A$

$$(\sin^2 A + \cos^2 A) + 3\sin^2 A \cos^2 A + 4$$

$= 1 + 4 = 5$ **Ans.** $k = 5$

40. Hint $\sec\theta = x + \frac{1}{4x}$

$$\Rightarrow \sec^2\theta = x^2 + \left(\frac{1}{4x}\right)^2 + 2 \cdot x \cdot \frac{1}{4x}$$

$$\Rightarrow \sec^2\theta - 1 = x^2 + \frac{1}{16x^2} - \frac{1}{2} \Rightarrow \tan^2\theta = \left(x - \frac{1}{4x}\right)^2$$

$$\Rightarrow \quad \tan\theta = \pm\left(x - \frac{1}{4x}\right)$$ **Ans.** $2x, 1/2x$

41. Hint $\text{LHS} = \sqrt{\sec^2\theta + \text{cosec}^2\theta}$

$$= \sqrt{1 + \tan^2\theta + 1 + \cot^2\theta}$$

$$= \sqrt{(\tan\theta)^2 + (\cot\theta)^2 + 2\tan\theta\cot\theta}$$

42. (i) **Hint** Use $\sec^2\theta = 1 + \tan^2\theta$.

Ans. $\left(\frac{x}{a}\right)^2 = 1 + \left(\frac{y}{b}\right)^2$

(ii) **Hint** $\cos\theta = \frac{x-k}{a}$ and $\sin\theta = \frac{y-h}{b}$

Ans. $\left(\frac{x-k}{a}\right)^2 + \left(\frac{y-h}{b}\right)^2 = 1$

43. Hint Do same as Example 27.

44. Hint $\cos\beta = \frac{\cos\alpha}{m}$ and $\sin\beta = \frac{\cos\alpha}{n}$

$\because \quad \cos^2\beta + \sin^2\beta = 1$

45. Hint Put $\cos 60^\circ = \frac{1}{2}$, $\sin 30^\circ = \frac{1}{2}$, $\tan 60^\circ = \sqrt{3}$,

$\cot 45^\circ = 1$ and $\sec 30^\circ = \frac{2}{\sqrt{3}}$

46. Hint LHS

$$= \sqrt{\cos\alpha\,\text{cosec}\,(90^\circ - \alpha) - \cos\alpha \sin(90^\circ - \alpha)}$$

$$= \sqrt{\cos\alpha\sec\alpha - \cos\alpha\cos\alpha}$$

$$= \sqrt{1 - \cos^2\alpha} = \sqrt{\sin^2\alpha}$$

47. Hint Use $\sin(90^\circ - \theta) = \cos\theta$, $\cos(90^\circ - \theta) = \sin\theta$ and $\sin^2\theta + \cos^2\theta = 1$. **Ans.** 2

48. Hint Use $\sin(90 - \theta) = \cos\theta$, $\cos(90 - \theta) = \sin\theta$

and $\sin^2\theta + \cos^2\theta = 1$ **Ans.** $\frac{4}{7}$

49. Hint $\frac{\tan\theta\cos\theta}{\sin\theta} + \frac{\tan 50^\circ}{\tan 50^\circ} - (\cos^2 20^\circ + \sin^2 20^\circ)$

$= 1 + 1 - 1$ **Ans.** 1

50. Hint Use complementary angles formula and also use identity $\sin^2 A + \cos^2 A = 1$.

Ans. (i) 0 (ii) 2 (iii) 0 (iv) 5 (v) 0

51. Hint We have, $\frac{\sec^2(90^\circ - \theta) - \cot^2\theta}{2(\sin^2 25^\circ + \sin^2 65^\circ)}$

$$+ \frac{2\cos^2 60^\circ \tan^2 28^\circ \tan^2 62^\circ}{\sin 30^\circ \cos 60^\circ}$$

$$= \frac{\text{cosec}^2\theta - \cot^2\theta}{2[\sin^2(90^\circ - 65^\circ) + \sin^2 65^\circ]}$$

$$+ \frac{2\cos 60^\circ \tan^2(90^\circ - 62^\circ)\tan^2 62^\circ}{\sin 30^\circ}$$

$$= \frac{1}{2(\cos^2 65^\circ + \sin^2 65^\circ)} + \frac{2 \cdot \frac{1}{2}\cot^2 62^\circ \tan^2 62^\circ}{1/2}$$

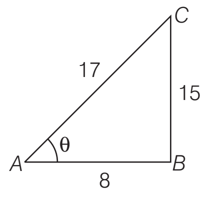
Ans. 5/2

52. Hint $\text{LHS} = \cos\theta\sin\theta - \frac{\sin\theta\sin\theta\cos\theta}{\text{cosec}\,\theta}$

$$- \frac{\cos\theta\cos\theta\sin\theta}{\sec\theta} + \sec\theta$$

$$= \cos\theta\sin\theta - \sin^3\theta\cos\theta - \cos^3\theta\sin\theta + \sec\theta$$
$$= \cos\theta\sin\theta - \sin\theta\cos\theta(\sin^2\theta + \cos^2\theta) + \sec\theta$$

53. Hint Here, $\sin A = \frac{15}{17}$, $\tan A = \frac{15}{8}$, $\cos A = \frac{8}{17}$



54. Hint $\text{LHS} = 1 + \tan^2 A \tan^2 B + 2\tan A \tan B$

$$+ \tan^2 A + \tan^2 B - 2\tan A \tan B$$

$$= \tan^2 A(1 + \tan^2 B) + 1(1 + \tan^2 B)$$
$$= (1 + \tan^2 B)(1 + \tan^2 A)$$

55. Hint Given equation is quadratic in terms of $\cot\theta$.

So, find the roots, i.e. $\cot\theta = \frac{1}{\sqrt{3}}$ and $\cot\theta = \sqrt{3}$.

Ans. $\frac{10}{3}$

56. Do same as Example 12.

57. Hint $(\sin\theta + 2\cos\theta)^2 = 1^2$

$$\Rightarrow \quad \sin^2\theta + 4\cos^2\theta + 2\sin\theta\cos\theta = 1$$

$$\Rightarrow \quad \sin^2\theta + 4(1 - \sin^2\theta) + 2\sin\theta\cos\theta$$

$$= \cos^2\theta + \sin^2\theta$$

$$\Rightarrow \quad 4\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta = 4$$

58. Hint Put $\sin\theta = \dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$ and $\cos\theta = \dfrac{1}{\sqrt{1+\tan^2\theta}}$

59. Hint Put $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

60. Hint $\sec^2 A - \dfrac{\sin^2 A(1-2\sin^2 A)}{\cos^2 A(2\cos^2 A - 1)}$

$$= \sec^2 A - \frac{\tan^2 A[(1-2(1-\cos^2 A)]}{(2\cos^2 A - 1)}$$

61. Hint Put $\cot A = \dfrac{\cos A}{\sin A}, \sec\theta = \dfrac{1}{\cos A}$

62. Hint LHS

$$= \left[\frac{1}{\dfrac{1}{\cos^2\theta} - \cos^2\theta} + \frac{1}{\dfrac{1}{\sin^2\theta} - \sin^2\theta}\right]\sin^2\theta\cos^2\theta$$

$$= \left[\frac{\cos^2\theta}{1-\cos^4\theta} + \frac{\sin^2\theta}{1-\sin^4\theta}\right]\sin^2\theta\cos^2\theta$$

$$= \left[\frac{\cos^2\theta}{(1+\cos^2\theta)(1-\cos^2\theta)} + \frac{\sin^2\theta}{(1-\sin^2\theta)(1+\sin^2\theta)}\right] \times \sin^2\theta\cos^2\theta$$

$$= \left[\frac{\cos^2\theta}{(1+\cos^2\theta)\sin^2\theta} + \frac{\sin^2\theta}{\cos^2\theta(1+\sin^2\theta)}\right] \times \sin^2\theta\cos^2\theta$$

63. Hint LHS $= \dfrac{\left(1 + \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}\right)(\sin\theta - \cos\theta)}{\dfrac{1}{\cos^3\theta} - \dfrac{1}{\sin^3\theta}}$

64. Hint $(\sin^8 A - \cos^8 A) = [(\sin^4 A)^2 - (\cos^4 A)^2)]$

$$= (\sin^4 A + \cos^4 A)(\sin^4 A - \cos^4 A)$$

$$= [(\sin^2 A + \cos^2 A)^2 - 2\sin^2\cos^2 A] \times (\sin^2 A + \cos^2 A)(\sin^2 A - \cos^2 A)$$

$$= (1 - 2\sin^2 A\cos^2 A)(\sin^2 A - \cos^2 A)$$

65. Hint Substitute the values of p and q in LHS.

66. Do same as Example 28 (ii).

67. Hint Put $\sec^2\theta = \tan^2\theta + 1$ and $\operatorname{cosec}^2\theta = \cot^2\theta + 1$.

68. Hint Put the values of x, y and z in LHS and simplify it.

69. Hint Put the values of l and m in the given equation.

70. Hint Put $\tan\theta = \dfrac{\sin\theta}{\cos\theta}, \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

71. Hint $\cot B = \dfrac{a}{\tan A}$ and $\operatorname{cosec} B = \dfrac{b}{\sin A}$

We know that, $\operatorname{cosec}^2 B - \cot^2 B = 1$

$$\Rightarrow \quad \frac{b^2}{\sin^2 A} - \frac{a^2}{\tan^2 A} = 1$$

72. Hint Given, $\operatorname{cosec}\theta - \sin\theta = m$ and $\sec\theta - \cos\theta = n$

$$\Rightarrow \quad \frac{1}{\sin\theta} - \sin\theta = m \text{ and } \frac{1}{\cos\theta} - \cos\theta = n$$

$$\Rightarrow \quad \frac{1-\sin^2\theta}{\sin\theta} = m \quad\text{and}\quad \frac{1-\cos^2\theta}{\cos\theta} = n$$

$$\Rightarrow \quad \frac{\cos^2\theta}{\sin\theta} = m \quad\text{and}\quad \frac{\sin^2\theta}{\cos\theta} = n$$

73. Do same as Example 26.

74. Hint

$$\frac{\sec\theta\sec\theta - \tan\theta\tan\theta + \sin^2(90° - 35°) + \sin^2 35°}{\tan 10°\tan 20°\tan 60°\tan(90° - 20°)\tan(90° - 10°)}$$

$$= \frac{\sec^2\theta - \tan^2\theta + \cos^2 35° + \sin^2 35°}{\tan 10°\tan 20°(\sqrt{3})\cot 20°\cot 10°}$$

Ans. $\dfrac{2}{\sqrt{3}}$

75. Hint $\dfrac{\cos^2 35° + \sin^2 35°}{\operatorname{cosec}^2 15° - \cot^2 15°} + \sqrt{3}$

$$\left(\tan 13°\tan 23° \times \frac{1}{\sqrt{3}} \times \cot 23°\cot 13°\right)$$

Ans. 2

76. Hint Using complementary angle formulae and identity $\sin^2 A + \cos^2 A = 1$.

77. Hint $\dfrac{\operatorname{cosec}^2\theta - \cot^2\theta}{2(\sin^2 25° + \cos^2 25°)}$

$$- \frac{2 \times \frac{1}{4} \times \tan^2 32° \times \cot^2 32°}{2(\sec^2 33° - \operatorname{cosec}^2 33°)} = \frac{1}{2} - \frac{1}{4}$$

Ans. $\dfrac{1}{4}$

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1 Prove that $\sqrt{\sec^2\theta+\operatorname{cosec}^2\theta}=\tan\theta+\cot\theta$.

2 Prove that $(1+\cot\theta-\operatorname{cosec}\theta)(1+\tan\theta+\sec\theta)=2$.

2017

3 Evaluate without using trigonometric tables,
$\sin^2 28^\circ+\sin^2 62^\circ+\tan^2 38^\circ-\cot^2 52^\circ+\frac{1}{4}\sec^2 30^\circ$.

4 Prove that $\frac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}=\tan\theta$.

2016

5 Without using trigonometric tables, evaluate
$$\operatorname{cosec}^2 57^\circ-\tan^2 33^\circ+\cos 44^\circ\operatorname{cosec} 46^\circ-\sqrt{2}\cos 45^\circ-\tan^2 60^\circ.$$

6 Prove that $\frac{\cos A}{1+\sin A}+\tan A=\sec A$.

2015

7 Prove that $\frac{\cos A}{1-\tan A}+\frac{\sin A}{1-\cot A}=\cos A+\sin A$

Or

Prove that $\frac{\sin\theta}{1-\cot\theta}+\frac{\cos\theta}{1-\tan\theta}=\cos\theta+\sin\theta$

2014

8 Without using trigonometric tables, evaluate
$\sin^2 34^\circ+\sin^2 56^\circ+2\tan 18^\circ\tan 72^\circ-\cos^2 30^\circ$.

9 Prove that $(\sin\theta+\cos\theta)(\tan\theta+\cot\theta)=\sec\theta+\operatorname{cosec}\theta$.

2013

10 Without using trigonometric table, evaluate
$3\cos 80^\circ\operatorname{cosec} 10^\circ+2\sin 59^\circ\sec 31^\circ$.

11 Show that $\sqrt{\frac{1-\cos A}{1+\cos A}}=\frac{\sin A}{1+\cos A}$.

2012

12 Without using trigonometric table, evaluate
$\cos^2 26^\circ+\cos 64^\circ\sin 26^\circ+\frac{\tan 36^\circ}{\cot 54^\circ}$.

13 Prove that $\frac{\tan^2\theta}{(\sec\theta-1)^2}=\frac{1+\cos\theta}{1-\cos\theta}$.

2011

14 Prove that
$(\operatorname{cosec} A-\sin A)(\sec A-\cos A)\sec^2 A=\tan A$.

15 Without using trigonometric table, evaluate
$$2\left(\frac{\tan 35^\circ}{\cot 55^\circ}\right)^2+\left(\frac{\cot 55^\circ}{\tan 35^\circ}\right)^2-3\left(\frac{\sec 40^\circ}{\operatorname{cosec} 50^\circ}\right)$$

* All these questions are completely covered in chapter either as solved examples or in chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. If $\sin\alpha$ and $\cos\alpha$ are the roots of the equation $px^2 + qx + r = 0$, then the value of $p^2 - q^2 + 2rp$ is

(a) 1 (b) -1 (c) 0 (d) None of these

2. In ΔABC right angled at B, if $BC = 5$ cm and $AC - AB = 1$ cm, then the value of $\frac{\cos^2 C}{1 - \sin C}$ is

(a) $\frac{12}{13}$ (b) $\frac{13}{12}$ (c) $\frac{25}{13}$ (d) $\frac{13}{25}$

3. If $\operatorname{cosec}\theta - \sin\theta = x^3$ and $\sec\theta - \cos\theta = y^3$, then the value of $x^2y^2(x^2 + y^2)$ is

(a) 0 (b) 1 (c) 2 (d) 3

4. If $T_n = \sin^n\theta + \cos^n\theta$, then the value of $\frac{T_3 - T_5}{T_1}$ is

(a) $\sin^2\theta\cos^2\theta$ (b) $\frac{\cos^2\theta}{\sin^2\theta}$ (c) $\sin^2\theta - \cos^2\theta$ (d) $\cos^3\theta + \sin^2\theta$

5. If $\sin\theta + \sin^2\theta = 1$, then the value of $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta + 2\cos^4\theta + 2\cos^2\theta - 2$ is

(a) 0 (b) 1 (c) 2 (d) 3

6. The value of the expression $1 - \frac{\sin^2 y}{1 + \cos y} + \frac{1 + \cos y}{\sin y} - \frac{\sin y}{1 - \cos y}$ is

(a) 0 (b) 1 (c) $\sin y$ (d) $\cos y$

7. If $(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C) = x$, then the value(s) of x is/are

(a) ± 1 (b) 0 (c) ± 2 (d) 1

8. If $f(x) = \cos^2 x + \sec^2 x$, then $f(x)$

(a) ≥ 1 (b) ≤ 1 (c) ≥ 2 (d) ≤ 2

9. If ABC is a right angled triangle, then find the relation between

$$\tan\left(\frac{A - B - C}{2}\right) \text{ and } -\tan\left(\frac{A + B - C}{2}\right).$$

(a) Equal (b) Unequal (c) Sum of these equal to 1 (d) None of these

10. If $a\sec\theta + b\tan\theta + c = 0$ and $p\sec\theta + q\tan\theta + r = 0$, then $(br - qc)^2 - (pc - ar)^2$ is equal to

(a) $(ap - bq)^2$ (b) $(aq - bp)^2$ (c) $(ap - bq)$ (d) $(aq - bp)$

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 415.

CHAPTER

19

Heights and Distances

Chapter Objectives

- Important Terms Related to Heights and Distances
- Problems Based on Heights and Distances

Measuring of heights and distances is the important application of Trigonometry. Sometimes, we need to find the height of different objects like tower, building, tree, etc., distance of a ship from a light-house, width of river and angle subtended by any object at a given point, etc., it is difficult to measure them directly. In this chapter, we will study the use of trigonometry in measuring the heights and distances of such cases.

Important Terms Related to Heights and Distances

In solving problems related to heights and distances of various objects, we need to define few terms as discussed below

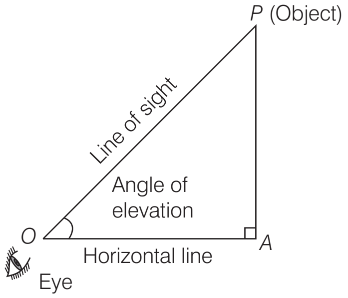
Line of Sight

The line of sight is the line drawn from the eye of an observer to the point in the object viewed by the observer.

Angle of Elevation

The angle of elevation of an object viewed, is the angle formed by the line of sight with the horizontal, when it is above the horizontal level, i.e. the case when we raise our head to look at object.

Let P be the position of the object above the horizontal line OA and O be the eye of the observer. Then, OP is the **line of sight** and $\angle AOP$ is called **angle of elevation**, because the observer has to elevate (raise) his/her line of sight from the horizontal OA to see the object P.



Some important points about angle of elevation are given below

(i) A plane level parallel to Earth's surface is called the horizontal plane level and a line drawn parallel to horizontal plane is called a horizontal line.

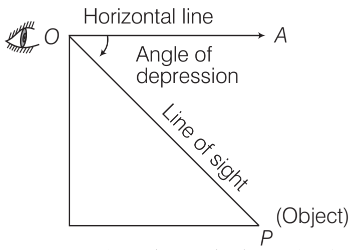
(ii) If the observer moves towards the perpendicular line (tower/building), then angle of elevation increases and if the observer moves away from the perpendicular line (tower/building), then angle of elevation decreases.

(iii) If the height of tower is doubled and the distance between the observer and foot of the tower is also doubled, then the angle of elevation remains same.

(iv) If the angle of elevation of Sun, above a tower decreases, then the length of shadow of a tower increases and *vice-versa*.

Angle of Depression

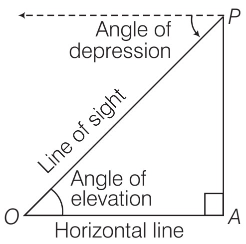
The angle of depression of an object viewed, is the angle formed by the line of sight with the horizontal, when it is below the horizontal level, i.e. the case when we lower our head to look at the object.



Let P be the position of the object below the horizontal level OA and O be the eye of the observer. Then, OP is the **line of sight** and $\angle AOP$ is called an **angle of depression**, because the observer has to depress (lower) his/her line of sight from the horizontal OA to see the object P.

Some important points about angle of depression are given below

(*i*) The angle of elevation of a point P as seen from a point O is always equal to the angle of depression of O as seen from P.



(*ii*) The angles of elevation and depression are always acute angles.

(*iii*) In solving problems, observer is represented by a point and object is represented by a line segment or a point.

Problems Based on Heights and Distances

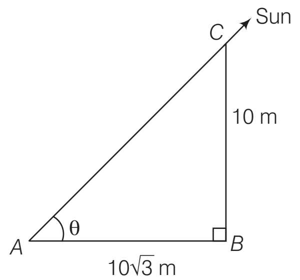
We can solve different types of problems related to heights and distances by applying trigonometric ratios in right angled triangle formed with the help of given information. e.g. The height of a tower or a tree can be determined without climbing over it, width of a river can be measured without crossing it.

It can be easily understood with the help of following examples.

Example 1. Find the angular elevation of the Sun, when the shadow of 10 m long pole is $10\sqrt{3}$ m.

Sol. Let BC be the pole, AB be its shadow and the angle of elevation of the Sun be θ.

Then, $\angle BAC = \theta$, $\angle ABC = 90°$, $AB = 10\sqrt{3}$ m and $BC = 10$ m



In right angled ΔABC, $\tan\theta = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{BC}{AB}$

$\Rightarrow \quad \tan\theta = \dfrac{10}{10\sqrt{3}}$ [given]

$\Rightarrow \quad \tan\theta = \dfrac{1}{\sqrt{3}}$

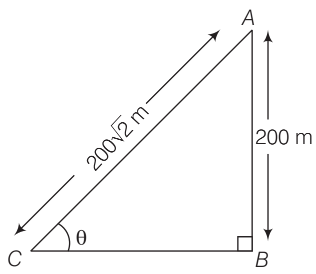
$\Rightarrow \quad \tan\theta = \tan 30° \quad \left[\because \tan 30° = \dfrac{1}{\sqrt{3}}\right]$

$\therefore \quad \theta = 30°$

Hence, the angular elevation of the Sun is 30°.

Example 2. A tower is 200 m high. Find the angle of elevation of its top from a point $200\sqrt{2}$ m away from its top.

Sol. Let height of the tower be $AB = 200$ m and the distance between its top and a point be $AC = 200\sqrt{2}$ m.



Also, let $\angle ACB = \theta$.

In right angled ΔABC,

$$\sin\theta = \frac{AB}{AC} \qquad \left[\because \sin\theta = \frac{\text{perpendicular}}{\text{hypotenuse}}\right]$$

$$= \frac{200}{200\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\theta = \sin 45° \qquad \left[\because \sin 45° = \frac{1}{\sqrt{2}}\right]$$

$$\therefore \quad \theta = 45°$$

Hence, the required angle of elevation of its top from a point is 45°.

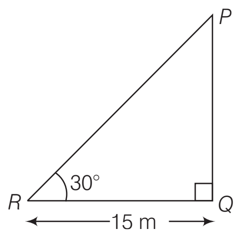
Example 3. From a point 15 m away from the foot of a tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower.

Sol. From the figure, $RQ = 15$ m and $\angle PRQ = 30°$.

In right angled ΔPQR, we have

$$\tan 30° = \frac{PQ}{RQ} \qquad \left[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}\right]$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{PQ}{15} \qquad \left[\because \tan 30° = \frac{1}{\sqrt{3}}\right]$$



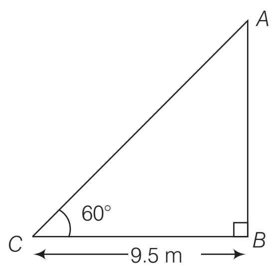
$$\Rightarrow \quad PQ = \frac{15}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{15\sqrt{3}}{3} \qquad \text{[rationalising]}$$

$$\therefore \quad PQ = 5\sqrt{3} = 5 \times 1.732 = 8.66 \text{ m}$$

Hence, the height of the tower is 8.66 m.

Example 4. The angle of elevation of a ladder leaning against a wall is 60° and the foot of the ladder is 9.5 m away from the wall. Find the length of the ladder.

Sol. Let BC be the horizontal ground and AC be the ladder leaning against the wall AB.



Then, $\angle ABC = 90°$, $\angle ACB = 60°$

and $BC = 9.5$ m

In right angled ΔABC, we have

$$\cos 60° = \frac{BC}{AC} \qquad \left[\because \cos\theta = \frac{\text{base}}{\text{hypotenuse}}\right]$$

$$\Rightarrow \quad \frac{1}{2} = \frac{9.5}{AC} \qquad \left[\because \cos 60° = \frac{1}{2}\right]$$

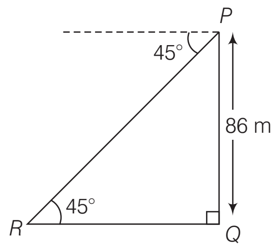
$$\Rightarrow \quad AC = 9.5 \times 2$$

$$\therefore \quad AC = 19 \text{ m}$$

Hence, the length of the ladder is 19 m.

Example 5. From the top of the building 86 m high, the angle of depression of a point on the level ground is 45°. Calculate the distance of the point from the foot of the building.

Sol. Given, $PQ = 86$ m and $\angle PRQ = 45°$



In right angled ΔPQR, we have

$$\tan 45° = \frac{PQ}{QR} \qquad \left[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}\right]$$

$$\Rightarrow \quad 1 = \frac{86}{QR} \qquad [\because \tan 45° = 1]$$

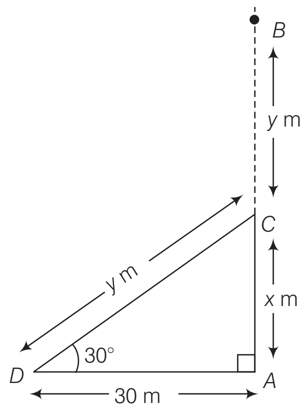
$$\therefore \quad QR = 86 \text{ m}$$

Hence, the distance of a point from the foot of the building is 86 m.

Example 6. In a violent storm, a tree got bent by the wind. The top of the tree meets the ground at an angle of 30°, at a distance of 30 m from the root. At what height from the bottom, did the tree get bent? What was the original height of the tree?

Sol. Let AB be the original height of the tree, which has got bent at a point C. After getting bent, let the part CB take the position CD, meeting the ground at D.

Then, $AD = 30$ m, $\angle ADC = 30°$, $\angle DAC = 90°$ and $CD = CB$.



Let $AC = x$ m and $CD = CB = y$ m.

In right angled ΔDAC, we have

$$\tan 30° = \frac{AC}{AD} \qquad \left[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}\right]$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{x}{30} \left[\because \tan 30° = \frac{1}{\sqrt{3}}, AC = x \text{ and } AD = 30 \text{ m}\right]$$

$$\Rightarrow x = 30 \times \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = 10\sqrt{3} \text{ m [rationalising]} \quad ...(i)$$

and $$\sec 30° = \frac{CD}{AD} \qquad \left[\because \sec\theta = \frac{\text{hypotenuse}}{\text{base}}\right]$$

$$\Rightarrow \frac{2}{\sqrt{3}} = \frac{y}{30} \left[\because \sec 30° = \frac{2}{\sqrt{3}}, CD = y \text{ and } AD = 30 \text{ m}\right]$$

$$\Rightarrow y = 30 \times \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = 20\sqrt{3} \text{ m [rationalising]} \quad ...(ii)$$

Since, $AB = AC + CB$

$\therefore$ $AB = x + y = 10\sqrt{3} + 20\sqrt{3}$ [from Eqs. (i) and (ii)]

$= 30\sqrt{3}$ m

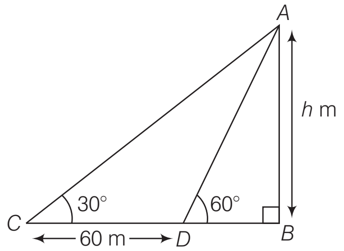
Hence, the tree got bent at a height of $10\sqrt{3}$ m from the ground and the original height of the tree was $30\sqrt{3}$ m.

Example 7. A man observes the angle of elevation of the top of a building to be 30°. He walks towards it in a horizontal line through its base. On covering 60 m, the angle of elevation changes to 60°. Find the height of the building correct to the nearest metres. ***[2011]***

Sol. Let AB be the building of height h m. Initially, the man is at the point C, such that the angle of elevation of the point A is 30°.

On walking 60 m, he reaches point D, such that the angle of elevation of the point A is 60°.

Also, $AB \perp CB$, i.e. $\angle ABC = 90°$



In right angled ΔABC,

$$\tan 30° = \frac{AB}{BC} \qquad \left[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}\right]$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{BC} \qquad \left[\because \tan 30° = \frac{1}{\sqrt{3}}\right]$$

$$\Rightarrow BC = h\sqrt{3} \text{ m} \quad ...(i)$$

In ΔABD, $\tan 60° = \dfrac{AB}{BD} \Rightarrow \sqrt{3} = \dfrac{h}{BD}$

$$\Rightarrow BD = \frac{h}{\sqrt{3}} \text{ m} \quad ...(ii)$$

Now, $BC - BD = CD$

$$\Rightarrow \sqrt{3}h - \frac{h}{\sqrt{3}} = 60 \qquad \text{[from Eqs. (i) and (ii)]}$$

$$\Rightarrow 3h - h = 60\sqrt{3}$$

$$\Rightarrow h = \frac{60\sqrt{3}}{2} = 30 \times 1.732 \qquad [\because \sqrt{3} = 1.732]$$

$$\therefore h = 51.96 \text{ m} \approx 52 \text{ m}$$

Hence, the height of the building is approximately 52 m.

Example 8. The angle of elevation of the top of an unfinished tower from a point distant 120 m from its base is 45°. How much higher must the tower be raised, so that its angle of elevation from the same point may be 60°?

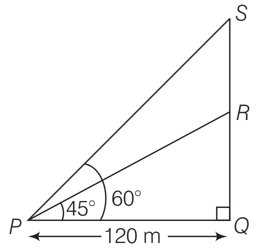
Sol. Given, $PQ = 120$ m, $\angle SPQ = 60°$ and $\angle RPQ = 45°$

In ΔSQP, we have

$$\tan 60° = \frac{SQ}{PQ} \qquad \left[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}\right]$$

$$\Rightarrow \sqrt{3} = \frac{SQ}{120} \qquad [\because \tan 60° = \sqrt{3}]$$

$$\Rightarrow SQ = 120\sqrt{3} \text{ m} \quad ...(i)$$



In ΔRQP, we have

$$\tan 45° = \frac{RQ}{PQ}$$

$$\Rightarrow 1 = \frac{RQ}{120} \qquad [\because \tan 45° = 1]$$

$$\Rightarrow RQ = 120 \text{ m} \quad ...(ii)$$

Now, the height to be raised

$= SQ - RQ$

$= 120\sqrt{3} - 120$ [from Eqs. (i) and (ii)]

$= 120(\sqrt{3} - 1)$

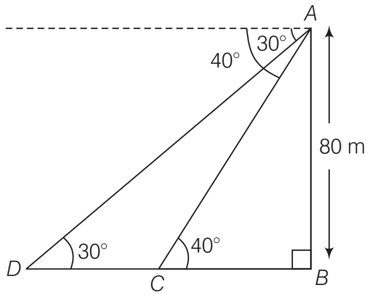
$= 120(1.732 - 1)$ $[\because \sqrt{3} = 1.732]$

$= 120 \times 0.732 = 87.84$ m

Hence, the height to be raised is 87.84 m.

Example 9. As observed from the top of a 80 m tall light-house, the angles of depression of two ships on the same side of the light-house in horizontal line with its base are 30° and 40°, respectively. Find the distance between the two ships. Give your answer correct to the nearest metres. ***[2012]***

Sol. Given, height of a light-house, $AB = 80$ m.



Let C and D be the positions of two ships, such that $\angle ADB = 30°$ and $\angle ACB = 40°$.

In right angled ΔABC, we have

$$\tan 40^\circ = \frac{AB}{BC} \qquad \left[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}\right]$$

$$\Rightarrow \tan(90^\circ - 50^\circ) = \frac{80}{BC}$$

$$\Rightarrow \cot 50^\circ = \frac{80}{BC} \qquad [\because \tan(90^\circ - \theta) = \cot\theta]$$

$$\Rightarrow BC = \frac{80}{\cot 50^\circ} = 80 \times \tan 50^\circ \qquad \left[\because \frac{1}{\cot\theta} = \tan\theta\right]$$

$$= 80 \times 1.1918$$

[from natural tangent table, $\tan 50^\circ = 1.1918$]

$$\therefore \quad BC = 95.34 \text{ m} \qquad \ldots\text{(i)}$$

In right angled ΔABD, we have

$$\tan 30^\circ = \frac{AB}{BD} \Rightarrow \frac{1}{\sqrt{3}} = \frac{80}{DC + BC} \qquad \left[\because \tan 30^\circ = \frac{1}{\sqrt{3}}\right]$$

$$\Rightarrow DC + 95.34 = 80 \times \sqrt{3} \qquad \text{[from Eq. (i)]}$$

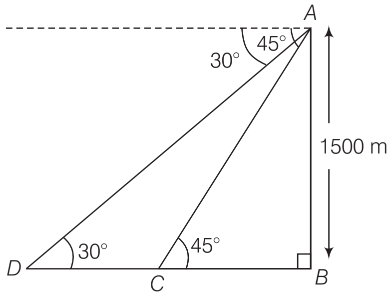
$$\Rightarrow \quad DC = 80 \times 1.732 - 95.34$$

$$\therefore \quad DC = 138.56 - 95.34 = 43.22 \text{ m}$$

Hence, the distance between two ships is 43.22 m.

Example 10. An aeroplane at an altitude of 1500 m finds that two ships are sailing towards it in the same direction. The angles of depression as observed from the aeroplane are 45° and 30°, respectively. Find the distance between the two ships. ***[2016]***

Sol. Let the distance between two ships be $DC = x$ m and height of the aeroplane from the ground is $AB = 1500$ m



In right angled ΔABC,

$$\tan 45^\circ = \frac{AB}{BC} \qquad \left[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}\right]$$

$$\Rightarrow \quad 1 = \frac{1500}{BC} \qquad [\because \tan 45^\circ = 1]$$

$$\Rightarrow \quad BC = 1500 \text{ m}$$

In right angled ΔABD,

$$\tan 30^\circ = \frac{AB}{BD} = \frac{AB}{BC + DC} = \frac{1500}{1500 + x}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{1500}{1500 + x} \qquad \left[\because \tan 30^\circ = \frac{1}{\sqrt{3}}\right]$$

$$\Rightarrow \quad 1500 + x = 1500\sqrt{3}$$

$$\therefore \quad x = 1500(\sqrt{3} - 1) = 1500 \times (1.732 - 1)$$

$$= 1500 \times 0.732 = 1098 \text{ m}$$

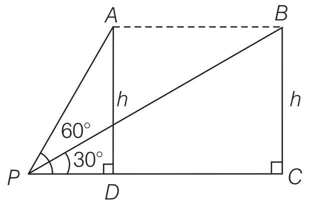
Hence, the distance between the two ships is 1098 m.

Example 11. The angle of elevation of an aeroplane from a point P on the ground is 60°. After 12 s from the same point P, the angle of elevation of the same plane changes to 30°. If the plane is flying horizontally at a speed of $600\sqrt{3}$ km/h, then find the height at which the plane is flying.

Sol. Let A be the position of plane, when the angle of elevation from point P on the ground is 60° and B be the position of plane, when the angle of elevation is 30°.

Given, speed of plane $= 600\sqrt{3}$ km/h

$$= \frac{600\sqrt{3} \times 1000}{60 \times 60} = \frac{500\sqrt{3}}{3} \text{ m/s} \qquad \left[\because 1 \text{ km/h} = \frac{1000}{60 \times 60} \text{ m/s}\right]$$



Distance covered by plane in 12 s

$$= \text{Speed} \times \text{Time} = \frac{500\sqrt{3}}{3} \times 12$$

$$\Rightarrow \text{Distance} = 2000\sqrt{3} \text{ m}$$

$$\therefore \quad AB = CD = 2000\sqrt{3} \text{ m}$$

Let the height of plane (AD) be h m and PD be x m.

In right angled ΔADP,

$$\tan 60^\circ = \frac{AD}{PD} \qquad \left[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}\right]$$

$$\Rightarrow \quad \frac{\sqrt{3}}{1} = \frac{h}{x} \qquad [\because \tan 60^\circ = \sqrt{3}]$$

$$\Rightarrow \quad h = \sqrt{3}x \Rightarrow x = \frac{h}{\sqrt{3}} \qquad \ldots\text{(i)}$$

In right angled ΔBCP,

$$\tan 30^\circ = \frac{BC}{PC} \Rightarrow \frac{1}{\sqrt{3}} = \frac{BD}{PD + CD} \qquad \left[\because \tan 30^\circ = \frac{1}{\sqrt{3}}\right]$$

$$\Rightarrow \quad \sqrt{3}h = x + 2000\sqrt{3} \qquad \left[\begin{aligned} \because \quad AD &= BC = h \text{ m}, \\ PD &= x \text{ m and} \\ AB &= CD = 2000\sqrt{3} \text{ m} \end{aligned}\right]$$

$$\Rightarrow \quad \sqrt{3}h = \frac{h}{\sqrt{3}} + 2000\sqrt{3} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \sqrt{3}h - \frac{h}{\sqrt{3}} = 2000\sqrt{3} \Rightarrow \frac{3h - h}{\sqrt{3}} = 2000\sqrt{3}$$

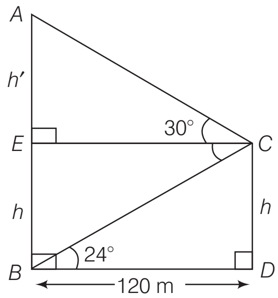
$$\Rightarrow \quad \frac{2h}{\sqrt{3}} = 2000\sqrt{3}$$

$$\Rightarrow \quad h = \frac{2000\sqrt{3} \times \sqrt{3}}{2} = 3000 \text{ m}$$

Hence, the height of plane is 3000 m or 3 km.

Example 12. The horizontal distance between two towers is 120 m. The angle of elevation of the top and angle of depression of the bottom of the first tower as observed from the second tower are 30° and 24°, respectively. Find the heights of the two towers. ***[2015]***

Sol. Let the heights of towers CD and AB be h m and $(h + h')$ m, respectively.



Join CA, CB, BD and draw $CE \perp AB$.

Then, $\quad BD = EC = 120$ m

In right angled ΔBDC,

$$\tan 24° = \frac{CD}{BD} = \frac{h}{120} \qquad \left[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}\right]$$

$\Rightarrow \qquad h = 120 \tan 24° = 120 \times 0.445 = 53.4$ m

In right angled ΔAEC, $\tan 30° = \dfrac{h'}{120}$

$\Rightarrow \qquad h' = 120 \tan 30° = 120 \times 0.577 = 69.24$ m

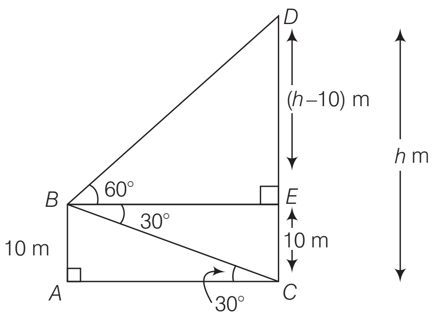
Hence, height of tower $AB = h + h'$

$= 53.4 + 69.24 = 122.64$ m

and height of tower $CD = h = 53.4$ m

Example 13. A man standing on the deck of a ship, which is 10 m above the water level, observes the angle of elevation of the top of the hill as 60° and the angle of depression of the base of the hill as 30°. Find the distance of the hill from the ship and the height of the hill.

Sol. Let AB be the height of the deck and CD be the hill.



Let the man be at B. Then, $AB = 10$ m.

Again, let $BE \perp CD$ and $AC \perp CD$.

Then, $\angle EBD = 60°$ and $\angle EBC = \angle BCA = 30°$

$CE = AB = 10$ m

Let $CD = h$ m. Then, $ED = (h - 10)$ m

In right angled ΔBED, we have

$$\cot 60° = \frac{BE}{ED} \qquad \left[\because \cot\theta = \frac{\text{base}}{\text{perpendicular}}\right]$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{BE}{(h-10)} \Rightarrow BE = \frac{(h-10)}{\sqrt{3}} \text{ m} \quad \left[\because \cot 60° = \frac{1}{\sqrt{3}}\right] \quad \text{...(i)}$$

In right angled ΔCAB, we have

$$\cot 30° = \frac{AC}{AB} \Rightarrow \sqrt{3} = \frac{AC}{10} \qquad [\because \cot 30° = \sqrt{3}]$$

$\Rightarrow \qquad AC = 10\sqrt{3}$ m $\qquad$ ...(ii)

But, $\qquad BE = AC$

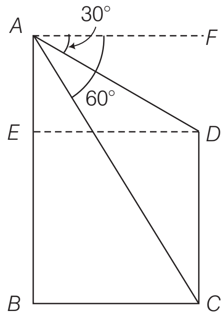
$\therefore \qquad \dfrac{(h-10)}{\sqrt{3}} = 10\sqrt{3} \qquad$ [from Eqs. (i) and (ii)]

$\Rightarrow \qquad (h - 10) = 30 \Rightarrow h = 40$ m

Hence, the distance of the ship from the hill is $10\sqrt{3}$ m and the height of the hill is 40 m.

Example 14. In the following figure, from the top of a building $AB = 60$ m high, the angles of depression of the top and bottom of a vertical lamp-post CD are observed to be 30° and 60°, respectively. Find

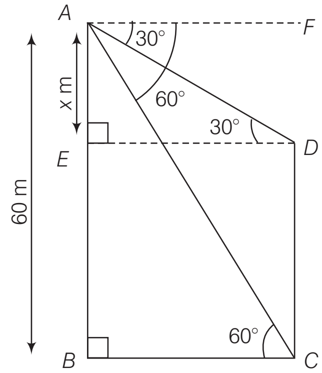
(i) the horizontal distance observed between AB and CD.



(ii) the height of the lamp-post. ***[2013]***

Sol.

(i) Given, height of building $AB = 60$ m and height of lamp-post is CD. Through point D, draw a horizontal line to meet AB at E.



Here, $\angle FAD = \angle EDA = 30°$

and $\quad \angle FAC = \angle BCA = 60° \qquad$ [alternate angles]

In right angled ΔABC, we have

$$\tan 60° = \frac{AB}{BC} \qquad \left[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}\right]$$

$$\Rightarrow \qquad \sqrt{3} = \frac{60}{BC} \qquad [\because \tan 60° = \sqrt{3}]$$

$$\Rightarrow \qquad BC = \frac{60}{\sqrt{3}} = \frac{60}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{60\sqrt{3}}{3} \qquad \text{[rationalising]}$$

$\therefore \qquad BC = 20\sqrt{3}$ m $\qquad$ ...(i)

Hence, the horizontal distance between AB and CD is $20\sqrt{3}$ m.

(ii) Let $AE = x$ m.

In right angled ΔAED, we have $\tan 30° = \dfrac{AE}{ED}$

$\Rightarrow \quad \dfrac{1}{\sqrt{3}} = \dfrac{x}{20\sqrt{3}} \quad \left[\because \tan 30° = \dfrac{1}{\sqrt{3}} \text{ and } BC = ED = 20\sqrt{3} \text{ m}\right]$

$\therefore \quad x = \dfrac{20\sqrt{3}}{\sqrt{3}} = 20$ m

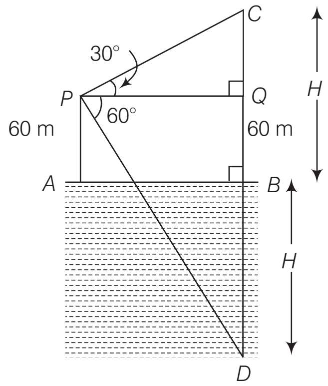
Now, height of the lamp-post, $CD = AB - AE$

$= 60 - 20 = 40$ m $\quad [\because AE = x = 20 \text{ m}]$

Hence, the height of the lamp-post is 40 m.

Example 15. The angle of elevation of a cloud from a point 60 m above a lake is 30° and the angle of depression of the reflection of the cloud in the lake is 60°. Find the height of the cloud.

Sol. Let AB be the surface of the lake and P be a point vertically above A, such that $AP = 60$ m.



Again, let C be the position of the cloud and D be its reflection in the lake.

Let the height of the cloud be H m.

Then, $BC = BD = H$ m

Draw $PQ \perp CD$.

Then, $\angle QPC = 30°$, $\angle QPD = 60°$,

$BQ = AP = 60$ m, $CQ = (H - 60)$ m

and $DQ = (H + 60)$ m

In right angled ΔCQP, we have

$\cot 30° = \dfrac{PQ}{CQ} \quad \left[\because \cot\theta = \dfrac{\text{base}}{\text{perpendicular}}\right]$

$\Rightarrow \quad \sqrt{3} = \dfrac{PQ}{(H - 60)} \quad [\because \cot 30° = \sqrt{3}]$

$\Rightarrow \quad PQ = (H - 60)\sqrt{3}$ m $\quad$ ...(i)

In right angled ΔDQP, we have

$\cot 60° = \dfrac{PQ}{DQ}$

$\Rightarrow \quad \dfrac{1}{\sqrt{3}} = \dfrac{PQ}{(H + 60)} \quad \left[\because \cot 60° = \dfrac{1}{\sqrt{3}}\right]$

$\Rightarrow \quad PQ = \dfrac{(H + 60)}{\sqrt{3}}$ m $\quad$ ...(ii)

From Eqs. (i) and (ii), we get

$(H - 60)\sqrt{3} = \dfrac{(H + 60)}{\sqrt{3}}$

$\Rightarrow \quad 3H - 180 = H + 60 \Rightarrow 2H = 240$

$\therefore \quad H = 120$ m

Hence, the height of the cloud is 120 m.

Example 16. From a window A, 10 m above the ground, the angle of elevation of the top C of a tower is $x°$, where $\tan x° = \dfrac{5}{2}$; and the angle of depression of the foot D of the tower is $y°$, where $\tan y° = \dfrac{1}{4}$.

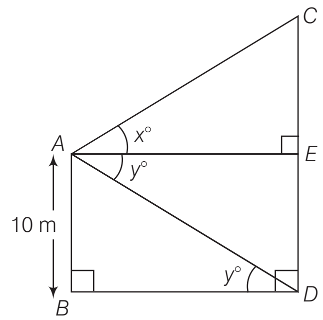
Calculate the height CD (in metres) of the tower. ***[2000]***

Sol. Given, $\tan x° = \dfrac{5}{2}$ and $\tan y° = \dfrac{1}{4}$ $\quad$ ...(i)

Draw $AE \perp CD$ and $BD \perp CD$.

Then, $\quad BD = 10$ m, $\angle CAE = x°$

and $\angle DAE = \angle ADB = y°$ $\quad$ [alternate angles]



In ΔAEC, we have

$\tan x° = \dfrac{CE}{AE} \quad \left[\because \tan\theta = \dfrac{\text{perpendicular}}{\text{base}}\right]$

[from Eq. (i)]

$\Rightarrow \quad \dfrac{5}{2} = \dfrac{CE}{AE}$ $\quad$ ...(ii)

In ΔABD, we have

$\tan y° = \dfrac{AB}{BD}$

$\Rightarrow \quad \dfrac{1}{4} = \dfrac{10}{AE} \quad [\because \text{from Eq. (i) and } BD = AE]$

$\Rightarrow \quad AE = 40$ m

On putting the value of $AE = 40$ in Eq. (ii), we get

$\Rightarrow \quad \dfrac{5}{2} = \dfrac{CE}{40} \Rightarrow CE = \dfrac{40 \times 5}{2} = 100$ m

Now, $\quad CD = CE + DE = CE + AB \quad [\because AB = DE]$

$= 100 + 10 = 110$ m $[\because AB = 10 \text{ m}, CE = 100 \text{ m}]$

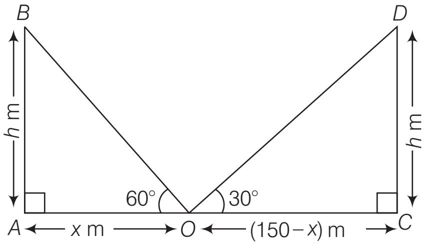
$\therefore \quad CD = 110$ m

Hence, the height of the tower is 110 m.

Example 17. Two pillars of equal heights stand on either side of a road, which is 150 m wide. At a point on the road between the pillars, the angles of elevation of the tops of the pillars are 60° and 30°. Find the height of each pillar and the position of the point on the road.

Sol. Let AB and CD be the given pillars and O be the point of observation on the road AC. Then, $\angle AOB = 60°$, $\angle COD = 30°$, $\angle OAB = 90°$, $\angle OCD = 90°$ and $AC = 150$ m

Let $AB = CD = h$ m and $OA = x$ m



Then, $OC = (150 - x)$ m

In right angled ΔOAB, we have

$$\tan 60° = \frac{AB}{OA} \qquad \left[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}\right]$$

$$\Rightarrow \sqrt{3} = \frac{h}{x} \Rightarrow h = \sqrt{3}x \qquad [\because \tan 60° = \sqrt{3}] \quad \text{...(i)}$$

In right angled ΔOCD, we have

$$\tan 30° = \frac{CD}{OC} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{(150-x)} \qquad \left[\because \tan 30° = \frac{1}{\sqrt{3}}\right]$$

$$\Rightarrow h = \frac{(150-x)}{\sqrt{3}} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\sqrt{3}x = \frac{(150-x)}{\sqrt{3}} \Rightarrow 3x = (150-x)$$

$$\Rightarrow 4x = 150 \Rightarrow x = \frac{150}{4} = \frac{75}{2} \Rightarrow x = 37.5 \text{ m}$$

On putting $x = \frac{75}{2}$ in Eq. (i), we get $h = \frac{75\sqrt{3}}{2}$ m

Hence, the height of each pillar is $\frac{75\sqrt{3}}{2}$ m and the point of observation is 37.5 m away from the first pillar AB.

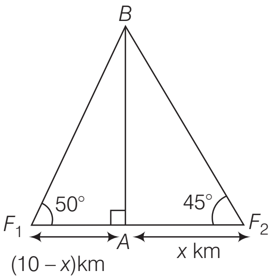
Example 18. A fire at a building B is reported on telephone to two fire stations F_1 and F_2 10 km apart from each other. F_1 observes that the fire is at an angle of 50° from it and F_2 observes that it is at an angle of 45° from it. Which station should send its team and how much distance it has to travel?

Sol. Let $AF_2 = x$ km, then $AF_1 = (10 - x)$ km

In ΔF_1BF_2, clearly $\angle F_1 > \angle F_2$

[$\because$ side opposite to greater angle of a triangle is smaller]

$\therefore$ F_1 should send their team.



In right angled ΔBAF_2,

$$\tan 45° = \frac{AB}{AF_2} \qquad \left[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}\right]$$

$$\Rightarrow 1 = \frac{AB}{AF_2} \qquad [\because \tan 45° = 1]$$

$$\Rightarrow AB = AF_2 = x \text{ km}$$

In right angled ΔBAF_1,

$$\tan 50° = \frac{AB}{AF_1}$$

$$\Rightarrow \tan 50° = \frac{x}{10-x}$$

$$\Rightarrow x = (10-x) \times 1.1918$$

$$\Rightarrow x = 11.918 - 1.1918x$$

$$\Rightarrow 2.1918x = 11.918$$

$$\Rightarrow x = \frac{11.918}{2.1918} = 5.44 \text{ km}$$

$$\Rightarrow F_1A = 10 - x = 10 - 5.44 = 4.56 \text{ km}$$

and $$\sec 50° = \frac{F_1B}{F_1A} \qquad \left[\because \sec\theta = \frac{\text{hypotenuse}}{\text{base}}\right]$$

$$\Rightarrow \frac{1}{\cos 50°} = \frac{F_1B}{4.56} \Rightarrow F_1B = \frac{4.56}{0.6428} = 7.094 \text{ km}$$

Hence, the team from F_1 has to travel 7.094 km.

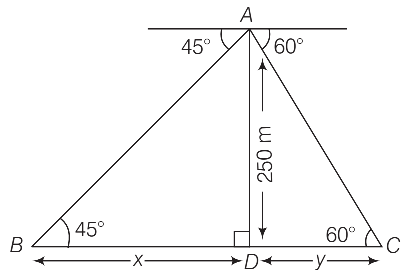
Example 19. An aeroplane at an altitude of 250 m observes the angles of depression of two boats on the opposite banks of a river to be 45° and 60°, respectively. Find the width of the river. Write the answer correct to the nearest whole number. ***[2014]***

Sol. Given, the height of aeroplane is $AD = 250$ m.

Let B and C be the positions of the boats, such that the angles of depression of two boats are 45° and 60°, respectively.

So, $\angle ABD = 45°$ and $\angle ACD = 60°$.

Let the distances of boats from the foot of perpendicular be $BD = x$ m and $DC = y$ m.



In right angled ΔADB, we have

$$\cot 45° = \frac{BD}{AD} = \frac{x}{250} \qquad \left[\because \cot\theta = \frac{\text{base}}{\text{perpendicular}}\right]$$

$$\Rightarrow 1 = \frac{x}{250} \qquad [\because \cot 45° = 1]$$

$$\Rightarrow x = 250 \text{ m}$$

In right angled ΔADC, we have

$$\cot 60° = \frac{DC}{AD} = \frac{y}{250} \Rightarrow \frac{1}{\sqrt{3}} = \frac{y}{250} \qquad \left[\because \cot 60° = \frac{1}{\sqrt{3}}\right]$$

$$\Rightarrow \qquad y = \frac{250}{\sqrt{3}} \text{ m}$$

Now, width of the river, $BC = BD + DC = x + y$

$$= 250 + \frac{250}{\sqrt{3}} = 250\left(1 + \frac{1}{\sqrt{3}}\right)$$

$$= 250\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)$$

$$= 250\left(\frac{1.732+1}{1.732}\right) \qquad [\because \sqrt{3} = 1.732]$$

$$= \frac{250 \times 2.732}{1.732} = 250 \times 1.577 = 394.25 \text{ m}$$

$$= 394 \text{ m (approx.)}$$

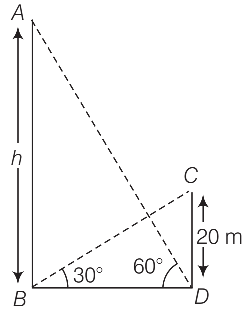
Hence, the width of the river is 394 m.

Example 20. The angle of elevation of the top of a hill at the foot of tower is 60° and the angle of elevation of the top of tower from the foot of hill is 30°. If the tower is 20 m high, then find

(i) the height of the hill.

(ii) the distance between the hill and the tower.

Sol. Let AB (= h m) be the height of the hill and CD (= 20 m) be the height of the tower.



Then, $\angle ADB = 60°$ and $\angle CBD = 30°$ [given]

In right angled ΔABD,

$$\tan 60° = \frac{AB}{BD} \qquad \left[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}\right]$$

$$\Rightarrow \qquad \sqrt{3} = \frac{h}{BD} \qquad [\because \tan 60° = \sqrt{3}]$$

$$\Rightarrow \qquad h = \sqrt{3}\, BD \qquad \ldots\text{(i)}$$

In right angled ΔCDB,

$$\tan 30° = \frac{CD}{BD} \Rightarrow \frac{1}{\sqrt{3}} = \frac{20}{BD} \qquad \left[\because \tan 30° = \frac{1}{\sqrt{3}}\right]$$

$$\Rightarrow \qquad BD = 20\sqrt{3} \text{ m} \qquad \ldots\text{(ii)}$$

(i) On putting the value of BD from Eq. (ii) in Eq. (i), we get

$$h = \sqrt{3}(20\sqrt{3}) = 60 \text{ m}$$

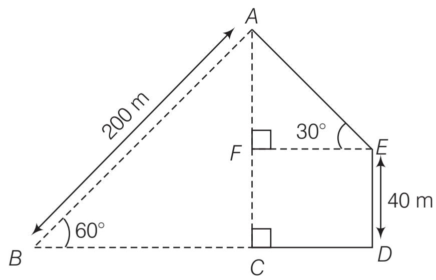
Hence, the height of the hill is 60 m.

(ii) From Eq. (ii), $BD = 20\sqrt{3} = 20 \times 1.732 = 34.64$ m

Hence, the distance between the hill and the tower is 34.64 m.

Example 21. A girl standing on the ground finds a bird flying at a distance of 200 m from her at an elevation of 60°. A man standing on the roof of 40 m high building, finds the angle of elevation of the same bird to be 30°. The girl and the man are on opposite sides of the bird. Find the distance of the bird from the man, correct to nearest metres.

Sol. Given, $AB = 200$ m, $FC = ED = 40$ m, $\angle ABC = 60°$ and $\angle AEF = 30°$.



In right angled ΔACB, we have

$$\sin 60° = \frac{AC}{AB} \qquad \left[\because \sin\theta = \frac{\text{perpendicular}}{\text{hypotenuse}}\right]$$

$$\Rightarrow \qquad \frac{\sqrt{3}}{2} = \frac{AC}{200} \qquad \left[\because \sin 60° = \frac{\sqrt{3}}{2}\right]$$

$$\Rightarrow \qquad AC = \frac{200\sqrt{3}}{2} = 100\sqrt{3}$$

$$= 100 \times 1.732 \qquad [\because \sqrt{3} = 1.732]$$

$$\therefore \qquad AC = 173.2 \text{ m}$$

Now, $\qquad AF = AC - FC$

$$= 173.2 - 40 = 133.2 \text{ m} \qquad \ldots\text{(i)}$$

In right angled ΔAFE, we have

$$\sin 30° = \frac{AF}{AE}$$

$$\Rightarrow \qquad \frac{1}{2} = \frac{133.2}{AE} \qquad \left[\sin 30° = \frac{1}{2} \text{ and from Eq. (i)}\right]$$

$$\Rightarrow \qquad AE = 133.2 \times 2 = 266.4 \text{ m}$$

$$\therefore \qquad AE = 266 \text{ m (approx.)}$$

Hence, the bird is 266 m away from the man.

CHAPTER EXERCISE

a 3 Marks Questions

1. What is the angle of elevation of the Sun, when the length of the shadow of a tree is equal to its vertical height?

2. The height of a tree is $\sqrt{3}$ times the length of its shadow. Find the angle of elevation of the Sun.

3. A tower is $100\sqrt{3}$ m high. Find the angle of elevation of its top from a point $100\sqrt{6}$ m away from its top.

4. The angle of elevation of the top of a tower from a point on the ground at a distance of 160 m from its foot, is found to be 60°. Find the height of the tower.

5. A ladder is placed along a wall, such that its upper end is resting against vertical wall. The foot of the ladder is 2.4 m away from the wall and the ladder is making an angle of 68° with the ground.
Find the height, upto which the ladder reaches.

6. If the length of a shadow casted by a pole is $1/\sqrt{3}$ times the length of the pole, then find the angle of elevation of the Sun.

7. A kite is flying at a height of 75 m from the level ground, attached to a string inclined at 60° to the horizontal. Find the length of the string to the nearest metres.

8. A vertical pole stands on the level ground. From a point on the ground, 25 m away from the foot of the pole, the angle of elevation of its top is found to be 60°. Find the height of the pole.

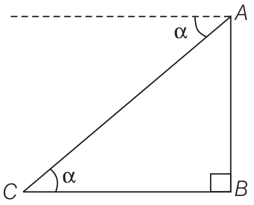
9. From the top of a cliff 108 m high, the angle of depression of a buoy is 28°. Calculate to the nearest metres, the distance of the buoy from the foot of the cliff.

10. A man standing on the top of a building observing a flag 9 m away from the building such that the angle of depression of the flag to be 28°. Find the height of the building.

11. From the top of a light-house 100 m high, the angle of depression of the ship is observed as 36°. Find the distance of a ship from the foot of the light-house.

12. The height of a tree is 10 m. It is bent by the wind in such a way that its top touches the ground and makes an angle of 60° with the ground. At what height from the bottom, did the tree get bent? [Take $\sqrt{3} = 1.732$]

13. In the following figure, not drawn to scale, AB is a tower. The angle of depression of C from A is α, where $\tan \alpha = 2/5$ and $BC = 200$ m. Calculate the height of the tower AB.



14. A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60°. When he moves 50 m away from the bank, he finds that the angle of elevation to be 30°. Find the height of the tree and the breadth of the river. ***[2003]***

15. Two climbers are at points A and B on a vertical cliff face. To an observer C, 40 m from the foot of the cliff, on the level ground, A is at an elevation of 48° and B of 57°. What is the distance between the climbers?

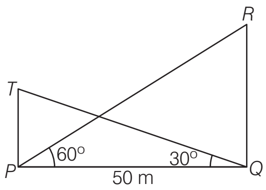
16. Two persons are standing on the opposite sides of a tower. They observe the angles of elevation of the top of the tower to be 30° and 38°, respectively. Find the distance between them, if the height of the tower is 50 m.

17. The angles of depression of two ships A and B as observed from the top of a light-house 60 m high are 60° and 45°, respectively. If two ships are on the opposite sides of the light-house, then find the distance between the two ships. Give your answer correct to the nearest whole number. ***[2017]***

b 4 Marks Questions

18. An observer 1.2 m tall and standing 28.2 m away from the tower. If the angle of elevation of the top of the tower from his eye is 60°, then what is the height of the tower?

19. The angle of elevation from a point P of the top of tower QR, 50 m high is 60° and that of the tower PT from a point Q is 30°. Find the height of the tower PT, connect to the nearest metre. ***[2018]***



20. A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is 60°. After sometime, the angle of elevation reduces to 30°. Find the distance travelled by the balloon during the interval.

21. The shadow of a vertical tower on level ground increases by 10 m, when the altitude of the Sun changes from angle of elevation 45° to 30°. Find the height of the tower, correct to one place of decimal. [Take $\sqrt{3} = 1.732$] ***[2006, 2002]***

22. The horizontal distance between two towers is 60 m. The angle of depression of the top of the first tower, when seen from the top of the second tower, is 30°. If the height of the second tower is 90 m, then find the height of the first tower.

23. A TV tower stands vertically on a bank of a river. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is 60°. From a point 20 m away this point on the same bank, the angle of elevation of the top of the tower is 30°. Find the height of the tower and width of the river.

24. A vertical pole and a vertical tower are on the same level ground. From the top of the pole, the angle of elevation of the top of the tower is 60° and the angle of depression of the foot of the tower is 30°. Find the height of the tower, if the height of the pole is 20 m. ***[2008]***

25. A statue 1.46 m tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point, the angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal. [Take $\sqrt{3} = 1.732$]

26. A man on the deck of a ship, 16 m above the water level, observes that the angles of elevation and depression respectively of the top and bottom of a cliff are 60° and 30°. Calculate the distance of the cliff from the ship.

27. At the foot of a mountain, the angle of elevation of its summit is 45°. After ascending 1000 m towards the mountain up a slope of 30° inclination, the elevation is found to be 60°. Find the height of the mountain.

28. From the top of a hill, the angles of depression of two consecutive kilometre stones due East are found to be 30° and 45°. Find the height of the hill and the distance of two stones from the foot of the hill. ***[2007]***

29. From the top of a light-house 100 m high, the angles of depression of two ships on opposite sides of it are 48° and 36°, respectively. Find the distance between the two ships to the nearest metres.

30. From the top of a church spire 96 m high, the angles of depression of two vehicles on road, at the same level as the base of the spire and on the same side of it, are α and β, where $\tan\alpha = 3/4$ and $\tan\beta = 1/3$. Calculate the distance between the vehicles.

31. From two points A and B on the same side of a building, the angles of elevation of the top of the building are 30° and 60°, respectively. If the height of the building is 10 m, then find the distance between A and B, correct to two decimal places.

32. A man on the top of a vertical tower observes a car moving at a uniform speed coming directly towards it. If it takes 12 min for the angle of depression to change from 30° to 45°, how soon after this, will the car reach the tower? Give your answer to nearest seconds.

33. A man on the top of a hill observes a truck at an angle of depression $x°$, where $\tan x° = \dfrac{1}{\sqrt{5}}$ and sees that it is moving towards the base of the hill. 10 min later, the angle of depression of the truck is found to be $y°$, where $\tan\ y° = \sqrt{5}$. Assuming that the truck moves at a uniform speed, determine how much more time it will take to reach the base of the tower?

34. A woman on a cliff observes a boat at an angle of depression of 30°, which is approaching the shore to the point immediately beneath the observer with a uniform speed. 6 min later, the angle of depression of the boat is found to be 60°. Find the time taken by the boat to reach the shore.

35. From the top of a tower, the angles of depression of two objects on the same side of the tower are found to be α and $\beta\,(\alpha > \beta)$, respectively. If the distance between the objects is a m, then show that the height of the tower is $h = \dfrac{a\tan\alpha\tan\beta}{\tan\alpha - \tan\beta}$ m.

36. The angle of elevation of a jet plane from a point A on the ground is 60°. After a flight of 30 s, the angle of elevation changes to 30°. If the jet plane is flying at a constant height of $3600\sqrt{3}$ m, then find the speed of the jet plane.

37. A bird is perched on the top of a tree 20 m high and its angle of elevation from a point on the ground is 45°. The bird flies-off horizontally straight away from the observe and in 1 s, the angle of elevation of the bird reduces to 30°. Find the speed of the bird.

38. An aeroplane when flying at a height of 4000 m from the ground, passes vertically above another aeroplane at an instant, when the angles of the elevation of the two planes from the same point on the ground are 60° and 45°, respectively. Find the vertical distance between the aeroplanes at that instant.

39. The radius of a circle is given as 15 cm and chord AB subtends an angle of 131° at the centre C of the circle, using trigonometry. Calculate

(i) the length of AB.

(ii) the distance of AB from the centre C.

40. The angle of elevation of a stationary cloud from a point 25 m above a lake is 30° and the angle of depression of its reflection in the lake is 60°. What is the height of the cloud above that lake-level?

41. The angle of elevation of the top of a hill at the foot of a tower is 60° and the angle of elevation of the top of the tower from the foot of hill is 30°. If the tower is 50 m high, then find the height of the hill.

42. The angle of elevation of the top B of a tower AB from a point X on the ground is 60°. At a point Y, 40 m vertically above X, the angle of elevation of the top is 45°. Find the height of the tower AB and the distance XB.

Hints and Answers

1. Do same as Example 1. **Ans.** 45°

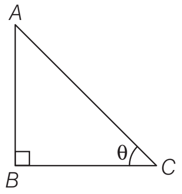
2. Do same as Example 1. **Ans.** 60°

3. Do same as Example 2. **Ans.** 45°

4. Do same as Example 3. **Ans.** 277.12 m

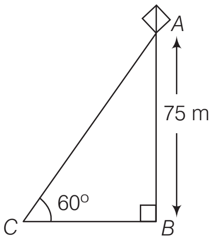
5. Do same as Example 4. **Ans.** 5.94 m

6. **Hint** Let AB be the pole and BC be its shadow.



Then, $BC = \frac{1}{\sqrt{3}} AB \Rightarrow \frac{BC}{AB} = \frac{1}{\sqrt{3}}$ **Ans.** 60°

7. **Hint** Let AB (=75 m) be the height of kite from the ground and $\angle ACB = 60°$.



Also, let length of the string be AC.

Now, in ΔABC,

$$\sin 60° = \frac{AB}{AC} \Rightarrow \frac{\sqrt{3}}{2} = \frac{75}{AC}$$

Ans. 87 m (approx.)

8. Do same as Example 3. **Ans.** 43.3 m

9. Do same as Example 5. **Ans.** 203 m

10. Do same as Example 5. **Ans.** 4.785 m

11. Do same as Example 5. **Ans.** 137.65 m

12. Do same as Example 6. **Ans.** 4.64 m

13. **Hint** In the given ΔABC,

$$\tan\alpha = \frac{AB}{BC}$$

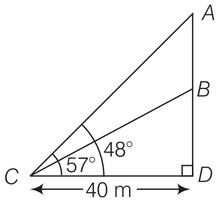
$$\Rightarrow \quad \frac{2}{5} = \frac{AB}{200}$$

Ans. 80 m

14. Do same as Example 7.

Ans. 43.3 m, 25 m

15. **Hint** Here, $CD = 40$ m, $\angle ACD = 57°$ and $\angle BCD = 48°$



Let distance between the climbus A and B

$$= AD - BD \quad \text{...(i)}$$

Now, in right angled ΔADC,

$$\tan 57° = \frac{AD}{CD}$$

$$\Rightarrow \quad 1.5399 = \frac{AD}{40}$$

$$\Rightarrow \quad AD = 61.596$$

and in right angled $\Delta\, BDC$, $\tan 48° = \dfrac{BD}{CD}$

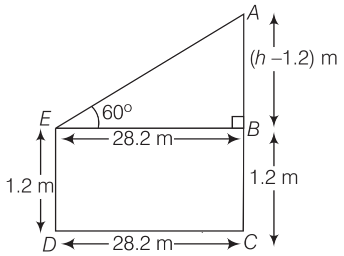
$$\Rightarrow \quad 1.1106 = \frac{BD}{40} \Rightarrow BD = 44.424$$

On substituting AD and BD in Eq. (i), we get the required result. **Ans.** 17.17 m

16. Do same as Example 18. **Ans.** 150.6 m

17. Do same as Example 19. **Ans.** 94

18. Hint In right angled ΔABE,



$$\tan 60° = \frac{AB}{EB} \Rightarrow \sqrt{3} = \frac{h - 1.2}{28.2}$$

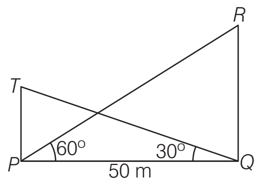
Ans. 50 m (approx.)

19. Hint In ΔPQR,

$$\tan 60° = \frac{QR}{PQ} \Rightarrow \sqrt{3} = \frac{50}{PQ}$$

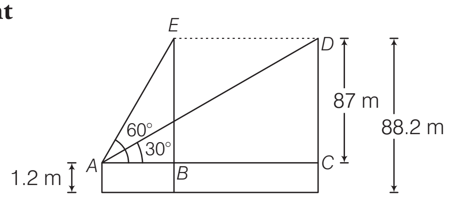
Similarly in PQT,

$$\tan 30° = \frac{PT}{PQ}$$



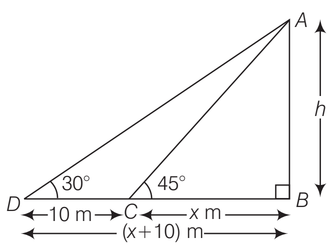
Ans. $PT = 16.66$ m

20. Hint



Ans. $BC = 80$ m

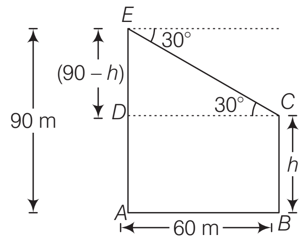
21. Hint In right angled ΔABC, $\tan 45° = \dfrac{AB}{CB}$



and in right angled $\Delta\, ABD$,

$$\tan 30° = \frac{AB}{BD} \Rightarrow \frac{1}{\sqrt{3}} = \frac{x}{x+10}$$ **Ans.** 13.66 m

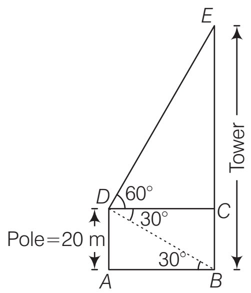
22. Hint $\tan 30° = \dfrac{ED}{CD}$



Ans. 55.36 m

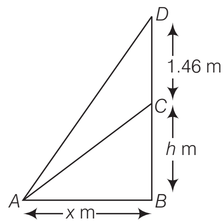
23. Do same as Example 7. **Ans.** $10\sqrt{3}$ m, 10 m

24. Hint



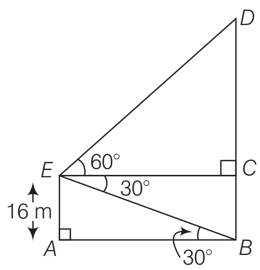
Ans. 80 m

25. Hint



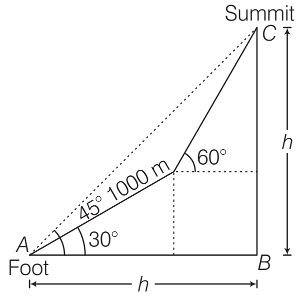
Ans. 2 m

26. Hint



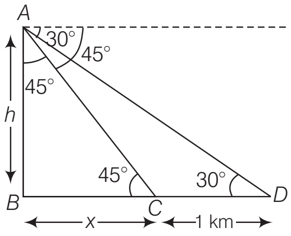
Ans. 27.7 m

27. Hint



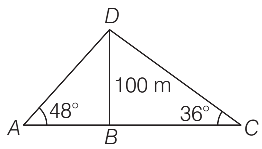
Ans. 1366 m

28. Hint



Ans. 1.365 km, 2.365 km

29. Hint

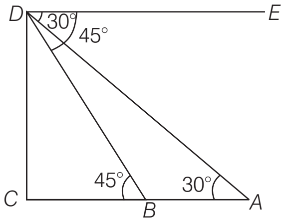


Ans. 228 m

30. Do same as Example 9. **Ans.** 160 m

31. Do same as Example 9. **Ans.** 11.55 m

32. Hint

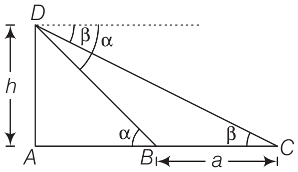


Further, solve as Example 10.
Also, use $AB = 12x$, where x is uniform speed.
Ans. 16 m, 23 s

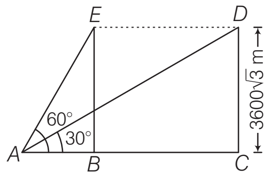
33. Do same as Q. 32. **Ans.** 150 s

34. Do same as Q. 32. **Ans.** 9 min

35. Hint



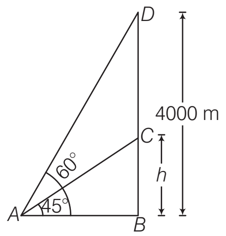
36. Hint



Further, solve as Example 11. **Ans.** 864 km/h

37. Do same as Q. 36. **Ans.** 14.64 m/s

38. Hint

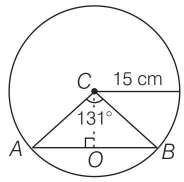


Ans. 1690.53 m

39. Hint Draw $OC \perp AB$

Then, $AO = OB$ and $AC = BC = 15$ cm

Also, $\angle ACO = \angle BCO = \dfrac{131°}{2}$



(i) Now, in right angled ΔAOC,

$$\sin\frac{131°}{2} = \frac{OA}{AC}$$

$$\Rightarrow \quad OA = 0.91 \times 15 = 13.65$$

$$\therefore \quad AB = 2OA$$

(ii) In right angled ΔAOC,

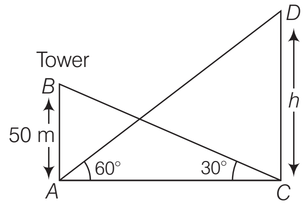
$$\cos\frac{131°}{2} = \frac{OC}{AC}$$

$$\Rightarrow \quad OC = 0.415 \times 15$$

Ans. (i) 27.3 cm (ii) 6.23 cm

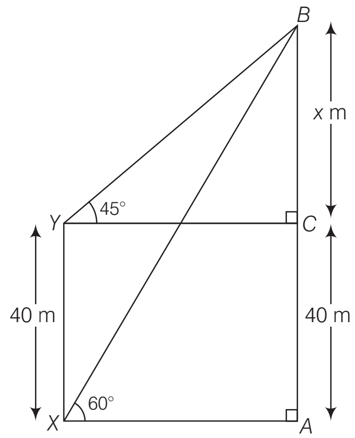
40. Do same as Example 15.
Ans. 50 m

41. Hint



Ans. 150 m

42. Hint



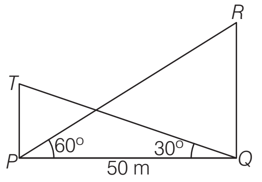
Ans. $AB = 20(\sqrt{3} + 3)$ m, $BX = (40 + 2\sqrt{3})$ m

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1 The angle of elevation from a point P of the top of tower QR, 50 m high is 60° and that of the tower PT from a point Q is 30°. Find the height of the tower PT, connect to the nearest metre.



2017

2 The angle of depression of two ships A and B as observed from the top of a light-house 60 m high are 60° and 45°, respectively. If the two ships are on the opposite sides of the light-house, then find the distance between the two ships. Give your answer correct to the nearest whole number.

2016

3 An aeroplane at an altitude of 1500 m finds that two ships are sailing towards it in the same direction. The angles of depression as observed from the aeroplane are 45° and 30° respectively. Find the distance between the two ships.

2015

4 The horizontal distance between two towers is 120 m. The angle of elevation of the top and angle of depression of the bottom of the first tower as observed from the second tower are 30° and 24°, respectively. Find the heights of the two towers.

2014

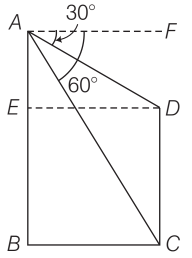
5 An aeroplane at an altitude of 250 m observes the angles of depression of two boats on the opposite banks of a river to be 45° and 60°, respectively. Find the width of the river. Write the answer correct to the nearest whole number.

2013

6 In the adjoining figure, from the top of a building $AB = 60$ m high, the angles of depression of the top and bottom of a vertical lamp-post CD are observed to be 30° and 60°, respectively. Find

(i) the horizontal distance observed between AB and CD.

(ii) the height of the lamp-post.



2012

7 As observed from the top of a 80 m tall light-house, the angles of depression of two ships on the same side of the light-house in horizontal line with its base are 30° and 40°, respectively. Find the distance between the two ships. Give your answer correct to the nearest metres.

2011

8 A man observes the angle of elevation of the top of a building to be 30°. He walks towards it in a horizontal line through its base. On covering 60 m, the angle of elevation changes to 60°. Find the height of the building correct to the nearest metres.

* All these questions are completely covered in chapter either as solved examples or in chapter exercise.

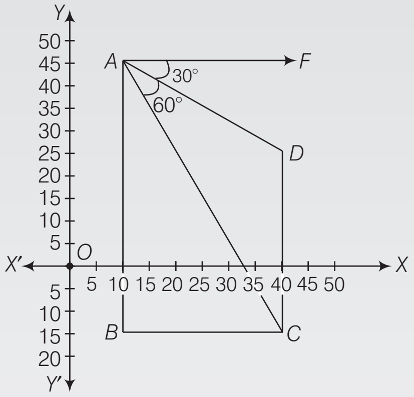
CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. A tower stands at the centre of a circular park. If A and B are two points on the boundary of the park, such that $AB = a$ m subtends an angle of 60° at the foot of the tower and the angle of elevation of the top of the tower from A or B is 30°, then the height of the tower is

(a) $\sqrt{3}\,a$ m (b) $a/\sqrt{3}$ m (c) $\frac{\sqrt{3}}{a}$ m (d) None of these

Directions (Q.Nos. 2- 4) *In the given figure, from the top of a building AB, 60 m high, the angles of depression of the top and the bottom of a vertical lamp-post CD are observed to be* 30° *and* 60°, *respectively.*



2. Find the horizontal distance between BA and CD.

(a) $60\sqrt{3}$ m (b) $40\sqrt{3}$ m (c) $20\sqrt{3}$ m (d) $10\sqrt{3}$ m

3. Find the height of the lamp-post CD.

(a) 60 m (b) 40 m (c) 20 m (d) 10 m

4. Find the radius of the circle, if Y-axis and AB are the tangent to the circle.

(a) 20 m (b) 15 m (c) 10 m (d) 5 m

5. First, plot the points $A(2, 4)$, $B(6, 4)$, $C(6, 2)$ and $D(2, 2)$ and join all adjacent points. A pole BE of height h is standing on point B. If angle of elevation of the top of a pole from point A is 30°, then the total area formed by the figure is

(a $8(\sqrt{3}+1)$ m^2 (b) $8(\sqrt{3}-1)$ m^2

(c) $\frac{8(\sqrt{3}+1)}{\sqrt{3}}$ m^2 (d) $\frac{8(\sqrt{3}-1)}{\sqrt{3}}$ m^2

6. If the angle of elevation of a cloud from a point h m above a lake is α and angle of depression of its reflection in the lake is β, then the height of the cloud is

(a) $\frac{h(\tan\beta+\tan\alpha)}{\tan\beta-\tan\alpha}$ m (b) $\frac{h(\tan\beta-\tan\alpha)}{\tan\beta+\tan\alpha}$ m

(c) $\frac{h\tan\beta+\tan\alpha}{\tan\beta-\tan\alpha}$ m (d) $\frac{h\tan\beta-\tan\alpha}{\tan\beta+\tan\alpha}$ m

7. A ladder rests against a vertical wall at an inclination α to the horizontal. If its foot is pulled away from the wall through a distance p, so that its upper end slides at distance q down the wall and the ladder makes an angle β to the horizontal, then $\frac{\cos\beta-\cos\alpha}{\sin\alpha-\sin\beta}$ is equal to

(a) q/p (b) p/q (c) pq (d) $1/pq$

8. A spherical balloon of radius r subtends an angle θ at the eye of the observer. If the angle of elevation of its centre is ϕ, then the height of the centre of balloon is

(a) $r\sin\phi/2\cos\theta$ (b) $r\sin\phi\ \operatorname{cosec}\theta$

(c) $r\sin\phi\ \operatorname{cosec}\theta/2$ (d) None of these

9. ABC is a triangular park with $AB = AC = 100$ m. A clock tower is situated at the mid-point of BC. The angles of elevation of the top of the tower at A and B are α and β respectively, where $\cot\alpha = 3.2$ and $\operatorname{cosec}\beta = 2.6$. Then, the height of the tower is

(a) 625 m (b) 25 m (c) 60 m (d) 100 m

10. The angle of elevation of the top of a tower (i.e. East) from a point A due South of it, is α, where $\tan\alpha = 6$ and that from B due West of it, is β, where $\tan\beta = 0.75$. If h is the height of the tower, then $AB = \lambda h$. Then, λ^2 is

(a) 65/36 (b) 36/65 (c) 16/9 (d) 9/16

11. The base of a cliff is circular. From the extremities of a diameter of the base, angles of elevation of the top of the cliff are 30° and 60°, respectively. If the height of the cliff is 500 m, then the diameter of the base of the cliff will be

(a) 2000 m (b) $\frac{2000}{\sqrt{3}}$ m (c) $2000\sqrt{3}$ m (d) $\frac{500}{\sqrt{3}}$ m

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 416.

CHAPTER

20

Statistics

Statistics is the study of collection, presentation analysis and interpretation of numerical data.

In the previous class, we have studied the classification of given data into ungrouped and grouped frequency distribution, representation of data (such as bar graphs, histograms, frequency polygon) and mean, median, for the ungrouped data.

In this chapter, we will study the measures of central tendency, i.e. mean, median and mode, from ungrouped data to that of grouped data along with graphical representation of data.

Chapter Objectives

- Mean of Grouped Data
- Median of Grouped Data
- Mode of Grouped Data
- Graphical Representation

Topic 1
Mean of Grouped Data

Mean or Arithmetic Mean or Average

The arithmetic mean of a set of observations is obtained by dividing the sum of the values of all observations by the total number of observations.

Thus, the mean of n observations $x_1, x_2, x_3, \dots, x_n$, is defined as

$$\text{Mean } (\bar{x}) = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

where, the Greek letter 'Σ' (sigma) means 'Summation.'

Let $x_1, x_2, \dots, x_n$ be n observations with respective frequencies $f_1, f_2, \dots, f_n$. This means observation x_1 occurs f_1 times, x_2 occurs f_2 times and so on.

Then, sum of the values of all the observations

$$= f_1 x_1 + f_2 x_2 + \dots + f_n x_n = \sum_{i=1}^{n} f_i x_i$$

and total number of observations

$$= f_1 + f_2 + \dots + f_n = \sum_{i=1}^{n} f_i$$

$$\therefore \quad \boxed{\text{Mean } (\bar{x}) = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}}$$

It can also be written as $\boxed{\bar{x} = \frac{\Sigma fx}{\Sigma f}}$.

Note $\bar{x}$ is read as 'x bar'.

Example 1. Find the mean of the following data.

x	10	30	50	70	89
f	7	8	10	15	10

Sol. Table for given data is

x_i	f_i	$f_i x_i$
10	7	70
30	8	240
50	10	500
70	15	1050
89	10	890
Total	$\Sigma f_i = 50$	$\Sigma f_i x_i = 2750$

Here, $\Sigma f_i = 50$ and $\Sigma f_i x_i = 2750$

$$\therefore \quad \text{Mean } (\bar{x}) = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{2750}{50} = 55$$

Hence, mean of the given data is 55.

Example 2. The mean of the following data is 14. Find the value of k.

x	5	10	15	20	25
f	7	k	8	4	5

Sol. Table for the given data is

x_i	f_i	$f_i x_i$
5	7	35
10	k	$10k$
15	8	120
20	4	80
25	5	125
Total	$\Sigma f_i = k + 24$	$\Sigma f_i x_i = 10\,k + 360$

Here, $\Sigma f_i = k + 24$ and $\Sigma f_i x_i = 10k + 360$

Given, mean = 14

$$\therefore \quad \frac{\Sigma f_i x_i}{\Sigma f_i} = 14 \Rightarrow \frac{10k + 360}{k + 24} = 14$$

$$\Rightarrow \quad 10k + 360 = 14(k + 24)$$

$$\Rightarrow \quad 10k + 360 = 14k + 336$$

$$\Rightarrow \quad 14k - 10k = 360 - 336$$

$$\Rightarrow \quad 4k = 24 \Rightarrow k = \frac{24}{4} = 6$$

Method of Calculating Mean of Grouped Data

To calculate the mean of grouped data, we have following three methods

1. Direct method
2. Assumed mean or shortcut method
3. Step-deviation method

The process of finding the mean by these methods are given below

1. Direct Method

In this method, we find the class marks of each class interval. These class marks would serve as the representative of whole class and are denoted by x_i. In general, for the ith class interval, we have the frequency f_i corresponding to the class mark x_i.

Example 3. The marks obtained by 30 students of Class X of a certain school in a Mathematics paper consisting of 100 marks are presented in table below

Class interval	10-25	25-40	40-55	55-70	70-85	85-100
Number of students	2	3	7	6	6	6

Find the mean of the marks obtained by the students.

Sol. The class marks for each class are given below

For 10-25, $x_1 = \frac{10 + 25}{2} = \frac{35}{2} = 17.5$

For 25-40, $x_2 = \frac{25 + 40}{2} = \frac{65}{2} = 32.5$

For 40-55, $x_3 = \frac{40 + 55}{2} = \frac{95}{2} = 47.5$

For 55-70, $x_4 = \frac{55 + 70}{2} = \frac{125}{2} = 62.5$

For 70-85, $x_5 = \frac{70 + 85}{2} = \frac{155}{2} = 77.5$

For 85-100, $x_6 = \frac{85 + 100}{2} = \frac{185}{2} = 92.5$

Now, the table for the given data is

Class interval	Frequency (f_i)	x_i	$f_i x_i$
10-25	2	17.5	35.0
25-40	3	32.5	97.5
40-55	7	47.5	332.5
55-70	6	62.5	375.0
70-85	6	77.5	465.0
85-100	6	92.5	555.0
Total	$\Sigma f_i = 30$		$\Sigma f_i x_i = 1860.0$

On putting the values of $\Sigma f_i x_i$ and Σf_i in the formula

$$\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i}, \text{ we get } \bar{x} = \frac{1860.0}{30} = 62$$

Hence, the mean marks obtained by the students are 62.

Example 4. Calculate the mean of the following distribution.

Class interval	0-10	10-20	20-30	30-40	40-50	50-60
Frequency	8	5	12	35	24	16

[2015]

Sol. The table for given data is

Class interval	Class marks (x_i)	Frequency (f_i)	$f_i x_i$
0-10	5	8	40
10-20	15	5	75
20-30	25	12	300
30-40	35	35	1225
40-50	45	24	1080
50-60	55	16	880
Total		$\Sigma f_i = 100$	$\Sigma f_i x_i = 3600$

$$\therefore \quad \text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{3600}{100} = 36$$

Example 5. If the mean of the following distribution is 24, find the value of 'a'.

Marks	0-10	10-20	20-30	30-40	40-50
Number of students	7	a	8	10	5

[2018]

Sol.

Marks	Mid value (X)	Number of students (f)	$f \cdot x$
0-10	5	7	35
10-20	15	a	$15a$
20-30	25	8	200
30-40	35	10	350
40-50	45	5	225
Total		$\Sigma f = 30 + a$	$\Sigma f = 810 + 15a$

$$\because \quad \text{Mean} = \frac{\Sigma fx}{\Sigma f}$$

$$\therefore \quad 24 = \frac{810 + 15a}{30 + a}$$

$$\Rightarrow \quad 24(30 + a) = 810 + 15a$$

$$\Rightarrow \quad 720 + 24a = 810 + 15a$$

$$\Rightarrow \quad 24a - 15a = 810 - 720$$

$$\Rightarrow \quad 9a = 90$$

$$\Rightarrow \quad a = \frac{90}{9} = 10$$

Hence, the required value of a is 10.

2. Assumed Mean or Shortcut Method

The cases in which numerical values of x_i and f_i are large and computation of product of x_i and f_i becomes tedious and time consuming, assumed mean method is used. In this method, first of all, one among x_i's is chosen as the **assumed mean** denoted by 'A'. After that, the difference d_i between A and each of the x_i's, i.e. $d_i = x_i - A$ is calculated.

Then, arithmetic mean is given by

$$\bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}, \text{ where } d_i = x_i - d.$$

It can also be written as $\bar{x} = A + \dfrac{\Sigma fd}{\Sigma f}$, where $d = x - A$.

Note We may take A to be that x_i which lies in the middle of $x_1, x_2, \ldots, x_n$.

Example 6. The following table gives the percentage distribution of female teachers in the primary schools of rural areas of various states and union territories of India

Percentage of female teachers	Number of states/UT
15-25	6
25-35	11
35-45	7
45-55	4
55-65	4
65-75	2
75-85	1

Find the mean percentage of female teachers by assumed mean method.

Sol. The class marks for each class are given below

For 15-25, $x_1 = \dfrac{15+25}{2} = \dfrac{40}{2} = 20$

For 25-35, $x_2 = \dfrac{25+35}{2} = \dfrac{60}{2} = 30$

For 35-45, $x_3 = \dfrac{35+45}{2} = \dfrac{80}{2} = 40$

For 45-55, $x_4 = \dfrac{45+55}{2} = \dfrac{100}{2} = 50$

For 55-65, $x_5 = \dfrac{55+65}{2} = \dfrac{120}{2} = 60$

For 65-75, $x_6 = \dfrac{65+75}{2} = \dfrac{140}{2} = 70$

For 75-85, $x_7 = \dfrac{75+85}{2} = \dfrac{160}{2} = 80$

Here, 50 lies in the centre of 20 , 30, ..., 80.

$\therefore$ Let $A = 50$.

Now, $d_1 = 20 - 50 = -30$, $d_2 = 30 - 50 = -20$

$d_3 = 40 - 50 = -10$, $d_4 = 50 - 50 = 0$

$d_5 = 60 - 50 = 10$, $d_6 = 70 - 50 = 20$, $d_7 = 80 - 50 = 30$

Now, let us make the following table

Percentage of female teachers	Class marks (x_i)	Number of States/UT (f_i)	$d_i = x_i - A$	$f_i d_i$
15-25	20	6	– 30	– 180
25-35	30	11	– 20	– 220
35-45	40	7	– 10	– 70
45-55	50	4	0	0
55-65	60	4	10	40
65-75	70	2	20	40
75-85	80	1	30	30
Total		$\Sigma f_i = 35$		$\Sigma f_i d_i = -360$

Here, $A = 50$, $\Sigma f_i d_i = -360$ and $\Sigma f_i = 35$

On putting these values in the formula $\bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$, we get

$$\bar{x} = 50 + \frac{(-360)}{35} = 50 - \frac{360}{35} = 50 - 10.29 = 39.71$$

Hence, the required mean percentage of female teachers is 39.71.

Example 7. Find the mean of the following frequency distribution using assumed mean method.

Class	2-8	8-14	14-20	20-26	26-32
Frequency	6	3	12	11	8

Sol. The table for the given data is

Class	Frequency (f_i)	x_i	$d_i = x_i - A$	$f_i d_i$
2-8	6	5	$5 - 17 = -12$	$6(-12) = -72$
8-14	3	11	$11 - 17 = -6$	$3(-6) = -18$
14-20	12	$17 = A$	$17 - 17 = 0$	$12 \times 0 = 0$
20-26	11	23	$23 - 17 = 6$	$11 \times 6 = 66$
26-32	8	29	$29 - 17 = 12$	$8 \times 12 = 96$
Total	$\Sigma f_i = 40$			$\Sigma f_i d_i = 72$

Here, $\Sigma f_i = 40$ and $\Sigma f_i d_i = 72$

$\therefore$ Mean $(\bar{x}) = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$

$$= 17 + \frac{72}{40} = 17 + 1.8$$

$$= 18.8$$

Example 8. Calculate the mean of the distribution given below using the shortcut method.

Marks	11-20	21-30	31-40	41-50	51-60	61-70	71-80
Number of students	2	6	10	12	9	7	4

[2014]

Sol. The table for the given data is

Marks	Number of students (f_i)	Mid value (x_i)	$d_i = x_i - A$	$f_i d_i$
11-20	2	15.5	–30	– 60
21-30	6	25.5	–20	– 120
31-40	10	35.5	–10	– 100
41-50	12	$45.5 = A$	0	0
51-60	9	55.5	10	90
61-70	7	65.5	20	140
71-80	4	75.5	30	120
Total	$\Sigma f_i = 50$			$\Sigma f_i d_i = 70$

Here, $\Sigma f_i = 50$ and $\Sigma f_i d_i = 70$

By using shortcut method,

$$\text{Mean marks} = A + \frac{\Sigma f_i d_i}{\Sigma f_i} = 45.5 + \frac{70}{50}$$

$$= 45.5 + 1.4 = 46.9$$

Hence, the mean marks of the given distribution is 46.9.

3. Step-deviation Method

Step-deviation method is used in the cases, where the deviation from the assumed mean are multiples of a common number. If the values of d_i for each class is a multiple of h (say), the calculation becomes simpler by taking $u_i = \frac{d_i}{h} = \frac{x_i - A}{h}$.

Generally, the value of h is equal to the difference between upper and lower class limits of a class or the difference between two consecutive class marks. The arithmetic mean is given by

$$\bar{x} = A + \left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right) \times h, \text{ where } u_i = \frac{x_i - A}{h}.$$

(i) The assumed mean method and step deviation method are just simplified forms of the direct method.

(ii) If the given data is discontinuous, there is no need to convert it into continuous data, while finding the mean. Because mid value x_i of each class interval would be same either class interval is continuous or not continuous.

Example 9. Find the mean of the following data by using step-deviation method.

Class	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	4	28	15	20	17	16

Sol. Here, class width, $h = 20 - 10 = 10$.

Now, let the assumed mean, $A = 35$.

Then, the table for the given data is

Class	x_i	f_i	$u_i = \frac{x_i - A}{h}$	$f_i u_i$
10-20	$\frac{10+20}{2} = 15$	4	$\frac{15-35}{10} = \frac{-20}{10} = -2$	−8
20-30	$\frac{20+30}{2} = 25$	28	$\frac{25-35}{10} = \frac{-10}{10} = -1$	−28
30-40	$\frac{30+40}{2} = 35 = A$	15	$\frac{35-35}{10} = \frac{0}{10} = 0$	0
40-50	$\frac{40+50}{2} = 45$	20	$\frac{45-35}{10} = \frac{10}{10} = 1$	20
50-60	$\frac{50+60}{2} = 55$	17	$\frac{55-35}{10} = \frac{20}{10} = 2$	34
60-70	$\frac{60+70}{2} = 65$	16	$\frac{65-35}{10} = \frac{30}{10} = 3$	48
Total		$\Sigma f_i = 100$		$\Sigma f_i u_i = 66$

Here, $\Sigma f_i u_i = 66$, $\Sigma f_i = 100$, $A = 35$ and $h = 10$

$$\therefore \text{ Mean } (\bar{x}) = A + \left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right) \times h$$

$$= 35 + \frac{66}{100} \times 10$$

$$= 35 + \frac{660}{100} = 35 + 6.6 = 41.6$$

Example 10. Calculate the mean of the following distribution using step-deviation method.

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of students	10	9	25	30	16	10

[2017]

Sol. Let the assumed mean A be 25 and $h = 10$.

Class intervals	Mid values (x_i)	Frequency (f_i)	$d_i = x_i - 25$	$u_i = \frac{x_i - A}{h}$	$f_i u_i$
0-10	5	10	− 20	− 2	− 20
10-20	15	9	− 10	− 1	− 9
20-30	25	25	0	0	0
30-40	35	30	10	1	30
40-50	45	16	20	2	32
50-60	55	10	30	3	30
Total		$N = \Sigma f_i = 100$			$\Sigma f_i u_i = 63$

Here, $A = 25$, $h = 10$, $\Sigma f_i = 100$ and $\Sigma f_i u_i = 63$

$$\therefore \text{ Mean} = A + \left\{\frac{\Sigma f_i u_i}{\Sigma f_i}\right\} \times h = 25 + \frac{63}{100} \times 10$$

$$= 25 + \frac{63}{10} = \frac{250 + 63}{10} = 31.3$$

Selection of Method for Calculating the Mean The value of mean obtained by all the three methods is the same. So, the choice of method to be used depends on the numerical values of x_i and f_i.

(i) If x_i and f_i are sufficiently small, then we use the direct method.

(ii) If x_i and f_i are numerically large numbers, then we can use assumed mean method or step-deviation method.

(iii) If the class sizes are unequal and x_i are numerically large, we can still apply the step-deviation method by taking h to be a suitable divisor of all the d_i's.

Example 11. In a class test, marks obtained by 120 students are given in the following frequency distribution. If it is given that mean is 59, then find the missing frequencies x and y.

Marks	Number of students
0-10	1
10-20	3
20-30	7
30-40	10
40-50	15
50-60	x
60-70	9
70-80	27
80-90	18
90-100	y

Sol. Let the assumed mean, $A = 55$ and width of class interval, $h = 10$.

The table for the given data is

Marks	Class mark (x_i)	Number of students (f_i)	$u_i = \frac{x_i - A}{h}$	$f_i u_i$
0-10	5	1	− 5	− 5
10-20	15	3	− 4	− 12
20-30	25	7	− 3	− 21
30-40	35	10	− 2	− 20
40-50	45	15	−1	− 15
50-60	$55 = A$	x	0	0
60-70	65	9	1	9
70-80	75	27	2	54
80-90	85	18	3	54
90-100	95	y	4	$4y$
Total		$N = \Sigma f_i = 90 + x + y$		$\Sigma f_i u_i = 44 + 4y$

Here, $N = 90 + x + y = 120$

$\Rightarrow \quad x + y = 30 \quad$ ...(i)

$\because$ Mean = 59 [given]

$$\Rightarrow \quad A + \left(\frac{\Sigma f_i u_i}{N}\right) \times h = 59$$

$$\Rightarrow \quad 55 + \left(\frac{44 + 4y}{120}\right) \times 10 = 59$$

$$\Rightarrow \quad \frac{11 + y}{3} = 4$$

$$\Rightarrow \quad 11 + y = 12$$

$$\Rightarrow \quad y = 1$$

On putting the value of y in Eq. (i), we get

$$x + 1 = 30 \Rightarrow x = 29$$

Hence, $x = 29$ and $y = 1$.

Topic Exercise 1

1. Find the mean of the following distribution by direct method.

Class interval	0-10	10-20	20-30	30-40	40-50
Number of workers	7	10	15	8	10

2. Find the mean of the following data by direct method.

Class	25-35	35-45	45-55	55-65	65-75
Frequency	6	10	8	12	4

3. The table below shows the daily expenditure on food of 50 households in a locality

Daily expenditure (in ₹)	0-100	100-200	200-300	300-400	400-500
Number of workers	6	9	15	12	8

Find the mean daily expenditure on food by direct method.

4. If the mean of given data is 50, find the value of p.

Class interval	0-20	20-40	40-60	60-80	80-100
Frequency	17	28	32	p	19

5. Find the missing frequency for the given frequency distribution table, if the mean of the distribution is 18.

Class interval	11-13	13-15	15-17	17-19	19-21	21-23	23-25
Frequency	3	6	9	13	f	5	4

6. Calculate the mean of the following data.

Class	4-7	8-11	12-15	16-19
Frequency	5	4	9	10

7. Find the mean of the following frequency distribution by assumed mean method.

Class	25-30	30-35	35-40	40-45	45-50	50-55	55-60
Frequency	14	22	16	6	5	3	4

[2011]

8. The data on number of patients attending a hospital in a month are given below

Number of patients	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Number of days	4	4	7	20	12	8	5

Find the average number of patients attending the hospital by using assumed mean method.

9. Using assumed mean method, find the mean of the following frequency distribution.

Class	63-65	66-68	69-71	72-74	75-77
Frequency	4	3	7	8	3

10. Find the mean of the following distribution by step-deviation method. *[2013]*

Class interval	20-30	30-40	40-50	50-60	60-70	70-80
Frequency	10	6	8	12	5	9

11. Find the mean of the following data by using step-deviation method.

Class	0-20	20-40	40-60	60-80	80-100
Frequency	17	28	32	24	19

12. The monthly pocket money of students of a class is given in the following frequency distribution

Pocket money (in ₹)	100-125	125-150	150-175	175-200	200-225
Number of students	14	8	12	5	11

Find the mean of pocket money using step-deviation method.

13. Calculate the mean of the scores of 20 students in a Mathematics test.

Marks	10-20	20-30	30-40	40-50	50-60
Number of students	2	4	7	6	1

14. If the mean of the distribution is 33.2 and the sum of all frequencies is 100, then find the missing frequencies f_1 and f_2.

Class	6-14	14-22	22-30	30-38	38-46	46-54	54-62	62-70
Frequency	11	21	f_1	15	14	8	f_2	6

Hints and Answers

1. Hint

Class interval	Class marks (x_i)	Number of workers (f_i)	$f_i x_i$
0-10	5	7	35
10-20	15	10	150
20-30	25	15	375
30-40	35	8	280
40-50	45	10	450
Total		$\Sigma f_i = 50$	$\Sigma f_i x_i = 1290$

$$\therefore \quad \text{Mean } (\bar{x}) = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{1290}{50}$$

Ans. 25.8

2. Do same as Q. 1. **Ans.** 49.5

3. Do same as Q. 1. **Ans.** 264

4. Do same as Example 5. **Ans.** $p = 24$

5. Do same as Example 5. **Ans.** $f = 8$

6. Hint Here, class interval are not continuous. But it does not affect mid-values (x_i).

So, we will solve it without making it continuous.

Class	Class marks (x_i)	Frequency (f_i)	$f_i x_i$
4-7	5.5	5	27.5
8-11	9.5	4	38
12-15	13.5	9	121.5
16-19	17.5	10	175
Total		$\Sigma f_i = 28$	$\Sigma f_i x_i = 362$

$$\therefore \quad \text{Mean } (\bar{x}) = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{362}{28}$$

Ans. 12.93

7. Do same as Example 7. **Ans.** 36.85

8. Do same as Example 7. **Ans.** 37.66

9. Hint Here, class interval are not continuous. We can solve it without making it continuous.

Class	Class marks (x_i)	Frequency (f_i)	$d_i = x_i - 70$	$f_i d_i$
63-65	64	4	−6	−24
66-68	67	3	−3	−9
69-71	$70 = A$	7	0	0
72-74	73	8	3	24
75-77	76	3	6	18
Total		$\Sigma f_i = 25$		$\Sigma f_i d_i = 9$

$$\therefore \quad \text{Mean } (\bar{x}) = A + \frac{\Sigma f_i d_i}{\Sigma f_i} = 70 + \frac{9}{25}$$ **Ans.** 70.36

10. Do same as Example 9. **Ans.** 49.6

11. Do same as Example 10. **Ans.** 50

12. Do same as Example 10. **Ans.** ₹ 158

13. Do same as Example 10. **Ans.** 35

14. Do same as Example 11. **Ans.** $f_1 = 16$, $f_2 = 9$

Topic 2
Median of Grouped Data

Median

Median is defined as the middle-most or the central observation, when the observations are arranged either in ascending or descending order of their magnitudes.

Median divides the arranged series into two equal parts, i.e. 50% of the observations lie below the median and the remaining are above the median.

Let n be the total number of observations and let they are arranged in ascending or descending order.

Median of the data depends on the number of observations (n).

***Case* I** If n is odd, then

$$\boxed{\textbf{Median} = \textbf{Value of} \left(\frac{n+1}{2}\right)\textbf{th observation}}$$

e.g. If five girls of different heights are made to stand in a row, in descending order of their heights, then the height of the third girl from either end is median height.

Since, $n = 5$ is odd.

$$\therefore \quad \text{Median} = \left(\frac{n+1}{2}\right)\text{th observation} = \frac{5+1}{2}$$

$$= \frac{6}{2} = 3\text{rd observation}$$

Height of the 3rd girl is median height

***Case* II** If n is even, then

$$\boxed{\begin{aligned}\textbf{Median} &= \textbf{Mean of the values of} \left(\frac{n}{2}\right)\textbf{th} \\ &\qquad \textbf{and} \left(\frac{n}{2}+1\right)\textbf{th observations} \\ &= \frac{1}{2} \times \textbf{Value of} \left[\left(\frac{n}{2}\right)\textbf{th} + \left(\frac{n}{2}+1\right)\textbf{th}\right] \textbf{observations}\end{aligned}}$$

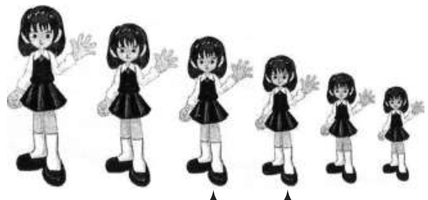
e.g. If six girls of different heights are made to stand in a row, in descending order of their heights, then the mean height of third and fourth girl from either end is the median height.

Since, $n = 6$ is even.

So, $\frac{n}{2} = \frac{6}{2} = 3$rd observation

and $\left(\frac{n}{2}+1\right) = \frac{6}{2} + 1 = \frac{6+2}{2} = 4$th observation

$\therefore$ Median = Mean of 3rd and 4th observations



Mean height of 3rd and 4th girl is median height

Example 1. Find the median of the first ten prime numbers.

Sol. First ten prime numbers in ascending order are 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29.

Here, $n = 10$ (even)

$$\therefore \text{Median} = \frac{1}{2} \times \text{Value of} \left[\left(\frac{n}{2}\right)\text{th} + \left(\frac{n}{2}+1\right)\text{th}\right]\text{observation}$$

$$= \frac{1}{2} \times \text{Value of} \left[\left(\frac{10}{2}\right)\text{th} + \left(\frac{10}{2}+1\right)\text{th}\right]\text{observation}$$

$$= \frac{1}{2}[\text{Value of 5th observation} + \text{Value of 6th observation}]$$

$$= \frac{1}{2}(11 + 13) = \frac{24}{2} = 12$$

Example 2. The mean of following numbers is 68. Find the value of x.

45, 52, 60, x, 69, 70, 26, 81 and 94

Hence, estimate the median. ***[2016]***

Sol. Given, mean $(\bar{x}) = 68$

$$\therefore \quad \bar{y} = \frac{\Sigma x}{n} = \frac{\text{Sum of the observations}}{\text{Number of observations}}$$

$$\Rightarrow \quad 68 = \frac{[45 + 52 + 60 + x + 69 + 70 + 26 + 81 + 94]}{9}$$

$$\Rightarrow \quad 68 = \frac{697 + x}{9} \Rightarrow 497 + x = 612$$

$$\therefore \quad x = 612 - 497 = 115$$

Now, for the median, we have to write the given data in ascending order 26, 45, 52, 60, 69, 70, 81, 94, 115

Here, $n = 9$ (odd)

$$\therefore \text{Median} = \left(\frac{n+1}{2}\right)\text{th term}$$

$$= \left(\frac{10}{2}\right)\text{th term} = 5\text{th term} = 69$$

Cumulative Frequency

The frequency of an observation in a data refers to how many times that observation occur in the data.

Cumulative frequency of a class is defined as the sum of all frequencies up to the given class.

The cumulative frequency is usually observed by constructing a **cumulative frequency table**.

Formation of cumulative frequency table can be understood with the help of an example which is given below

Example 3. The set of data given below shows the ages of participants in a certain summer camp. Draw a cumulative frequency table for the data.

Age (in years)	10	11	12	13	14	15
Frequency	3	18	13	12	7	27

Sol. The cumulative frequency of first observation is the same as its frequency, since there is no frequency before it.

Now, the cumulative frequency table is

Age (in years)	**Frequency**	**Cumulative frequency** (*cf*)
10	3	3
11	18	$3+18=21$
12	13	$21+13=34$
13	12	$34+12=46$
14	7	$46+7=53$
15	27	$53+27=80$

Cumulative Frequency Distribution

Cumulative frequency distribution is of two types

(i) Less than type (ii) More than type

Formation of these two distributions can be understood with the help of an example.

Example 4. Consider a grouped frequency distribution of marks obtained, out of 100, by 58 students, in a certain examination, as follows

Marks	Number of students
0-10	5
10-20	7
20-30	4
30-40	2
40-50	3
50-60	6
60-70	7
70-80	9
80-90	8
90-100	7

Form the cumulative frequency distribution of less than type and more than type.

Sol. **Cumulative frequency distribution of the less than type** Here, the number of students who have scored marks less than 10 are 5. The number of students who have scored marks less than 20 includes the number of students who have scored marks from 0-10 as well as the number of students who have scored marks from 10-20.

Thus, the total number of students with marks less than 20 is $5+7$, i.e. 12. So, the cumulative frequency of the class 10-20 is 12.

Similarly, on computing the cumulative frequencies of the other classes, i.e. the number of students with marks less than 30, less than 40, ..., less than 100, we get the distribution which is called the cumulative frequency distribution of the less than type.

Marks obtained	Number of students (Cumulative frequency)
Less than 10	5
Less than 20	$5+7=12$
Less than 30	$12+4=16$
Less than 40	$16+2=18$
Less than 50	$18+3=21$
Less than 60	$21+6=27$
Less than 70	$27+7=34$
Less than 80	$34+9=43$
Less than 90	$43+8=51$
Less than 100	$51+7=58$

Here, 10, 20, 30,..., 100 are the upper limits of the respective class intervals.

Cumulative frequency distribution of the more than type For this type of distribution, we make the table for the number of students with scores, more than or equal to 0, more than or equal to 10, more than or equal to 20 and so on. From the question, we observed that all 58 students have scored marks more than or equal to 0.

There are 5 students scoring marks in the interval 0-10, it shows that there are $58-5=53$ students getting more than or equal to 10 marks. In the same manner, the number of students scoring 20 marks or above $=53-7=46$ students, scoring 30 or above $=46-4=42$ students and so on.

Similarly, computing the cumulative frequencies of the other classes, i.e. the number of students with marks more than or equal to 40, more than or equal to 50,..., we get the distribution which is called the cumulative frequency distribution of the more than type.

Marks obtained	Number of students (Cumulative frequency)
More than or equal to 0	58
More than or equal to 10	$58-5=53$
More than or equal to 20	$53-7=46$
More than or equal to 30	$46-4=42$
More than or equal to 40	$42-2=40$
More than or equal to 50	$40-3=37$

Marks obtained	Number of students (Cumulative frequency)
More than or equal to 60	37 – 6 = 31
More than or equal to 70	31 – 7 = 24
More than or equal to 80	24 – 9 = 15
More than or equal to 90	15 – 8 = 7

Here, 0, 10, 20, 30,..., 90 are the lower limits of the respective class intervals.

Example 5. The following distribution gives the daily income of 50 workers of a factory

Daily income (in ₹)	100-120	120-140	140-160	160-180	180-200
Number of workers	12	14	8	6	10

Write the above distribution as 'less than type' cumulative frequency distribution.

Sol. We construct a cumulative frequency distribution of the less than type as follows

Daily income (in ₹)	Number of workers	Cumulative frequency (cf)
Less than 120	12	12
Less than 140	14	12 + 14 = 26
Less than 160	8	26 + 8 = 34
Less than 180	6	34 + 6 = 40
Less than 200	10	40 +10 = 50

Example 6. The following distribution gives cumulative frequencies of 'more than type'

Marks obtained (More than or equal to)	5	10	15	20
Number of students (Cumulative frequency)	30	23	8	2

Change the above data into a continuous grouped frequency distribution.

Sol. Given distribution is the more than type distribution. Here, we observe that, all 30 students have obtained marks more than or equal to 5. Further, since 23 students have obtained score more than or equal to 10. So, 30 – 23 = 7 students lie in the class 5-10. Similarly, we can find the other classes and their corresponding frequencies. Now, we construct the continuous grouped frequency distribution as

Class (Marks obtained)	Number of students
5-10	30 – 23 = 7
10-15	23 – 8 = 15
15-20	8 – 2 = 6
More than or equal to 20	2

Median for Discrete Data

A set of observations $x_1, x_2, x_3, \ldots, x_n$ with respective frequencies $f_1, f_2, f_3, \ldots, f_n$ is known as **discrete data.**

Method to Find the Median of Discrete Data

Firstly, we arrange the data in the ascending or descending order of x_i, then we find the cumulative frequencies of all the observations. Let n be the total number of observations (sum of frequencies), then median of the data depends on the number of observations (n).

If n is odd, then

$$\boxed{\textbf{Median} = \textbf{Value of} \left(\frac{n+1}{2}\right)\textbf{th observation}}$$

If n is even, then

$$\text{Median} = \text{Mean of} \left(\frac{n}{2}\right)\text{th and} \left(\frac{n}{2}+1\right)\text{th observation}$$

$$= \frac{1}{2} \times \textbf{ Value of} \left[\left(\frac{n}{2}\right)\textbf{th} + \left(\frac{n}{2}+1\right)\textbf{th}\right]\textbf{observation}$$

Here, for the value of observation, first look at the cumulative frequency just greater than (and nearest to) the position of required observations. Then, determine the corresponding value of the observation.

Example 7. Find the median of the following data.

Marks obtained	20	29	28	42	19	35	51
Number of students	3	4	5	7	9	2	3

Sol. Let us arrange the data in ascending order of x_i and make a cumulative frequency table.

Marks obtained (x_i)	**Number of students** (f_i)	***cf***
19	9	9
20	3	9 + 3 = 12
28	5	12 + 5 = 17
29	4	17 + 4 = 21
35	2	21 + 2 = 23
42	7	23 + 7 = 30
51	3	30 + 3 = 33

Here, $n = 33$ (odd)

$$\therefore \text{Median} = \text{Value of} \left(\frac{n+1}{2}\right)\text{th observation}$$

$$= \text{Value of} \left(\frac{33+1}{2}\right)\text{th observation}$$

$$= \text{Value of } 17\text{th observation}$$

Since, the corresponding value of 17th observation of cumulative frequency in x_i is 28.

Hence, the median is 28.

Example 8. Obtain the median for the following frequency distribution.

x	1	2	3	4	5	6	7	8	9
y	8	10	11	16	20	25	15	9	6

Sol. Here, the given data is in ascending order of x_i.

The cumulative frequency table for the given data is

x_i	f_i	cf
1	8	8
2	10	18
3	11	29
4	16	45
5	20	65
6	25	90
7	15	105
8	9	114
9	6	120

Here, $n = 120$ (even)

$\therefore$ Median

$$= \frac{1}{2}\left[\text{Value of } \left\{\left(\frac{n}{2}\right)\text{th} + \left(\frac{n}{2}+1\right)\text{th}\right\}\right]\text{observation}$$

$$= \frac{1}{2}\left[\text{Value of } \left\{\left(\frac{120}{2}\right)\text{th} + \left(\frac{120}{2}+1\right)\text{th}\right\}\right]\text{observation}$$

$$= \frac{1}{2}[\text{Value of 60th observation} + \text{Value of 61th observation}]$$

Both 60th and 61th observations lie in the cumulative frequency 65 and its corresponding value of x is 5.

$\therefore$ Median $= \frac{1}{2}(5+5) = 5$

Median for Grouped Data

In a grouped data, we may not find the middle observation by looking at the cumulative frequencies, since the middle observation will be some value in a class interval, so it is necessary to find the value inside a class that divides the whole distribution into two halves.

For this, we find the cumulative frequencies of all the classes and $n/2$, where n = number of observations. Now, locate the class whose cumulative frequency is greater than (and nearest to) $n/2$ and this class is called the **median class.** After finding the median class, use the following formula for calculating the median.

$$\boxed{\text{Median} = l + \left\{\frac{(n/2) - cf}{f}\right\} \times h}$$

where, l = lower limit of median class

n = number of observations

cf = cumulative frequency of the class preceding the median class

f = frequency of the median class

and h = class width (assuming class sizes to be equal).

Example 9. 200 surnames were randomly picked up from a local telephone directory and the frequency distribution of the number of letters in English alphabets in the surnames was obtained as follows

Number of letters	0-5	5-10	10-15	15-20	20-25
Number of surnames	20	60	80	32	8

Find the median of the above data.

Sol. The cumulative frequency table of given data is

Number of letters	Number of surnames (f_i)	Cumulative frequency (cf)
0-5	20	20
5-10	60	$20 + 60 = 80$ (cf)
10-15	$80 = f$	$80 + 80 = 160$
15-20	32	$160 + 32 = 192$
20-25	8	$192 + 8 = 200$
Total	$n = 200$	

Here, $n = 200 \Rightarrow \frac{n}{2} = \frac{200}{2} = 100$

Since, the cumulative frequency just greater than 100 is 160 and the corresponding class interval is 10-15.

$\therefore$ Median class = 10-15, $l = 10$, $cf = 80$, $h = 5$ and $f = 80$

Now, median $= l + \left\{\frac{\frac{n}{2} - cf}{f}\right\} \times h$

$$= 10 + \left\{\frac{100-80}{80}\right\} \times 5 = 10 + \left(\frac{20}{80}\right) \times 5$$

$$= 10 + 1.25 = 11.25$$

Example 10. A survey regarding the heights (in cm) of 51 boys of class X of a school was conducted and the following data was obtained

Heights (in cm)	**Number of boys**
Less than 140	4
Less than 145	11
Less than 150	29
Less than 155	40
Less than 160	46
Less than 165	51

Find the median height.

Sol. To calculate the median height, we need to convert the given data in the continuous grouped frequency distribution.

Given distribution is of less than type and 140, 145, 150,...,165 gives the upper limits of the corresponding class intervals. So, the classes should be below 140, 140-145, 145-150, ..., 160-165.

Clearly, the frequency of class interval below 140 is 4, since there are 4 boys with height less than 140. For the frequency of class interval 140-145 subtract the number of boys having height less than140 from the number of boys having height less than 145.

Thus, the frequency of class interval 140 - 145 is 11 – 4 = 7. Similarly, we can calculate the frequencies of other class intervals and get the following table

Class interval	Frequency	Cumulative frequency (cf)
Below 140	4	4
140-145	$11-4=7$	11
145-150	$29-11=18$	29
150-155	$40-29=11$	40
155-160	$46-40=6$	46
160-165	$51-46=5$	51

Here, $n=51 \Rightarrow \frac{n}{2}=\frac{51}{2}=25.5$

Since, the cumulative frequency just greater than 25.5 is 29 and the corresponding class interval is 145-150.

$\therefore$ Median class = 145 – 150

Now, $l=145,\ f=18,\ cf=11$ and $h=5$

$$\therefore \quad \text{Median} = l+\left\{\frac{\frac{n}{2}-cf}{f}\right\}\times h$$

$$=145+\left\{\frac{25.5-11}{18}\right\}\times 5$$

$$=145+\frac{72.5}{18}=145+4.03$$

$$=149.03$$

Example 11. Find the missing frequencies in the following frequency distribution table, if $n=100$ and median is 32.

Marks obtained	0-10	10-20	20-30	30-40	40-50	50-60	Total
Number of students	10	?	25	30	?	10	100

Sol. Given, median = 32 and $n=\Sigma f=100$

Let f_1 and f_2 be the frequencies of the class interval 10-20 and 40-50, respectively.

Since, sum of frequencies = 100

$\therefore \quad 10+f_1+25+30+f_2+10=100$

$\Rightarrow \quad f_1+f_2=100-75$

$\Rightarrow \quad f_1+f_2=25$

$\Rightarrow \quad f_2=25-f_1 \qquad$ ...(i)

Now, the cumulative frequency table for given distribution is

Class interval	Frequency (f)	Cumulative frequency (cf)
0-10	10	10
10-20	f_1	$10+f_1$
20-30	25	$35+f_1$ (cf)
30-40	$30=f$	$65+f_1$
40-50	f_2	$65+f_1+f_2$
50-60	10	$75+f_1+f_2$
Total	$n=f_1+f_2+75$	

Here, $n=100 \Rightarrow \frac{n}{2}=50$

Given, median = 32, which belongs to the class 30-40.

So, the median class is 30 – 40.

Then, $l=30,\ h=10,\ f=30$ and $cf=35+f_1$

$$\therefore \quad \text{Median} = l+\left\{\frac{\frac{n}{2}-cf}{f}\right\}\times h$$

$$\Rightarrow \quad 32=30+\left\{\frac{50-35-f_1}{30}\right\}\times 10$$

$$\Rightarrow \quad 32-30=\frac{15-f_1}{3} \Rightarrow 2\times 3=15-f_1$$

$$\Rightarrow \quad f_1=15-6=9$$

On putting the value of f_1 in Eq. (i), we get

$$f_2=25-9=16$$

Hence, the missing frequencies are $f_1=9$ and $f_2=16$.

Quartiles

Quartiles of a ranked set of data values are the three points that divide the data set into four equal groups, each group comprising a quarter of the data.

Lower (First) Quartile

The lower (first) quartile is the median of the lower half of the data set. It is denoted by Q_1.

$$\text{Lower quartile, } Q_1=\begin{cases}\frac{n+1}{4}\text{th observation, if } n \text{ is odd}\\ \frac{n}{4}\text{th observation, if } n \text{ is even}\end{cases}$$

Upper (Third) Quartile

The upper (third) quartile is the median of the upper half of a data set. It is denoted by Q_3.

$$\text{Upper quartile, } Q_3=\begin{cases}\frac{3(n+1)}{4}\text{th observation, if } n \text{ is odd}\\ \frac{3n}{4}\text{th observation, if } n \text{ is even}\end{cases}$$

Inter Quartile Range

The difference between the upper quartile (Q_3) and the lower quartile (Q_1) is called the inter quartile range.

Thus, the inter quartile range $= Q_3 - Q_1$

and the semi-inter quartile range $= \dfrac{Q_3 - Q_1}{2}$.

Note (i) The middle quartile (Q_2) is called the median.

(ii) The semi-inter quartile range is also known as **quartile deviation**.

Example 12. From the following data, compute the lower, middle and upper quartiles.

Age (in years)	20	30	40	50	60	70	80
Number of members	3	61	132	153	140	51	3

Sol. The cumulative frequency table is as given below

Age (in years)	**Number of members**	**Cumulative frequencies**
20	3	3
30	61	64
40	132	196
50	153	349
60	140	489
70	51	540
80	3	543
	$n = 543$	

Lower Quartile

We have, $\dfrac{n+1}{4} = \dfrac{543+1}{4} = \dfrac{544}{4} = 136$

The cumulative frequency just greater than $\dfrac{n+1}{4}$ is 196 and the corresponding value of the variable is 40.

$\therefore$ Q_1 = Lower Quartile = 40 yr

Middle Quartile (Median)

We have, $\dfrac{n+1}{2} = \dfrac{543+1}{2} = \dfrac{544}{2} = 272$

Cumulative frequency just greater than $\dfrac{n+1}{2}$ is 349 and the corresponding value of the variable is 50.

$\therefore$ Q_2 = Middle Quartile = 50 yr

Upper Quartile

We have, $\dfrac{3(n+1)}{4} = \dfrac{3\times(543+1)}{4} = \dfrac{3\times 544}{4} = 408$

Cumulative frequency just greater than $\dfrac{3(N+1)}{4}$ is 489 and the corresponding value of the variable is 60.

$\therefore$ Q_3 = Upper Quartile = 60 yr

Topic Exercise 2

1. The median of the observations 11, 12, 14, $(x-2), (x+4), (x+9)$, 32, 38 and 47 arranged in ascending order is 24. Find the value of x and hence find the mean. *[2013]*

2. Write the frequency distribution table for the following data.

Marks	Below 10	Below 20	Below 30	Below 40	Below 50	Below 60
Number of students	0	15	20	30	35	40

3. Given below is a cumulative frequency distribution of 'more than type'

Marks	Number of students
More than or equal to 60	11
More than or equal to 50	23
More than or equal to 40	43
More than or equal to 30	58
More than or equal to 20	72
More than or equal to 10	82

Change the above data into a continuous grouped frequency distribution.

4. The number of students absent in a school was recorded everyday for 147 days and the raw data was presented in the form of the following frequency table

Number of students	Number of days
5	1
6	5
7	11
8	14
9	16
10	13
11	10
12	70
13	4
15	1
18	1
20	1

Find the median of the above data.

5. Find the median of the following frequency distribution.

x	1	2	3	4	5	6	7	8	9	10
y	2	3	6	8	16	12	10	13	9	15

6. Calculate the median from the following distribution.

Class	5-10	10-15	15-20	20-25	25-30	30-35	35-40	40-45
Frequency	5	6	15	16	5	4	2	2

7. The maximum bowling speed (in km/h) of 33 players at a cricket coaching centre are given below

Speed (in km/h)	85-100	100-115	115-130	130-145
Number of players	11	9	8	5

Calculate the median bowling speed.

8. Find the median for the following data.

Height (in cm) [Less than]	120	140	160	180	200
Number of students	12	26	34	40	50

9. The weights (in kg) of 45 students of a class are given in the following distribution table. Determine the value of weight x which is such that the number of students having weight less than x kg is same as the number of students having weight more than x kg.

Weight (in kg)	Cumulative frequency
Below 45	5
Below 50	11
Below 55	15
Below 60	22
Below 65	38
Below 70	45

10. Compute the median marks for the following data.

Marks	Number of students
0 and above	50
10 and above	46
20 and above	40
30 and above	20
40 and above	10
50 and above	3
60 and above	0

11. If median of the following frequency distribution is 24, then find the missing frequency x.

Age (in years)	0-10	10-20	20-30	30-40	40-50
Number of persons	5	25	x	18	7

12. If median of the following frequency distribution is 35, then find the value of x.

Class interval	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	2	3	x	6	5	3	2

13. If the median of the following distribution is 58 and the sum of all the frequencies is 140. Find the values of x and y.

Variable	15-25	25-35	35-45	45-55	55-65	65-75	75-85	85-95
Frequency	8	10	x	25	40	y	15	7

14. If median of the number of patients attending a hospital is 36, then find the missing frequencies f_1 and f_2 in the following frequency distribution, when it is given that total number of days is 100.

Number of patients	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Number of days	5	12	f_1	f_2	15	11	14

15. During Medical check up of 200 students of school, their weights were recorded as follows

Weight (in kg)	30-39	40-49	50-59	60-69	70-79	80-89
Number of students	5	22	63	74	30	6

Find the median weight of students.

Hints and Answers

1. Hint Here, number of observations, $n = 9$, which is odd.

So, we use the formula,

$$\text{Median} = \left(\frac{n+1}{2}\right)\text{th observation} = 5\text{th observation}$$

$$\Rightarrow \quad 24 = x + 4 \qquad [\because \text{median} = 24, \text{given}]$$

$\therefore$ Mean

$$= \frac{11 + 12 + 14 + x - 2 + x + 4 + x + 9 + 32 + 38 + 47}{9}$$

$$= \frac{165 + 3x}{9}$$ **Ans.** 20 and 25

2.

Marks	Number of students
10-20	15
20-30	20 – 15 = 5
30-40	30 – 20 = 10
40-50	35 – 30 = 5
50-60	40 – 35 = 5

3. Do same as Example 6.

Ans.

Marks	**Number of students**
10-20	10
20-30	14
30-40	15
40-50	20
50-60	12
60 or more	11

4. Do same as Example 7. **Ans.** 12

5. Do same as Example 8. **Ans.** 6.5

6. Do same as Example 9. **Ans.** 20.47
7. Do same as Example 9. **Ans.** 109.72 km/h
8. Do same as Example 10. **Ans.** 138.57 cm
9. **Hint** Here, we are required to find median weight of students and do same as Example 10. **Ans.** 60.16 kg
10. **Hint** To find median marks, we convert the given data into continuous grouped frequency distribution

Marks	Number of students	Cumulative frequency
0-10	50 – 46 = 4	4
10-20	46 – 40 = 6	10
20-30	40 – 20 = 20	30
30-40	20 – 10 = 10	40
40-50	10 – 3 = 7	47
50-60	3 – 0 = 3	50

Here, $\frac{n}{2} = \frac{50}{2} = 25$

$\therefore$ Median class = 20-30

Now, median $= l + \frac{\frac{n}{2} - cf}{f} \times h = 20 + \frac{25-10}{20} \times 10$

Ans. 27.5

11. **Hint**

Age (in years)	Number of persons	Cumulative frequency
0-10	5	5
10-20	25	30
20-30	x	$30 + x$
30-40	18	$48 + x$
40-50	7	$55 + x$

Since, the median is 24. Therefore, the median class is 20-30.

$$\therefore \quad \text{Median} = l + \left(\frac{\frac{n}{2} - cf}{f} \right) \times h$$

$$\Rightarrow 24 = 20 + \frac{\frac{55+x}{2} - 30}{x} \times 10$$

$$\Rightarrow \quad 4 = \frac{x-5}{2x} \times 10 \Rightarrow 8x = 10x - 50 \Rightarrow x = \frac{50}{2} = 25$$

12. Do same as Q. 11. **Ans.** $x = 5$
13. Do same as Q. 11. **Ans.** $x = 15, \ y = 20$
14. Do same as Example 11. **Ans.** $f_1 = 18, \ f_2 = 25$
15. **Hint** Convert the given distribution into continuous grouped frequency distribution and then find the median. **Ans.** 60.85 kg

Topic 3
Mode of Grouped Data

Mode

The observation, which occurs most frequently among the given observations, i.e. the value of the observation having maximum frequency is called **mode**. e.g. Mode of the numbers 2, 3, 4, 4, 6, 6, 6, 6, 7 and 9 is 6 because it is repeated maximum number of times, i.e. 4 times.

Sometimes, it is possible that more than one value may have the same maximum frequency. In this situation, the data is said to be **multimodal**. Grouped data can also be multimodal but here we will discuss the problems having single mode only. In case of ungrouped data, the mode is the observation with maximum frequency.

Example 1. In a class of 72 students, marks obtained by the students in a class test (out of 10) are given below

Marks obtained (Out of 10)	1	2	3	4	6	7	9	10
Number of students	3	5	12	18	23	8	2	1

Find the mode of the data.

Sol. The mode of the given data is 6 as it has the maximum frequency, i.e. 23.

Modal Class

In a grouped frequency distribution, it is not possible to determine the mode by looking at the frequencies. So, here we first locate a class with the maximum frequency. This class is called modal class. e.g.

Class interval	0-10	10-20	20-30	30-40	40-50	50-60
Number of students	2	9	14	20	22	8

Here, the highest frequency is of the class 40-50, which is 22. Hence, the modal class is 40-50.

Mode of Grouped Data

In grouped data, mode is a value lies in the modal class and it is given by the formula,

$$\text{Mode} = l + \left\{ \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \right\} \times h$$

where, l = lower limit of the modal class
h = size of the class intervals (assuming all class sizes to be equal)
f_1 = frequency of the modal class
f_0 = frequency of the class preceding the modal class
f_2 = frequency of the class succeeding the modal class

(i) Mode can also be calculated for a grouped data with unequal class sizes.

(ii) Mode can be less, equal or more than the mean of the data.

(iii) It depends upon the situation whether we have to calculate mean or mode. Since, mean is used to find the average value of data and mode is used to find the value having maximum frequency. e.g. If we are interested in finding the average marks obtained by the students, then mean is required and if we are interested in finding the average of the marks obtained by most of the students then mode is required.

Example 2. An NGO working for welfare of cancer patients, maintained its records as follows

Age of patients (in years)	0-20	20-40	40-60	60-80
Number of patients	35	315	120	50

Find the mode.

Sol. Here, the maximum frequency is 315 and the class corresponding to this frequency is 20-40. So, the modal class is 20-40.

Age of patients (in years)	0-20	20-40	40-60	60-80
Number of patients	$35 = f_0$	$315 = f_1$	$120 = f_2$	50

$\therefore\ l = 20, f_1 = 315, f_0 = 35, f_2 = 120$ and $h = 20$

Now, mode $= l + \left(\dfrac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$

$= 20 + \dfrac{315 - 35}{2 \times 315 - 35 - 120} \times 20$

$= 20 + \dfrac{280}{475} \times 20$

$= 20 + 11.79 = 31.79$

Hence, the average age of maximum number of patients is 31.79.

Example 3. If the mode of the following series is 54, then find the value of f.

Class interval	0-15	15-30	30-45	45-60	60-75	75-90
Frequency	3	5	f	16	12	7

Sol. Here, the given mode is 54, which lies between 45-60. Therefore, the modal class is 45-60.

$\therefore\ l = 45, f_1 = 16, f_0 = f, f_2 = 12$ and $h = 15$

$\because$ Mode $= l + \left(\dfrac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$

$\therefore\quad 54 = 45 + \dfrac{16 - f}{32 - f - 12} \times 15$

$\Rightarrow\quad 9 = \dfrac{16 - f}{20 - f} \times 15$

$\Rightarrow\quad 9(20 - f) = 15(16 - f)$

$\Rightarrow\quad 180 - 9f = 240 - 15f$

$\Rightarrow\quad 6f = 240 - 180 = 60$

$\Rightarrow\quad f = 10$

Hence, the required value of f is 10.

Relationship between Mean, Median and Mode

There is an empirical relationship between the three measures of central tendency, which is given by

Mode = 3(Median) − 2(Mean)

or Mean $= \dfrac{3\text{ (Median)} - \text{Mode}}{2}$ or Median $= \dfrac{\text{Mode} + 2\text{ (Mean)}}{3}$

(i) The mean is the most frequently used measure of central tendency, because it takes into account all the observations and lies between the extremes. Extreme values in the data affect the mean.

(ii) In problems, where individual observations are not important and we wish to find out a typical observation, the median is more appropriate. e.g. finding average wage in a country etc.

(iii) In situations, which require establishing the most frequent value or most popular item, the mode is the best choice. e.g. finding the most popular TV programme being watched, etc.

Example 4. If median = 137 units and mean = 137.05 units, then find the mode.

Sol. Given, median = 137 units and mean = 137.05 units

We know that, Mode = 3 (Median) − 2(Mean)
= 3(137) − 2(137.05) = 411 − 274.10 = 136.90

Hence, the value of mode is 136.90 units.

Topic Exercise 3

1. Find the mode of the given data.

Marks	0-20	20-40	40-60	60-80
Frequency	15	6	18	10

2. For the following data, find the mode.

Class	1-3	3-5	5-7	7-9	9-11
Frequency	14	16	4	4	2

3. The following table shows the age distribution of cases of a certain disease admitted during a year in a particular hospital

Age (in years)	Number of cases
0-10	5
10-20	8
20-30	7
30-40	12
40-50	28
50-60	20
60-70	10
70-80	10

Find the average age for which maximum case occurred.

4. The weight of coffee in 70 packets are shown in the following table

Weight (in grams)	Number of packets
200-201	12
201-202	26
202-203	20
203-204	9
204-205	2
205-206	1

Determine the modal weight.

5. The mode of the following data is 36. Find the missing frequency x in it.

Class	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	8	10	x	16	12	6	7

6. The monthly consumption of electricity of some consumers is given below as a distribution

Monthly consumption (in units)	Number of consumers
90-120	20
120-150	15
150-180	x
180-210	75
210-240	50

Find the missing frequency (x), if mode of distribution is given to be 200 units.

7. Some surnames were picked up from a local telephone directory and the frequency distribution of the number of letters of the English alphabets was obtained as follows

Number of letters	1-4	4-7	7-10	10-13	13-16	16-19
Number of surnames	10	25	35	x	12	8

If it is given that mode of the distribution is 8, then find the missing frequency (x).

8. Find the mode of the following data.

Class interval	0-9	10-19	20-29	30-39	40-49	50-59
Frequency	12	15	21	17	19	6

9. Given below is the frequency distribution of the heights of players in a school

Height (in cm)	Number of students
160-163	18
164-167	116
168-171	142
172-175	129
176-179	80
180-183	15

10. On sports day of a school, agewise participation of students is shown in the following distribution

Age (in years)	5-7	7-9	9-11	11-13	13-15	15-17	17-19
Number of students	x	15	18	30	50	48	x

Find the mode of the data. Also, find missing frequencies when sum of frequencies is 181.

11. In a certain distribution, mean and median are 9.5 and 10, respectively. Find the mode of the distribution, using an empirical relation?

12. Mode and mean of a data are $12k$ and $15k$, respectively. Find the median of the data.

Hints and Answers

1. Do same as Example 2. **Ans.** 52

2. Do same as Example 2. **Ans.** 3.28

3. Do same as Example 2. **Ans.** 46.67

4. Do same as Example 2. **Ans.** 201.7 g

5. Do same as Example 3.
Ans. $x = 10$

6. Do same as Example 3.
Ans. $x = 25$

7. Do same as Example 3.
Ans. $x = 15$

8. **Hint** Here, the given frequency distribution is not continuous.
So, first we make it continuous by subtracting 0.5 from lower limit and adding 0.5 to upper limit.

Class interval	Continuous class interval	Frequency
0-9	– 0.5 - 9.5	12
10-19	9.5 - 19.5	15
20-29	19.5 - 29.5	21 (Modal class)
30-39	29.5 - 39.5	17
40-49	39.5 - 49.5	19
50-59	49.5 - 59.5	6

$\therefore \quad l = 19.5,\ f_1 = 21,\ f_0 = 15,\ f_2 = 17$ and $h = 10$

Now, mode $= 19.5 + \dfrac{21 - 15}{42 - 15 - 17} \times 10$

Ans. 25.5

9. Do same as Q. 8.

Ans. 170.17

10. Hint Here, the maximum frequency is 50. So, the modal class is 13-15.

$\therefore \quad l = 13,\ f_1 = 50,\ f_0 = 30,\ f_2 = 48$ and $h = 2$

Now, mode $= 13 + \dfrac{50 - 30}{100 - 30 - 48} \times 2 = 13 + \dfrac{20}{22} \times 2$

Since, sum of frequencies $= 181$

$\therefore \quad x + 15 + 18 + 30 + 50 + 48 + x = 181$

Ans. 14.82; 10

11. Do same as Example 4. **Ans.** 11

12. Hint Median $= \dfrac{\text{Mode} + 2(\text{Mean})}{3}$

$= \dfrac{12k + 2 \times 15k}{3} = \dfrac{42k}{3}$ **Ans.** 14 k

Topic 4

Graphical Representation

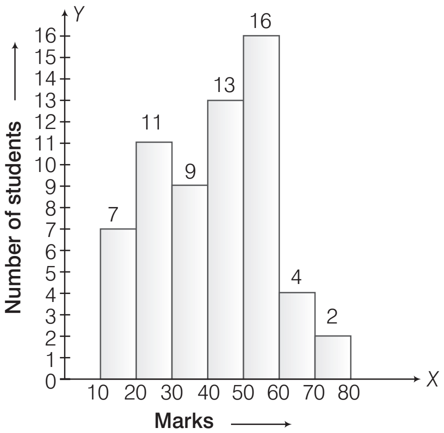
Histogram

A histogram is the graphical representation of a grouped frequency distribution in exclusive form with continuous classes in the form of rectangles with class intervals as bases and the corresponding frequencies as heights. There is no gap between any two consecutive rectangles.

Example 1. Represent the following frequency distribution by means of a histogram.

Marks	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Number of students	7	11	9	13	16	4	2

Sol. We represent the class intervals along the X-axis on a suitable scale and the corresponding frequencies along the Y-axis on a suitable scale. We construct rectangles with class intervals as bases and the corresponding frequencies as heights. Thus, we obtain a histogram as shown below

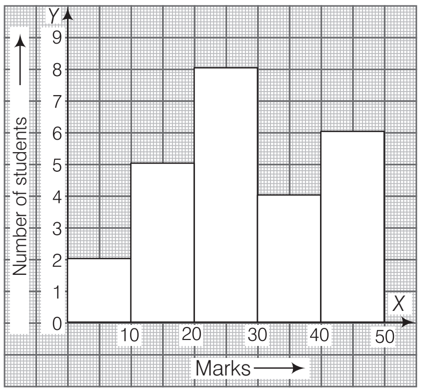


Example 2. The histogram adjacent represents the scores obtained by 25 students in a Mathematics mental test. Use the data to

(i) frame a frequency distribution table.

(ii) calculate mean.

(iii) determine the modal class. ***[2016]***



Sol. (i) From the given histogram, it is clear that the frequency of each class would be height of rectangle corresponding to each class. Therefore, the frequency distribution table is as follows

Scores	Number of students
0-10	2
10-20	5
20-30	8
30-40	4
40-50	6
Total	25

(ii) To calculate mean, we have following table

Class	f_i	Class mark (x_i)	$f_i x_i$
0-10	2	5	10
10-20	5	15	75
20-30	8	25	200
30-40	4	35	140
40-50	6	45	270
Total	$\Sigma f_i = 25$		$\Sigma f_i x_i = 695$

$\therefore$ Mean $(\bar{x}) = \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{695}{25} = 27.8$

(iii) We know that in a frequency distribution, the class which have maximum frequency is called modal class.

$\therefore$ Modal class is 20-30.

[$\because$ class 20-30 has frequency 8, which is maximum amongst all given frequencies]

Calculation of Mode from Histogram

In a continuous frequency distribution, the mode can be determined from the histogram of the given (continuous) frequency distribution.

Method to Determine the Mode

Following steps are used to determine the mode

Step I Draw histogram for the given data.

Step II Inside the highest rectangle (which represents the class with maximum frequency, i.e. modal class), draw two straight lines from the corners of the rectangles on both sides of the highest rectangle to the opposite corners of the highest rectangle.

Step III Through the point of intersection of the two straight lines drawn in Step II, draw a vertical line to meet the X-axis at the point M (say). The variate at the point M is the required mode.

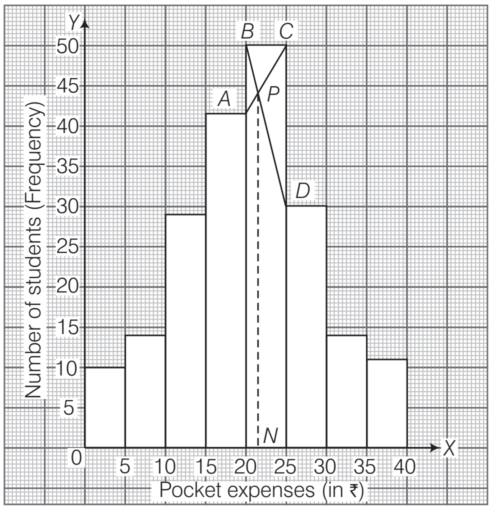
Note If in a problem, frequency distribution is discontinuous, first convert it into continuous distribution and then find the mode as explained above.

Example 3. Use a graph paper for this question, the daily pocket expenses of 200 students in a school are given below.

Pocket expenses (in ₹)	0-5	5-10	10-15	15-20	20-25	25-30	30-35	35-40
Number of students (Frequency)	10	14	28	42	50	30	14	12

Draw a histogram representing the above distribution and estimate the mode from the graph. ***[2014]***

Sol. In the highest rectangle, draw two straight lines AC and BD which intersect at P. Through point P, draw a vertical line to meet the X-axis at N. The abscissa of the point represents 21. Hence, the required mode is 21.



Example 4. For the following distribution, draw a histogram.

Weight (in kg)	44-47	48-51	52-55	56-59	60-63	64-67
Number of shops	20	28	36	16	8	4

From the histogram estimate the mode.

Sol. We observe that, the given frequency distribution is not a continuous frequency distribution. In order to make it continuous, we subtract $\dfrac{h}{2}$ from the lower limit of each class and add $\dfrac{h}{2}$ to the upper limit of each class, where $h = 48 - 47 = 1$.

The continuous frequency distribution for the given data is as follows

Weight (in kg)	43.5-47.5	47.5-51.5	51.5-55.5	55.5-59.5	59.5-63.5	63.5-67.5
Number of students	20	28	36	16	8	4

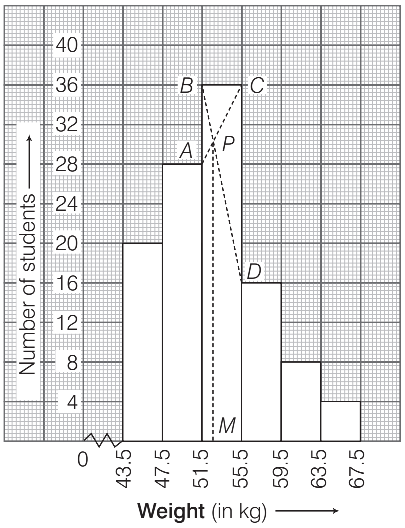
In order to draw the histogram representing this data, let us choose the following scale.

Along X-axis 1 cm = 4 kg

Along Y-axis 1 cm = 4 students

On this scale, we mark points 43.5, 47.5, 51.5, 55.5, 59.5, 63.5 and 67.5 on X-axis and points 20, 28, 36, 16, 8 and 4 on Y-axis. Now, we construct rectangles with class intervals as bases and corresponding frequencies as heights.

The histogram so obtained is shown in the following graph.



We observe that the rectangle constructed on the class 51.5-55.5 is the modal class bar. In this bar, we draw two line segments AC and BD by joining upper corners of this bar to the upper corners of adjoining pre and post bars as shown in the graph. Let P be the point of intersection of AC and BD. Through P, draw a vertical line to meet the X-axis at M. The abscissa of M is 52.7. So, mode = 52.7 kg.

An Ogive Curve

The term 'ogive' is pronounced as 'ojeev' and is derived from the word ogee. An ogive is a shape consisting of a concave arc flowing into a convex arcs. So, forming an S-shaped curve with vertical ends.

Graphical representation of cumulative frequency distribution is of two types, less than type ogive and more than type ogive. The process of graphical representation of less than type cumulative frequency distribution are given below.

Less Than Type Ogive

It is the graph drawn between the upper limits and cumulative frequencies of a distribution. Here, we mark the points with upper limit as x-coordinate and corresponding cumulative frequency as y-coordinate and join them by a free hand smooth curve. This type of graph is cumulated upward.

Example 5. The following table gives the height of trees

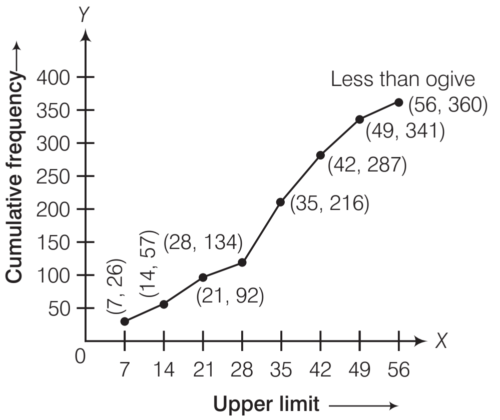
Height (Less than)	7	14	21	28	35	42	49	56
Number of trees	26	57	92	134	216	287	341	360

Draw 'less than ogive'.

Sol. Given distribution is cumulative frequency distribution of less than type.

Now, we mark the upper limits along X-axis and cumulative frequencies along Y-axis, on the graph paper.

Then, plot the points (7, 26), (14, 57), (21, 92), (28, 134), (35, 216), (42, 287), (49, 341) and (56, 360). Join all these points by a freehand smooth curve to obtain an ogive of less than type.



Median and Quartiles from Ogives

In a continuous frequency distribution, the median and the quartiles (lower and upper) can be estimated from the ogive of the given (continuous) frequency distribution.

Method to Determine the Median and Quartiles

Step I Construct cumulative frequency table. Let n be the sum of frequencies.

Step II Draw ogive for the given distribution.

Step III

(a) **To find the median** (M_e) Locate a point along Y-axis representing frequency equal to $n/2$.

Through this point, draw a horizontal line to meet the ogive and through this point of the ogive, draw a vertical line to meet the X-axis at the point M (say). The variate at the point M is the required median.

The class in which the median lies is called the **median class**.

(b) **To find the lower quartile** (Q_1) Locate a point along Y-axis representing frequency equal to $n/4$.

Through this point, draw a horizontal line to meet the ogive and through this point of the ogive, draw a vertical line to meet the X-axis at the point N(say). The variate at the point N is the required lower quartile.

(c) **To find the upper quartile** (Q_3) Locate a point along Y-axis representing frequency equal to $\frac{3n}{4}$.

Now, proceed as in Step III (b) to find the required upper quartile.

Example 6. The weight of 50 workers is given below.

Weight (in kg)	50-60	60-70	70-80	80-90	90-100	100-110	110-120
Number of workers	4	7	11	14	6	5	3

Draw an ogive of the given distribution using a graph sheet. Take 2 cm = 10 kg on one axis and 2 cm = 5 workers along the other axis. Use a graph to estimate the following.

(i) The upper and lower quartiles.

(ii) If weighing 95 kg and above is considered overweight, then find the number of workers who are overweight. *[2015]*

Sol. The cumulative frequency table for the continuous distribution is shown below

Weight (in kg)	50-60	60-70	70-80	80-90	90-100	100-110	110-120
Frequency (f)	4	7	11	14	6	5	3
Cumulative frequency (cf)	4	11	22	36	42	47	50

On the graph paper, we plot the following points $A(60, 4)$, $B(70, 11)$, $C(80, 22)$, $D(90, 36)$, $E(100, 42)$, $F(110, 47)$ and $G(120, 50)$, join all these points by a free hand drawing.

The required ogive is shown on the graph paper given below.

(i) **Lower quartile** $n = 50$, so $\frac{n}{4} = \frac{50}{4} = 12.5$

Taking point F_1 on Y-axis representing frequency = 12.5 and corresponding value of X-axis is 71.3.

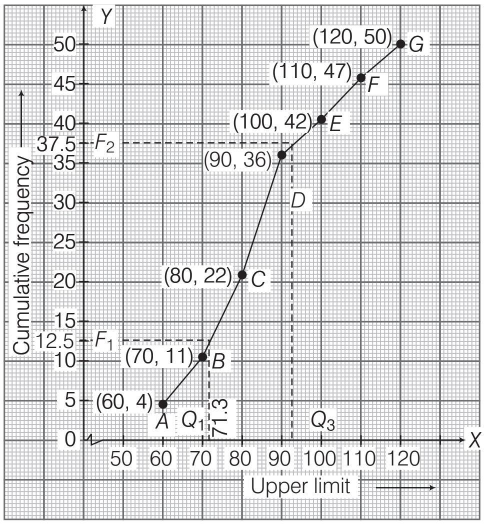
$\therefore \; Q_1 = 71.3$

Upper quartile $n = 50$, so $\frac{3n}{4} = \frac{3 \times 50}{4} = \frac{150}{4} = 37.5$

Taking point F_2 on Y-axis representing frequency = 37.5 and corresponding value of X-axis is 92.5.

$\therefore \; Q_3 = 92.5$

(ii) On the ogive, the point $(95, y)$ corresponds to $y = 40$. So, cf is 40. Therefore, the number of variates above 95 kg is $50 - 40 = 10$.



Example 7. The marks obtained by 100 students in a Mathematics test are given below

Marks	Number of students
0-10	3
10-20	7
20-30	12
30-40	17
40-50	23
50-60	14
60-70	9
70-80	6
80-90	5
90-100	4

Draw an ogive for the given distribution on a graph sheet. (use a scale of 2 cm = 10 units on both axes). Use the ogive to estimate the (i) median. (ii) lower quartile.

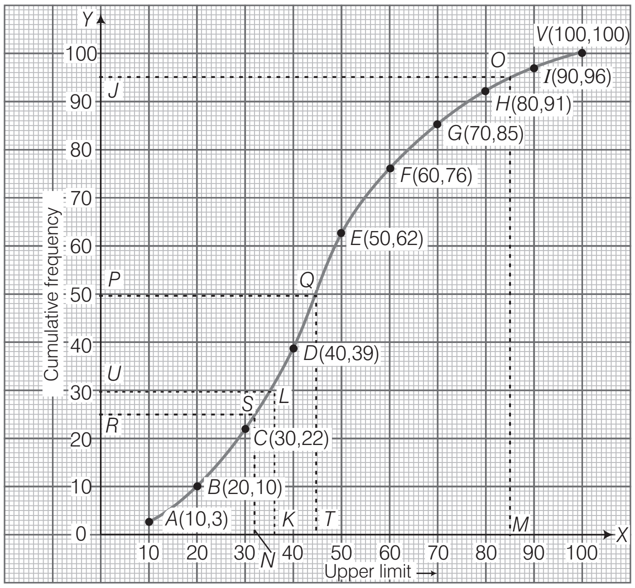
(iii) number of students who obtained more than 85% marks in the test.

(iv) number of students who did not pass in the test, if the pass percentage was 35. *[2014]*

Sol. The cumulative frequency table for the given continuous distribution is given below

Marks	Number of students	Cumulative frequency (cf)
0-10	3	3
10-20	7	10
20-30	12	22
30-40	17	39
40-50	23	62
50-60	14	76
60-70	9	85
70-80	6	91
80-90	5	96
90-100	4	100

On the graph paper, we plot the following points A (10, 3), B (20, 10), C (30 , 22), D (40, 39), E (50, 62), F (60, 76), G (70, 85), H (80, 91), I (90, 96) and V (100, 100). Join all these points by a free hand drawing. The required ogive is shown on the graph paper given below



Here, number of students $(n) = 100$, which is even.

(i) Let P be the point on Y-axis representing frequency

$$= \frac{n}{2} = \frac{100}{2} = 50.$$

Through P, draw a horizontal line to meet the ogive at point Q. Through Q, draw a vertical line to meet the X-axis at T. The abscissa of the point T represents 43 marks. Hence, the median marks is 43.

(ii) Let R be the point on Y-axis representing frequency

$$= \frac{n}{4} = \frac{100}{4} = 25.$$

Through R, draw a horizontal line to meet the ogive at point S. Through S, draw a vertical line to meet the X-axis at N. The abscissa of the point N represents 31 marks. Hence, the lower quartile = 31 marks.

(iii) 85% marks = 85% of 100 = 85 marks.

Let the point M on X-axis represents 85 marks.

Through M, draw a vertical line to meet the ogive at the point O. Through O draw a horizontal line to meet the Y-axis at point J. The ordinate of point J represents 95 students.

$\therefore$ Number of students who obtained more than 85% in the test = 100 – 95 = 5

(*iv*) 35% marks = 35% of 100 = 35

Let the point K on X-axis represents 35 marks.

Through K, draw a vertical line to meet the ogive at the point L. Through L, draw a horizontal line to meet the Y-axis at point U. The ordinate of point U represents 30 students on Y-axis. Hence, the number of students, who did not pass in the test is 30.

Note If the student solve all the results directly, then they do not get full marks.

Example 8. The daily wages of 80 workers in a project are given below.

Wages (in ₹)	400-450	450-500	500-550	550-600	600-650	650-700	700-750
No. of workers	2	6	12	18	24	13	5

Use a graph paper to draw an ogive for the above distribution. (Use a scale of 2 cm = ₹ 50 on X-axis and 2 cm = 10 workers on Y-axis). Use your ogive to estimate.

(i) The median wage of the workers.

(ii) The lower quartile wage of workers.

(iii) The number of workers who earn more than ₹ 625 daily. ***[2017]***

Sol. Let us first prepare the following cumulative frequency table

Wages (in ₹)	Number of workers	Cumulative frequency
400-450	2	2
450-500	6	8
500-550	12	20
550-600	18	38
600-650	24	62
650-700	13	75
700-750	5	80

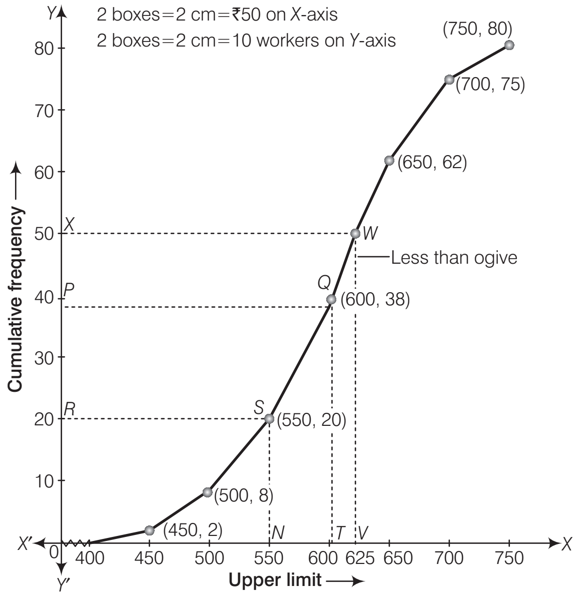
In order to draw an ogive for the given distribution, we assume a class 350-400 before the first class interval with zero frequency.

Now, we mark the upper class limits (including the imagined class) along X-axis and the cumulative frequency along Y-axis on the following scale.

2 cm = ₹ 50 on X-axis, 2 cm = 10 workers on Y-axis

On the scale, we plot the points (400, 0), (450, 2), (500, 8), (550, 20), (600, 38), (650, 62), (700, 75) and (750, 80) with the help of following graph.

Join all these points by a free hand drawing the required ogive is shown on the graph.



(i) Let P be the point on Y-axis representing frequency

$$= \frac{n}{2} = \frac{80}{2} = 40 \qquad \left(40 \times \frac{2}{10} = 8\right)$$

Through point P, draw a horizontal line to meet the ogive at point Q. Through Q draw a vertical line to meet the X-axis at T. The abscissa of the point T represents ₹ 605 wages.

Hence, the median marks is ₹ 605.

(ii) Let R be the point on Y-axis representing frequency

$$= \frac{n}{4} = \frac{80}{4} = 20 \qquad \left(20 \times \frac{2}{10} = 4\right)$$

Through R, draw a horizontal line to meet the ogive at S. Through S, draw a vertical line to meet the X-axis at N. The abscissa of the point N represents ₹ 550 wages. Hence, the lower quartile = ₹ 550

(iii) The number of workers who earn more than ₹ 625 are 80 – 50 = 30 (according to graph).

Example 9. The table shows the distribution of the scores obtained by 160 shooters in a shooting competition. Use a graph sheet and draw an ogive for the distribution. [Take 2 cm = 10 scores on the X-axis and 2 cm = 20 shooters on the Y-axis].

Scores	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Number of shooters	9	13	20	26	30	22	15	10	8	7

Use your graph to estimate the following.

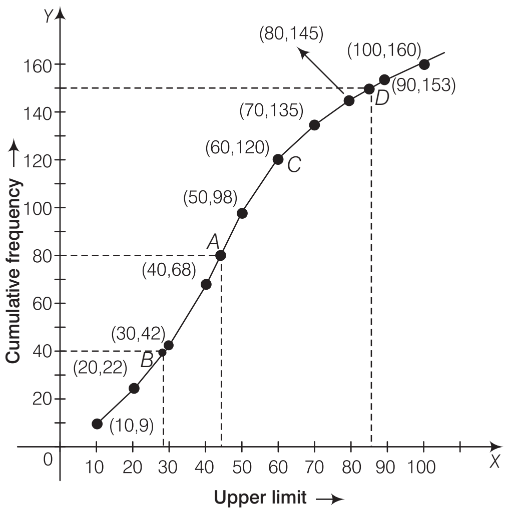
(i) The median (ii) The inter quartile range

(iii) The number of shooters who obtained a score of more than 85%. *[2016]*

Sol. The table is as follows

Scores	Number of shooters	Cumulative frequency (Less than type)
0-10	9	9
10-20	13	22
20-30	20	42
30-40	26	68
40-50	30	98
50-60	22	120
60-70	15	135
70-80	10	145
80-90	8	153
90-100	7	160
Total	$N = 160$	

Take 2 cm = 10 scores on the X-axis
and 2 cm = 20 shooters on the Y-axis.
Draw the graph between upper limit of the class intervals and cumulative frequency, we have the following diagram



$$\therefore \qquad \frac{n}{2} = \frac{160}{2} = 80$$

(i) Draw a line $y = 80$, parallel to X-axis which cuts the graph at point A. Draw perpendicular from point A on the X-axis. This line, where cuts the X-axis, will be median.
$\therefore$ From the graph, we see that the perpendicular line meets X-axis at $x = 44$.

$\therefore$ Median = 44

(ii) Now, to find first quartile $\frac{n}{4} = \frac{160}{4} = 40$

Draw a line $y = 40$, parallel to X-axis which cuts the ogive at point B. Draw a perpendicular line from point B on the X-axis which cuts the X-axis at $x = 29$.

$\therefore$ First quartile $(Q_1) = 29$

To find third quartile $\frac{3n}{4} = \frac{160 \times 3}{4} = 120$

Draw a line $y = 120$, parallel to X-axis, which cuts the ogive at point c. Now, draw a perpendicular line from point c on the X-axis which cuts the X-axis at $x = 60$.

$\therefore$ Third quartile $(Q_3) = 60$

$\therefore$ Inter quartile range $= Q_3 - Q_1 = 60 - 29 = 31$

(iii) Draw a line $x = 85$, parallel to Y-axis, which cuts the ogive at point D parallel to X-axis which cuts Y-axis at $y = 148$.

$\therefore$ Number of shooters above 85% = 160 − 148 = 12

Topic Exercise 4

1. For the following frequency distribution draw a histogram. Hence, calculate the mode. *[2004]*

Class	0-5	5-10	10-15	15-20	20-25	25-30
Frequency	2	7	18	10	8	5

2. Draw a histogram from the following frequency distribution and find the mode from the graph.

Class	0-5	5-10	10-15	15-20	20-25	25-30
Frequency	2	5	18	14	8	5

[2013]

3. Find the mode of following data, using a histogram.

Class	0-10	10-20	20-30	30-40	40-50
Frequency	5	12	20	9	4

4. The following table shows the expenditure of 60 boys on books.

Expenditure (in ₹)	20-25	25-30	30-35	35-40	40-45	45-50
No. of students	4	7	23	18	6	2

Find the mode of their expenditure.

5. Using a graph paper draw a histogram for the given distribution showing the number of runs scored by 50 batsman. Estimate the mode of the data. *[2018]*

Runs scored	Number of batsman
3000-4000	4
4000-5000	18
5000-6000	9
6000-7000	6
7000-8000	7
8000-9000	2
9000-10000	4

6. The marks of 200 students in a test are given below

Marks %	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89
No. of students	7	11	20	46	57	37	15	7

Draw the histogram and find the mode.

7. Draw a histogram for the distributions and estimate their modes.

Marks %	30-39	40-49	50-59	60-69	70-79	80-89	90-99
No. of students	14	26	40	92	114	78	36

8. Find the mode of the following frequency distribution by drawing a histogram.

Mid values	12	18	24	30	36	42	48
Frequency	20	12	8	24	16	8	12

9. Find the median for the following distribution graphically.

Class interval	0-10	10-20	20-30	30-40	40-50
Frequency	5	7	10	8	5

10. Estimate the median for the given data by drawing ogive.

Class	0-10	10-20	20-30	30-40	40-50
Frequency	4	9	15	14	8

11. A frequency distribution of lengths of sticks in a bundle is given below

Class mark (in cm)	5	15	25	35	45	55	65
Frequency	10	17	12	9	11	18	7

Prepare a classified frequency distribution. Display it by an ogive and use it to find the median length of the sticks.

12. Use graph paper for this question.
A survey regarding height (in cm) of 60 boys belonging to class 10 of a school was conducted. The following data was recorded

Height (in cm)	Number of boys
135-140	4
140-145	8
145-150	20
150-155	14
155-160	7
160-165	6
165-170	1

Taking 2 cm = height of 10 cm along one axis and 2 cm = 10 boys along the other axis draw an ogive of the above distribution. Use the graph to estimate the following.

(i) the median (ii) lower quartile

(iii) if above 158 cm is considered as the tall boys of the class. Find the number of boys in the class who are tall. ***[2018]***

13. The marks of 200 students in a test is given below

Marks %	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89
No. of students	7	11	20	46	57	37	15	7

Draw the ogive and find the

(i) median.

(ii) number of students who scored more than 35%. marks.

14. The marks obtained by 200 students in an examination are given below.

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No. of students	5	11	10	20	28	37	40	29	14	6

Draw an ogive for the given distribution taking 2 cm = 10 marks on one axis and 2 cm = 20 students on the other axis. Using the graph, determine

(i) the median marks.

(ii) the number of students who failed, if minimum marks required to pass is 40.

(iii) if scoring 85 and more marks is considered as grade one, find the number of students who secured grade one in the examination. ***[2011]***

15. The marks obtained by 120 students in a test are given below.

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No. of students	5	9	16	22	26	18	11	6	4	3

Draw an ogive for the given distribution on a graph sheet.
Use suitable scale for ogive to estimate

(i) the median.

(ii) the number of students who obtained more than 75% marks in the test.

(iii) the number of students who did not pass the test, if minimum marks required to pass is 40. ***[2013, 12]***

16. Attempt this question on graph paper.

Age (in years)	5-15	15-25	25-35	35-45	45-55	55-65	65-75
Number of casualties	6	10	15	13	24	8	7

(i) Construct the 'less than' cumulative frequency curve for the above data, using 2 cm = 10 yr, on one axis and 2 cm = 10 casualties on the other.

(ii) From your graph determine (a) median (b) lower quartile.

17. From the following cumulative frequency table, draw ogive and then use it to find

Marks (Less than)	10	20	30	40	50	60	70	80	90	100
Cumulative Frequency	5	24	37	40	42	48	70	77	79	80

(i) median (ii) lower quartile (iii) upper quartile.

18. The weights of 60 boys are given in the following distribution table

Weight (in kg)	37	38	39	40	41
No. of boys	10	14	18	12	6

Find the

(i) median (ii) lower quartile

(iii) upper quartile (iv) inter quartile range.

19. The following distribution represents the height of 160 students of a school

Height (in cm)	140-145	145-150	150-155	155-160	160-165	165-170	170-175	175-180
No. of students	12	20	30	38	24	16	12	8

Draw an ogive for the given distribution taking 2 cm = 5 cm of height on one axis and 2 cm = 20 students on the other axis. Using the graph, determine

(i) the median height.

(ii) the inter quartile range.

(iii) the number of students, whose height is above 172 cm. ***[2012]***

20. The monthly income of a group of 320 employees in a company is given below.

Monthly income	**Number of employees**
6000-7000	20
7000-8000	45
8000-9000	65
9000-10000	95
10000-11000	60
11000-12000	30
12000-13000	5

Draw an ogive of the given distribution on a graph sheet taking 2 cm = ₹ 1000 on one axis and 2 cm = 50 employees on the other axis. From the graph determine

(i) the median wage.

(ii) the number of employees whose income is below ₹ 8500.

(iii) if the salary of a senior employee is above ₹ 11500, then find the number of senior employees in the company.

(iv) the upper quartile. ***[2010]***

Hints and Answers

1. Do same as Example 3. **Ans.** 13

2. Do same as Example 3. **Ans.** 14

3. Do same as Example 3. **Ans.** 24

4. Do same as Example 3. **Ans.** 34

5. Do same as Example 3. **Ans.** 4600

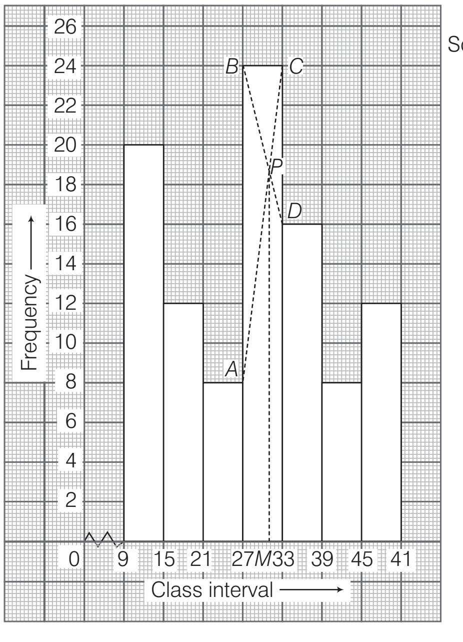
6. Do same as Example 4. **Ans.** 53.5

7. Do same as Example 4. **Ans.** 73

8. **Hint** Here, mid-values of the class-intervals are given. So, we find class limits by subtracting and adding $\frac{h}{2}$ to the mid values, where $h = 18 - 12 = 6$.

The continuous frequency distribution obtained from the given distribution is as follows

Class intervals	9-15	15-21	21-27	27-33	33-39	39-45	45-51
Frequency	20	12	8	24	16	8	12



Scale : Along *X*-axis, 1 cm = 6 units, Along *Y*-axis, 1 cm = 4 units

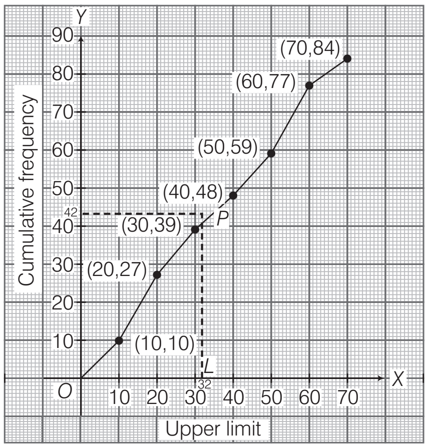
Ans. 30.9

9. Do same as Example 7 (i). **Ans.** 26

10. Do same as Example 7 (i). **Ans.** 28

11. Hint

Class mark (in cm)	Class interval	Frequency	Cumulative frequency
5	0-10	10	10
15	10-20	17	27
25	20-30	12	39
35	30-40	9	48
45	40-50	11	59
55	50-60	18	77
65	60-70	7	84



Here, $\quad N = 84 \Rightarrow \frac{N}{2} = \frac{84}{2} = 42$

From 42 on Y-axis draw a line parallel to X-axis meeting the curve at P.

From P, draw a perpendicular on X-axis meeting it at L, which is 32. **Ans.** 32 cm

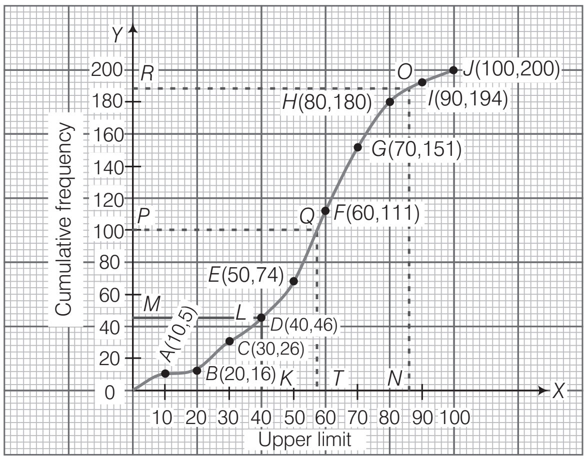
12. Do same as Example 7.

Ans. (i) 149.5 cm (ii) 145.5 (iii) 7

13. Do same as Example 7 (i) and (ii).

Ans. (i) 52.5% (ii) 172

14. Do same as Example 7 (i).



(i) **Hint** Here, $n = 200$, which is even. **Ans.** 57

(ii) **Hint** The minimum marks required to pass is 40, let the point K on X-axis represents 40 marks. Through K, draw a vertical line to meet the ogive at the point L. Through L, draw a horizontal line to meet the Y-axis at point M. The ordinate of point M represents 46 students on Y-axis. Hence, the number of students who failed is 46, if minimum marks required to pass is 40. **Ans.** 46

(iii) **Hint** Scoring 85 and more marks is considered as grade one.

Let point N on X-axis represents 85 marks. Through N, draw a vertical line to meet the ogive at the point O. Through O draw a horizontal line to meet the Y-axis at point R. The ordering of point P represents 190 marks.

$\therefore$ Number of students who secured grade one in the examination $= 200 - 190 = 10$. **Ans.** 10

15. Do same as Example 7.

Ans. (i) 43.5 marks (ii) 10 students (iii) 52 students

16. Do same as Example 9.

Ans. (ii) (a) 43 (b) 29

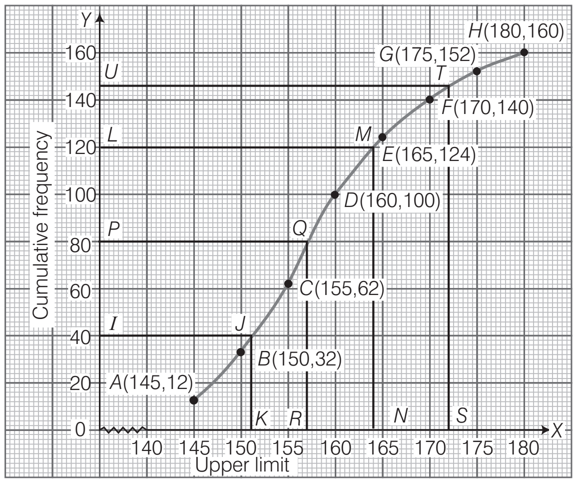
17. Do same as Example 9.

Ans. (i) 40 (ii) 18 (iii) 66

18. Do same as Example 9.

Ans. (i) 39 (ii) 38 (iii) 40 (iv) 2

19. Do same as Example 9.



Ans. (i) 157.3 (ii) 172 cm (iii) 13

20. Do same as Example 8.

Ans. (i) ₹ 15.70 (ii) ₹ 95 (iii) ₹ 20 (iv) ₹ 10200

CHAPTER EXERCISE

a 3 Marks Questions

1. The marks obtained by 12 students in a monthly test are 11, 19, 7, 13, 18, 21, 9, 5, 20, 17, 16, 21. Find
(i) the mean of their marks.
(ii) the mean of their marks, when the marks of each student are increased by 3.

2. Calculate the mean for the following distribution.

Pocket money (in ₹)	60	70	80	90	100	110	120
No. of students	2	6	13	22	24	10	3

3. The table below gives the distribution of villages under different heights from sea level in a certain region. Compute the mean height of the region.

Heights (in m)	200	600	1000	1400	1800	2200
Number of villages	142	265	560	271	89	16

4. Marks obtained by 40 students in a short assessment are given below, where a and b are missing data.

Marks	5	6	7	8	9
No. of students	6	a	16	13	b

If the mean of the distribution is 7.2, find a and b.

5. The mean of the following distribution is 23.4. Find the value of p.

Class intervals	0-8	8-16	16-24	24-32	32-40	40-48
Frequency	5	3	10	p	4	2

6. The following table gives the daily wages of workers in a factory

Daily wages (in ₹)	100-150	150-200	200-250	250-300	300-350	350-400
Frequency	6	15	20	10	7	2

Calculate the mean by using
(i) Direct method. (ii) Shortcut method.

7. The following are the marks obtained by 70 boys in a class test

Marks	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No. of boys	10	12	14	12	9	7	6

Calculate the mean by
(i) shortcut method. (ii) step-deviation method.

8. A study of the yield of 150 tomato plants resulted in the following record.

Tomatoes per plant	1-5	6-10	11-15	16-20	21-25
No. of plants	20	50	46	22	12

Calculate the mean of the member of tomatoes per plant.

9. Consider the following distribution of daily wages of 50 workers of a factory.

Daily wages (in ₹)	100-120	120-140	140-160	160-180	180-200
No. of workers	12	14	8	6	10

Find the mean daily wages of the workers of the factory by using an appropriate method.

10. Apply step-deviation method to find the AM of the following frequency distribution.

Variate (x)	5	10	15	20	25	30	35	40	45	50
Frequency (f)	20	43	75	67	72	45	39	9	8	6

11. The mean of the following frequency distribution is 25.8 and the sum of all the frequencies is 50. Find the values of p and q.

Class	0-10	10-20	20-30	30-40	40-50
Frequency	7	p	15	q	10

12. The following table gives the life-time in days of 100 electricity tubes of a certain make.

Life-time (in days)	Number of tubes
Less than 50	8
Less than 100	23
Less than 150	55
Less than 200	81
Less than 250	93
Less than 300	100

Find the mean life-time of electricity tubes.

13. Find the mean marks of students from the following cumulative frequency distribution.

Marks	Number of students	Marks	Number of students
0 and above	80	60 and above	28
10 and above	77	70 and above	16
20 and above	72	80 and above	10
30 and above	65	90 and above	8
40 and above	55	100 and above	0
50 and above	43		

14. From the following data,
25, 10, 40, 88, 45, 60, 77, 36, 18, 95, 56, 65, 70, 38 and 83
find (i) median (ii) upper quartile (iii) inter quartile range.

15. Find the mean, median and mode of the following distribution.

8, 10, 7, 6, 10, 11, 6, 13, 10 *[2009]*

16. A boy scored the following marks in various class tests during a term, each test being marked out of 20.
15, 17, 16, 7, 10, 12, 14, 16, 19, 12, 16
(i) What are his modal marks?
(ii) What are his median marks?
(iii) What are his mean marks?

17. Find the mean, median and mode of the following marks obtained by 16 students in a class test, marked out of 10 marks.
0, 0, 2, 2, 3, 3, 3, 4, 5, 5, 5, 5, 6, 6, 7, 8

18. Find the mode and median of the following frequency distribution. *[2012]*

x	10	11	12	13	14	15
f	1	4	7	5	9	3

19. The marks obtained by 30 students in a class assessment of 5 marks is given below

Marks	0	1	2	3	4	5
No. of students	1	3	6	10	5	5

Calculate the mean, median and mode of the above distribution. *[2015]*

20. Draw a histogram to represent the following data and estimate the mode from the graph.

Marks obtained	0-10	10-20	20-30	30-40	40-50	50-60
No. of students	4	10	6	8	5	9

21. Construct a histogram for the frequently distribution and estimate the mode from the graphs.

Class interval	1-10	11-20	21-30	31-40	41-50
Frequency	11	23	30	20	16

22. Draw a histogram for the following data and estimate the mode from the graph.

Class marks	12.5	17.5	22.5	27.5	32.5	37.5
Frequency	7	12	20	28	8	11

23. Construct a frequency distribution table and a cumulative frequency curve (ogive) for the following distribution.

Marks obtained	Less than 10	Less than 20	Less than 30	Less than 40	Less than 50
No. of students	8	22	48	60	75

24. Construct an ogive for the following distribution by less than method.

Cost of living	Under 140	Under 150	Under 160	Under 170	Under 180	Under 190	Under 200
Cumulative frequency	0	4	12	30	42	48	52

25. The marks obtained by 15 students in a class test are 12, 14, 07, 9, 23, 11, 8, 13, 11, 19, 16, 24, 17, 3 and 20. Find
(i) the mean of their marks.
(ii) the median of their marks when the marks of each student are increased by 4.
(iii) the mean of their marks when 2 marks are deducted from the marks of each student.

b 4 Marks Questions

26. Find the upper quartile, lower quartile, inter quartile range and semi-inter quartile range for the frequency distribution.

Variate	10	11	12	13	14	15	16	17	18	19	20
Frequency	1	2	3	1	2	4	2	1	1	2	1

27. (i) Using step-deviation method, calculate the mean marks of the following distribution.

Class interval	50-55	55-60	60-65	65-70	70-75	75-80	80-85	85-90
Frequency	5	20	10	10	9	6	12	8

(ii) State the modal class.

28. The marks of 10 students of a class in an examination arranged in ascending order is as follows

$$13, 35, 43, 46, x, x+4, 55, 61, 71, 80$$

If the median marks is 48, find the value of x. Hence, find the mode of the given data. *[2017]*

29. The monthly profits (in rupees) of 100 shops are distributed as follows

Profit (in ₹)	0-100	100-200	200-300	300-400	400-500	500-600
No. of shops	13	18	27	20	17	6

Find the modal value graphically and check this value by direct calculation.

30. By drawing an ogive, estimate the median for the following frequency distribution.

Weight (kg)	10 - 15	15 - 20	20 - 25	25 - 30	30 - 35
No. of boys	11	25	12	5	2

31. The table given below shows the monthly salary of some employees of a company, if lowest salary being ₹ 6500.

Salary (in ₹)	Under 7000	Under 7500	Under 8000	Under 8500	Under 9000	Under 9500	Under 10000
No. of employees	10	18	22	25	17	10	8

(i) Use the table to form its cumulative frequency curve.
Take 2 cm = ₹ 500, starting the origin at ₹ 6500 on the X-axis and 2 cm = 10 employees on the Y-axis.
(ii) Use the graph to write down the median salary in rupees.

32. Using a graph paper, draw an ogive for the following distribution that shows a record of the weight in kilograms of 200 students.

Weight	40-45	45-50	50-55	55-60	60-65	65-70	70-75	75-80
Frequency	5	17	22	45	51	31	20	9

Use your ogive to estimate the following.
(i) The percentage of students weighting 55 kg or more.
(ii) The weight above which the heaviest 30% of the students fall.
(iii) The number of students who are
(a) underweight and (b) overweight, if 55.70 kg is considered as standard weight. *[2005]*

33. Draw an ogive of the following distribution and use it to find

Class interval	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	20	35	15	40	13	17	10

(i) the median.
(ii) the lower quartile.
(iii) the upper quartile.
(iv) the inter quartile range.
(v) the semi-inter quartile range.

34. Attempt this question on graph paper. Marks obtained by 200 students in examination are given below

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No. of students	5	10	14	21	25	34	36	27	16	12

Draw an ogive for the given distribution, taking 2 cm = 10 marks on one axis and 2 cm = 20 students on the other axis. From the graph, find
(i) the median.
(ii) the upper quartile.
(iii) number of students scoring above 65 marks.
(iv) if 10 students qualify for merit scholarship, find the minimum marks required to qualify.

35. The table below shows the distribution of the scores obtained by 120 shooters in a shooting competition.

Scores obtained	Number of shooters
0-10	5
10-20	9
20-30	16
30-40	22
40-50	26
50-60	18
60-70	11
70-80	6
80-90	4
90-100	3

Using a graph sheet, draw an ogive for the distribution.

Use your ogive to estimate
(i) the median.
(ii) the inter quartile range.
(iii) the number of shooters who obtained more than 75% scores.

36. The daily wages of 160 workers in a building project are given below.

Wages (in ₹)	0 - 10	10 - 20	20-30	30-40	40-50	50-60	60-70	70-80
No. of workers	12	20	30	38	24	16	12	8

Using a graph paper, draw an ogive for the above distribution.

Use your ogive to estimate
(i) the median wage of the workers.
(ii) the upper quartile wage of the workers.
(iii) the lower quartile wage of the workers.
(iv) the percentage of workers, who earn more than ₹ 45 a day. *[2006]*

Hints and Answers

1. (i) **Hint** Mean of marks

$$= \frac{\text{Sum of marks of all students}}{\text{Number of students}} = \frac{177}{12}$$

Ans. 14.75

(ii) **Hint** When the marks of each student are increased by 3, then the sum of their marks increases by 12×3, i.e. by 36.

$\therefore$ New sum of marks of all students

$$= 177 + 36 = 213$$

$\therefore$ New mean of marks

$$= \frac{\text{New sum of marks}}{\text{Number of students}} = \frac{213}{12}$$

Ans. 17.75

Note The new mean of marks also increases by 3.

2. Do same as Example 1 of Topic 1. **Ans.** ₹ 92.75

3. Do same as Example 1 of Topic 1. **Ans.** 984.51

4. Hint

Marks (x_i)	Number of students (f_i)	$f_i x_i$
5	6	30
6	a	$6a$
7	16	112
8	13	104
9	b	$9b$
Total	$\Sigma f_i = 35 + a + b$	$\Sigma f_i x_i = 246 + 6a + 9b$

$\therefore \quad 40 = 35 + a + b \quad [\because \Sigma f_i = 40]$

$\Rightarrow \quad a + b = 5 \quad \ldots(i)$

$\Rightarrow \quad 7.2 = \dfrac{246 + 6a + 9b}{40} \quad [\because \bar{x} = 7.2]$

$\Rightarrow \quad 2a + 3b = 14 \quad \ldots(ii)$

Ans. 1 and 4

5. Do same as Example 5 of Topic 1.
Ans. 16

6. (i) Do same as Example 3 of Topic 1.
Ans. 227.5

(ii) Do same as Example 7 of Topic 1.
Ans. 227.5

7. (i) Do same as Example 7 of Topic 1.
Ans. 61.14

(ii) Do same as Example 9 of Topic 1.
Ans. 61.14

8. Hint We shall use step deviation method. Construct the table as under, taking assumed mean $A = 13$.
Here, c (width of each class) = 5.

Now, further proceed as Example 6 of Topic 1.
Ans. 11.534

Note The frequency distribution is discontinuous.

9. Do same as Example 9 of Topic 1.
Ans. 145.20

10. Hint Here, $n = 10 - 5 = 5$ and do same as Example 9 of Topic 1.
Ans. 22.214

11. Do same as Example 11 of Topic 1.
Ans. $p = 10$ and $q = 18$

12. Hint Convert into the class intervals and their corresponding frequences. Do same as Example 9 of Topic 1.
Ans. 142

13. Hint Convert into the class intervals and their corresponding frequences and do same as Example 9 of Topic 1.
Ans. 51.75 marks

14. (i) **Hint** Median $= \left(\dfrac{n+1}{2}\right)$th observation. **Ans.** 56

(ii) **Hint** Upper quartile $= \dfrac{3(n+1)}{4}$th observation.

Ans. 77

(iii) **Hint** For inter quartile range $= Q_3 - Q_1$. **Ans.** 41

15. Hint Mean $= \dfrac{8 + 10 + 7 + 6 + 10 + 11 + 6 + 13 + 10}{9}$

For the median, firstly arrange the given data in ascending order.

6, 6, 7, 8, 10, 10, 10, 11, 13

Here, $n = 9$, which is odd.

$\therefore$ Median $= \left(\dfrac{n+1}{2}\right)$th observation

$= \left(\dfrac{9+1}{2}\right)$th observation $\quad$ [put $n = 9$]

$\because$ Mode is the observation, which is repeated maximum number of times.
In the given distribution, 10 is repeated maximum number of times.

Ans. Mean = 9; median = 10 and mode = 10

16. Do same as Q. 15.
Ans. (i) 12 (ii) 15 (iii) 4

17. Do same as Q. 15.
Ans. 4, 4.5, 5

18. Hint The table for cumulative frequency is shown below

x	f	cf
10	1	1
11	4	5
12	7	12
13	5	17
14	9	26
15	3	29
Total	$n = 29$	

Here, $n = 29$, which is odd.

$\therefore$ Median $= \left(\frac{n+1}{2}\right)$th observation $= 15$th observation

We know that mode is the highest occurring frequency of an observation.

From the table, we see that highest frequency is 9, so corresponding observation is 14.

Ans. Median $= 13$ and Mode $= 14$

19. Hint For mean marks, do same as Example 1 of Topic 1.

For median, do same as Example 7 of Topic 2.

For modal class, do same as Example 1 of Topic 3.

Ans. (i) 3 (ii) 3 (iii) 3

20. Do same as Example 3 of Topic 4. **Ans.** 16

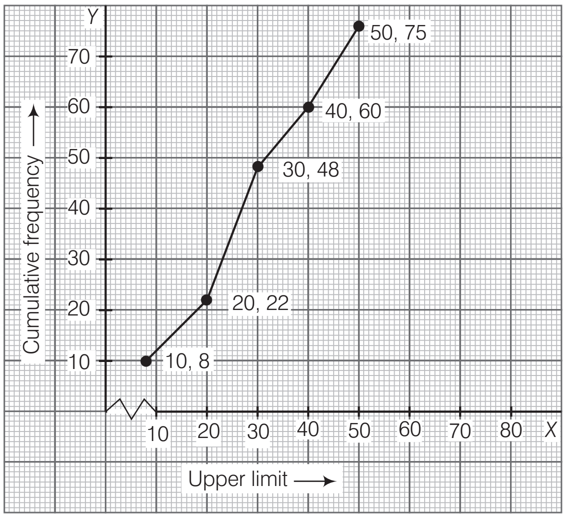
21. Do same as Example 4 of Topic 4. **Ans.** 24.6

22. Do same as Q. 8. of Topic Exercise 4. **Ans.** 26.5

23. Do same as Example 5 of Topic 4.

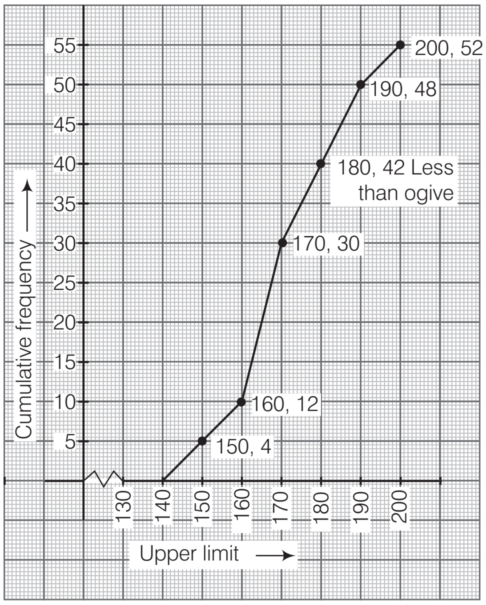
Ans.

Marks	No. of students
0-10	8
10-20	14
20-30	26
30-40	12
40-50	15



24. Hint Use scale : Along X-axis, 1 cm $= 10$ units and along Y-axis, 1 cm $= 5$ units

Ans.



25. Do same as Q. 1 and Q. 14.

Ans. (i) 13.8 (ii) 17.8 (iii) 11.8

26. Do same as Example 12 of Topic 2.

Ans. 16, 12, 4, 2

27. Do same as Example 10 of Topic 1.

Ans. (i) 69 (ii) 55-60

28. Hint Here, number of observations $= 10$ (even)

$\therefore$ Median

$$= \frac{\text{Value of } \left(\frac{10}{2}\right)\text{th observation} + \text{Value of } \left(\frac{10}{2}+1\right)\text{th observation}}{2}$$

$$= \frac{\text{Value of 5th observation} + \text{Value of 6th observation}}{2}$$

$$= \frac{x + x + 4}{2} = \frac{2x+4}{2} = x + 2$$

$\because$ Median $= 48$ [given]

$\therefore$ $x + 2 = 48 \Rightarrow x = 46$

$\therefore$ 5th observation $= 46$ and 6th observation $= x + 4 = 46 + 4 = 50$

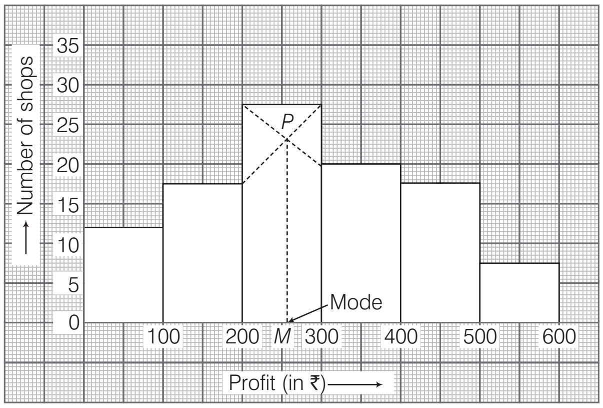
Now, the given data is

13, 35, 43, 46, 46, 50, 55, 61, 71, 80.

Since, 46 appear maximum number of times.

Ans. 46

29. Do same as Example 3 of Topic 4.



Direct calculation We observe that the class 200-300 has the maximum frequency. Therefore, 200-300 is the modal class, such that

$l = 200, h = 100,\ f = 27,\ f_1 = 18$ and $f_2 = 20$

$$\therefore \text{ Mode} = l + \frac{f - f_1}{2f - f_1 - f_2} \times h$$

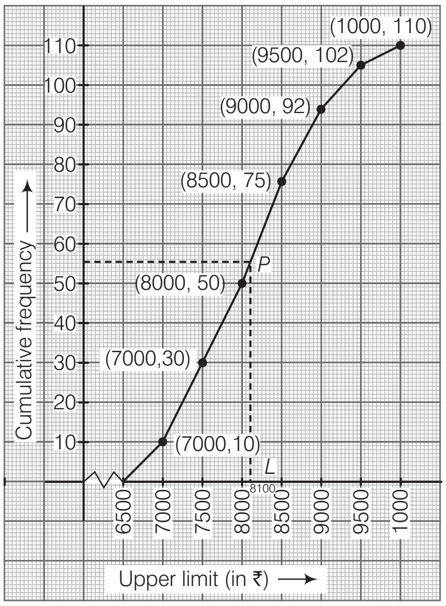
$$= 200 + \frac{27 - 18}{54 - 18 - 20} \times 100$$

Ans. Mode $= 256$, the value of mode is same by both method, i.e. mathematically and graphically.

30. Do same as Example 7 (i) of Topic 4. **Ans.** 18.4

31. Do same as Example 7 (i) of Topic 4.

(i)



(ii) **Hint** Here, $N = 110$

$$\Rightarrow \quad \frac{N}{2} = \frac{110}{2} = 55$$

From 55 on Y-axis, draw a line parallel to X-axis meeting the curve at P.

From P, draw a perpendicular on X-axis meeting it at L. L is the median.

Ans. Median = ₹ 8100

32. Do same as Example 7 (iii), (iv) of Topic 4.

Ans. (i) 78%

(ii) 65 kg

(iii) Do same as Example 6 (ii) of Topic 4.

Ans. 150

33. Do same as Example 9 of Topic 4.

Ans. (i) 32.5

(ii) 14.5

(iii) 41

(iv) 26.5

(v) **Hint** Semi-inter quartile range,

$$L = \frac{Q_3 - Q_1}{2}$$

$$= \frac{41 - 14.5}{2}$$

$$= \frac{26.5}{2}$$

Ans. 13.25

34. Do same as Example 9 of Topic 4.

Ans. (i) 57.30

(ii) 71.80

(iii) 72

(iv) 92

35. Do same as Example 9 of Topic 4.

Ans. (i) 43

(ii) 26.7

(iii) 10

36. Do same as Example 9 of Topic 4.

Ans. (i) 34.73

(ii) 48.33

(iii) 22.66

(iv) 30%

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1 If the mean of the following distribution is 24, find the value of '*a*'.

Marks	0-10	10-20	20-30	30-40	40-50
Number of students	7	a	8	10	5

2 Using a graph paper draw a histogram for the given distribution showing the number of runs scored by 50 batsman. Estimate the mode of the data.

Runs scored	Number of batsman
3000-4000	4
4000-5000	18
5000-6000	9
6000-7000	6
7000-8000	7
8000-9000	2
9000-10000	4

3 Use graph paper for this question. A survey regarding height (in cm) of 60 boys belonging to class 10 of a school was conducted. The following data was recorded

Height (in cm)	Number of boys
135-140	4
140-145	8
145-150	20
150-155	14
155-160	7
160-165	6
165-170	1

Taking 2 cm = height of 10 cm along one axis and 2 cm = 10 boys along the other axis draw an ogive of the above distribution. Use the graph to estimate the following.

(i) The median

(ii) Lower quartile

(iii) If above 158 cm is considered as the tall boys of the class, find the number of boys in the class who are tall.

2017

4 The marks of 10 students of a class in an examination arranged in ascending order is as follows

13, 35, 43, 46, x, $x+4$, 55, 61, 71, 80

If the median marks is 48, then find the value of x. Hence, find the mode of the given data.

5 The daily wages of 80 workers in a project are given below.

Wages (in ₹)	Number of workers
400-450	0
450-500	6
500-550	12
550-600	18
600-650	24
650-700	13
700-750	5

Use a graph paper to draw an ogive for the above distribution. (Use a scale of 2 cm = ₹ 50 on X-axis and 2 cm = 10 workers on Y-axis). Use your ogive to estimate

(i) the median wage of the workers.

(ii) the lower quartile wage of workers.

(iii) the number of workers who earn more than ₹ 625 daily.

6 Calculate the mean of the following distribution using step-deviation method.

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of students	10	9	25	30	16	10

2016

7 The table shows the distribution of the scores obtained by 160 shooters in a shooting competition. Use a graph sheet and draw an ogive for the distribution. (Take, 2 cm = 10 scores on the X-axis and 2 cm = 20 shooters on the Y-axis).

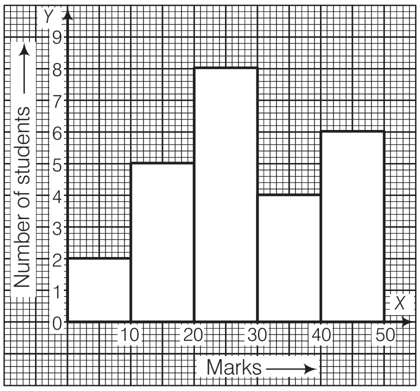
Scores	Number of shooters
0-10	9
10-20	13
20-30	20
30-40	26
40-50	30
50-60	22
60-70	15
70-80	10
80-90	8
90-100	7

Use your graph to estimate the following.

(i) The median (ii) The inter quartile range

(iii) The number of shooters who obtained a score of more than 85%.

8 The histogram adjacent represents the scores obtained by 25 students in a Mathematics mental test. Use the data to



(i) frame a frequency distribution table.
(ii) calculate mean.
(iii) determine the modal class.

9 The mean of following numbers is 68. Find the value of 'x'

45, 52, 60, x, 69, 70, 26, 81 and 94

Hence, estimate the median.

2015

10 Calculate the mean of the following distribution.

Class interval	0-10	10-20	20-30	30-40	40-50	50-60
Frequency	8	5	12	35	24	16

11 The marks obtained by 30 students in a class assessment of 5 marks is given below

Marks	0	1	2	3	4	5
Number of students	1	3	6	10	5	5

Calculate the mean, median and mode of the above distribution.

12 The weight of 50 workers is given below

Weight (in kg)	50-60	60-70	70-80	80-90	90-100	100-110	110-120
Number of workers	4	7	11	14	6	5	3

Draw an ogive of the given distribution using a graph sheet. Take 2 cm= 10 kg on one axis and 2 cm= 5 workers along the other axis. Use a graph to estimate the following.

(i) The upper and lower quartiles.
(ii) If weighing 95 kg and above is considered overweight, find the number of workers who are overweight.

2014

13 Calculate the mean of the distribution given below using the shortcut method.

Marks	11-20	21-30	31-40	41-50	51-60	61-70	71-80
Number of students	2	6	10	12	9	7	4

14 The marks obtained by 100 students in a Mathematics test are given below

Marks	Number of students
0-10	3
10-20	7
20-30	12
30-40	17
40-50	23
50-60	14
60-70	9
70-80	6
80-90	5
90-100	4

Draw an ogive for the given distribution on a graph sheet. (Use a scale of 2 cm = 10 units on both axes). Use the ogive to estimate the

(i) median.
(ii) lower quartile.
(iii) number of students who obtained more than 85% marks in the test.
(iv) number of students who did not pass in the test, if the pass percentage was 35.

15 Use a graph paper for this question, the daily pocket expenses of 200 students in a school are given below.

Pocket expenses (in ₹)	0-5	5-10	10-15	15-20	20-25	25-30	30-35	35-40
Number of students (Frequency)	10	14	28	42	50	30	14	12

Draw a histogram representing the above distribution and estimate the mode from the graph.

2013

16 The median of the observations 11, 12, 14, $(x-2)$, $(x+4)$, $(x+9)$, 32, 38 and 47 arranged in ascending order is 24. Find the value of x and hence find the mean.

17 Find the mean of the following distribution by step deviation method.

Class interval	20-30	30-40	40-50	50-60	60-70	70-80
Frequency	10	6	8	12	5	9

18 Draw a histogram from the following frequency distribution and find the mode from the graph.

Class	0-5	5-10	10-15	15-20	20-25	25-30
Frequency	2	5	18	14	8	5

19 The marks obtained by 120 students in a test are given below.

Marks	Number of students
0-10	5
10-20	9
20-30	16
30-40	22
40-50	26
50-60	18
60-70	11
70-80	6
80-90	4
90-100	3

Draw an ogive for the given distribution on a graph sheet.
Use suitable scale for ogive to estimate
(i) the median.
(ii) the number of students who obtained more than 75% marks in the test.
(iii) the number of students who did not pass the test, if minimum marks required to pass is 40.

2012

20 Find the mode and median of the following frequency distribution.

x	10	11	12	13	14	15
f	1	4	7	5	9	3

21 The following distribution represents the height of 160 students of a school.

Height (in cm)	Number of students
140-145	12
145-150	20
150-155	30
155-160	38
160-165	24
165-170	16
170-175	12
175-180	8

Draw an ogive for the given distribution taking 2 cm = 5 cm of height on one axis and 2 cm = 20 students on the other axis. Using the graph, determine
(i) the median height.
(ii) the inter quartile range.
(iii) the number of students, whose height is above 172 cm.

2011

22 Find the mean of the following frequency distribution by assumed mean method.

Class interval	Frequency
25-30	14
30-35	22
35-40	16
40-45	6
45-50	5
50-55	3
55-60	4

23 The marks obtained by 200 students in an examination are given below.

Marks	Number of students
0-10	5
10-20	11
20-30	10
30-40	20
40-50	28
50-60	37
60-70	40
70-80	29
80-90	14
90-100	6

Draw an ogive for the given distribution taking 2 cm = 10 marks on one axis and 2 cm = 20 students on the other axis. Using the graph, determine
(i) the median marks.
(ii) the number of students who failed, if minimum marks required to pass is 40.
(iii) if scoring 85 and more marks is considered as grade one, find the number of students who secured grade one in the examination.

* All these questions are completely covered in chapter either as solved examples or in topic/chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. Mean of a certain number of observations is m. If each observation is divided by $x (x \neq 0)$ and increased by y, then the mean of new observation is

(a) $\frac{m + xy}{x}$ (b) $\frac{m + xy}{y}$

(c) $\frac{m - xy}{x}$ (d) $\frac{m - xy}{y}$

2. If $x < y < 2x$, then the median and mean of x, y and $2x$ are 27 and 33, respectively. The mean of x and y is

(a) 25 (b) 25.1

(c) 25.2 (d) 25.5

3. In a colony, the average age of the boys is 14 yr and the average age of the girls is 17 yr. If the average age of the children in the colony is 15 yr, then the ratio of number of boys to that of girls is

(a) 1 : 2 (b) 2 : 1

(c) 1 : 1 (d) None of these

4. In a class of 20 students, 10 boys brought 11 books each and 6 girls brought 13 books each. Remaining students brought atleast one book each and no two students brought the same number of books. If the average number of books brought in the class is a positive integer, then the minimum number of books brought by the remaining students is

(a) 18 (b) 16 (c) 14 (d) 12

5. The mean expenditure of a person from Monday to Wednesday is ₹ 250 and the mean expenditure from Wednesday to Friday is ₹ 400. If he spend is ₹ 300 on Wednesday, then the mean expenditure of the person from Monday to Friday is

(a) ₹ 300 (b) ₹ 310

(c) ₹ 320 (d) ₹ 330

6. Observe the data given in three sets

P : 3, 5, 9, 12, x, 7, 2

Q : 8, 2, 1, 5, 7, 9, 3

R : 5, 9, 8, 3, 2, 7, 1

If the ratio between P's and Q's means is 7 : 5, then the ratio between P's and R's means is

(a) 5 : 7 (b) 6 : 7

(c) 7 : 5 (d) 7 : 6

7. If the ratio of mode and median of a certain data is 6 : 5, then the ratio of its mean and median is

(a) 10 : 9 (b) 9 : 10

(c) 10 : 8 (d) 8 : 10

8. If the range of 15, 14, x, 25, 30 and 35 is 23. Then, the least possible value of x is

(a) 8 (b) 10

(c) 12 (d) 14

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 419.

CHAPTER

21

Probability

In our daily life, we come across many statements having words 'probably, doubt, most probably, chances', etc., which involve an element of uncertainty.

e.g. (i) Most probably, it will rain today.

(ii) There is a 50–50 chance of India winning a toss in today's match.

Here, in statement (i), we are predicting rain today based on our past experience, when it was rained under similar conditions. Similar prediction is made in statement (ii).

The uncertainty or certainty of an even can be measured in terms of numbers varying from 0 to 1 by means of probability. In this chapter, we will study the concept of probability as a measure of certainty or uncertainty.

Chapter Objectives

- Some Basic Terms
- Sample Space
- Event
- Probability

Some Basic Terms

To understand probability we need to understand some basic terms related to it, which are discussed below

Experiment

An operation which produce some well–defined outcomes, is called an experiment e.g. Tossing of a coin, throwing a die, etc.

Random Experiment

An experiment, when repeated under identical conditions, do not produce the same outcome every time, but the outcomes produced is one out of the several possible outcomes is known as a **random** (or probabilistic) **experiment**.

e.g. Tossing a fair coin is a random experiment because if we toss a coin, then either a head or a tail will come up. But if we toss a coin again and again, then the outcome each time will not be the same. Here, head and tail are called **outcomes**.

Sample Space

A set of all possible outcomes of an experiment is called a **sample space**. It is denoted by S. Each element of a sample space is called **sample point**.

e.g. When a die is thrown, then sample space for this experiment is $S = \{1, 2, 3, 4, 5, 6\}$.

Example 1. What is the resulting sample space, if
(i) one coin is tossed?
(ii) two coins are tossed simultaneously?
(iii) three coins are tossed simultaneously?

Sol.
(i) When a coin is tossed, then possible outcomes are H and T.
$\therefore$ Sample space, $S = \{H, T\}$
(ii) When two coins are tossed, then possible outcomes are (HH), (HT), (TH) and (TT).
$\therefore$ Sample space, $S = \{(HH), (HT), (TH), (TT)\}$
(iii) When three coins are tossed, then possible outcomes are (HHH), (HHT), (HTH), (THH), (THT), (TTH), (HTT) and (TTT).
$\therefore$ Sample space, $S = \{(HHH), (HHT), (HTH), (THH), (THT), (TTH), (HHT), (TTT)\}$

Example 2. What is the resulting sample space, if
(i) one die is rolled?
(ii) two dice is rolled?

Sol.
(i) When a die is rolled, then possible outcomes are 1, 2, 3, 4, 5 and 6.
$\therefore$ Sample space = {1, 2, 3, 4, 5, 6}
(ii) When two dice are rolled, then possible outcomes are (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6) and so on.
$\therefore$ Sample space = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6) (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

Event

A possible outcomes or combination of outcomes is called an **event**. In other words, a subset of a sample space associated with a random experiment, is called an event. It is denoted by capital letters, like E, F, etc.

e.g. Getting head in the toss of a coin is an event.

If E is **an event** of getting head in the toss of a coin, then $E = \{H\}$.

Occurrence of an Event

An event E associated to a random experiment is said to be occur (or happen) in a trial, if the outcome of trial is one of the outcomes that favours E.

e.g. If a die is rolled and the outcome of a trial is 4, then we can say that each of the following events has happened (or occurred)
(i) getting a number greater than 2.
(ii) getting an even number.
(iii) getting a number less than 5.

Favourable Outcomes

The outcomes which ensure the occurrence of an event are called **favourable** (desired) **outcomes** to the event. e.g. The favourable outcomes to the occurrence of an even number, when a die is thrown, are 2, 4 and 6.

Equally Likely Outcomes

The outcomes of a random experiment are said to be equally likely, when each outcome is as likely to occur as the other, i.e. when we have no reason to believe that one is more likely to occur than the other.

e.g. When a die is thrown, all the six outcomes, i.e. 1, 2, 3, 4, 5 and 6 are equally likely to appear. So, the outcomes 1, 2, 3, 4, 5 and 6 are equally likely outcomes.

Types of Events

There are various types of events are given below

Elementary Event

An event having only one (favourable) outcome from the sample space is called an **elementary event**. e.g. In tossing a coin, the sample space $S = \{H, T\}$, so there are exactly two elementary events, namely $\{H\}$ and $\{T\}$.

Compound Event

A collection of two or more elementary events associated with an experiment is called **compound event**.

In other words, an event which has more than one (favourable) outcomes from the sample space, is called **compound event**. e.g. In the random experiment of tossing two coins simultaneously, the sample space, $S = \{HH, HT, TH, TT\}$ and if we define the event E as getting exactly one head, then it is a collection of elementary events (or outcomes) HT and TH, i.e. $E = \{HT, TH\}$, which is a compound event.

Complement of an Event

Let E be an event associated with an sample space S, then complement of E is the set of all those sample points of the sample space which are not in E and it is denoted by E' or $\overline{E}$.

In set form, $\overline{E}$ is written as $\{n : n \in S, n \notin E\}$. Also, E and $\overline{E}$ are called complementary events of each other.

e.g. In the random experiment of throwing a die, the sample space $S = \{1, 2, 3, 4, 5, 6\}$ and if we define the event E as getting the multiple of 3, then the complement of E, i.e. $\overline{E} = \{1, 2, 4, 5\}$.

Sure Event

An event, which is sure to occur, is called a **sure event** or a certain event. e.g. When we throw a die, then the event of getting a number less than 7 is a sure event.

Impossible Event

An event, which is impossible to occur, is called an **impossible event**. In other words, the events, which are unreliable or unlogical, are considered as impossible events.

e.g. In throwing a die, there are only six possible outcomes 1, 2, 3, 4, 5 and 6. Then, the event of getting a number less than 1 is an impossible event.

Probability

If E is an event that happen when an experiment is performed, then the probability of the event $P(E)$ is given by

$$P(E) = \frac{\text{Number of outcomes favourable to } E}{\text{Total number of possible outcomes}}$$

e.g. When a die is thrown, then there are six possible outcomes. So, sample space, $S = \{1, 2, 3, 4, 5, 6\}$ and total number of outcomes is 6. Let E be the event of getting 4, then favourable outcome is only one, i.e. 4.

So, probability of getting 4,

$$P(E) = \frac{\text{Number of outcomes favourable to } E}{\text{Total number of possible outcomes}} = \frac{1}{6}$$

Important Results Related to Probability

(i) The probability of happening of an event (E) always varies from 0 to 1.

(ii) The sum of all the probabilities of all possible outcomes of an experiment is 1.

(iii) Probability of an impossible event is zero.

(iv) Probability of a sure event is 1.

(v) $P(E) + P(\overline{E}) = 1$ where, E represents the occurrence of an event and $\overline{E}$ represents the non-occurrence of an event.

In other words, let E and $\overline{E}$ are complementary events.

Then, $P(E) + P(\overline{E}) = 1$

or $P(\overline{E}) = 1 - P(E)$

or $P(E) = 1 - P(\overline{E})$

Different Types of Problems Based on Single Event

There are different types of problems in which we need to find the probability of an single event. To solve these problems, we required the sample space related to given problem and favourable outcomes, related to given event which depend on the object/coin/die, etc., given in the problem. Let us discuss to find the probability for the problems of the following types

Type I Problems Based on Tossing a Coin

A coin has two sides, head (H) and tail (T). If we toss one coin or more than one coin, then

(i) Sample space of one coin $= \{H, T\}$

(ii) Sample space of two coins
$= \{(H, T), (T, H), (H, H), (T, T)\}$

(iii) Sample space of three coins
$= \{(H, H, H), (H, H, T), (H, T, H), (T, H, H),$
$(H, T, T), (T, H, T), (T, T, H), (T, T, T)\}$

Now, if given problem is related to tossing one or more than one (i.e. 2 or 3) coins simultaneously, then we can easily understand to find its probability with the help of following examples

Note (i) When we take a coin, we assume that it is to be 'fair'. Since, it is symmetrical, so there is no reason for it to come down more often on one side than the other. This property of the coin is said to be unbiased.

(ii) **Random toss of a coin** It is a phrase, which means that the coin, is allowed to fall freely without any bias or interference.

Example 3. Two unbiased coins are tossed simultaneously. Find the probability of getting (i) one head (ii) two tails (iii) exactly one tail (iv) atleast one head.

Sol. Here, two coins are tossed simultaneously.

So, sample space, $S = \{(H, T), (T, H), (H, H), (T, T)\}$

$\therefore$ Total number of possible outcomes = 4

(i) Let E_1 be the event of getting one head. Then, favourable outcomes related to E_1 are (H, T) and (T, H).

$\therefore$ Number of outcomes favourable to $E_1 = 2$

So, $P(E_1) = \dfrac{\text{Number of outcomes favourable to } E_1}{\text{Total number of possible outcomes}} = \dfrac{2}{4} = \dfrac{1}{2}$

(ii) Let E_2 be the event of getting two tails. Then, favourable outcomes related to E_2 are (T, T).

$\therefore$ Number of outcomes favourable to $E_2 = 1$.

So, $P(E_2) = \dfrac{1}{4}$.

(iii) Let E_3 be the event of getting exactly one tail. Then, favourable outcomes related to E_3 are (T, H) and (H, T).

$\therefore$ Number of outcomes favourable to $E_3 = 2$

So, $P(E_3) = \dfrac{2}{4} = \dfrac{1}{2}$.

(iv) Let E_4 be the event of getting atleast one head. Then, favourable outcomes related to E_4 are $(H, T), (T, H)$ and (H, H).

$\therefore$ Number of outcomes favourable to $E_4 = 3$

So, $P(E_4) = \dfrac{3}{4}$.

Example 4. Three unbiased coins are tossed together. Find the probability of getting

(i) all heads. (ii) two heads.
(iii) one head. (iv) atleast two heads.

Sol. Here, three coins are tossed simultaneously.

So, sample space,

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$\therefore$ Total number of possible outcomes = 8

(i) Let E_1 = Event of getting all heads = $\{HHH\}$

Then, number of outcomes favourable to $E_1 = 1$

$\therefore P$ (getting all heads) = $P(E_1)$

$$= \frac{\text{Number of outcomes favourable to } E_1}{\text{Total number of possible outcomes}} = \frac{1}{8}$$

(ii) Let E_2 = Event of getting two heads $= \{HHT, THH, HTH\}$

Then, number of outcomes favourable to $E_2 = 3$

$\therefore P$ (getting two heads) $= P(E_2) = \frac{3}{8}$

(iii) Let E_3 = Event of getting one head = $\{HTT, THT, TTH\}$

Then, number of outcomes favourable to $E_3 = 3$

$\therefore P$ (getting one head) $= P(E_3) = \frac{3}{8}$

(iv) Let E_4 = Event of getting atleast two heads $= \{HHT, HTH, THH, HHH\}$

Then, number of outcomes favourable to $E_4 = 4$

$\therefore P$ (getting atleast two heads) $= P(E_4) = \frac{4}{8} = \frac{1}{2}$

Type II Problems Based on Throwing a Die

A die has six faces marked with 1, 2, 3, 4, 5 and 6. If we have more than one die, then all dice are considered as distinct, if not otherwise stated. If we throw one (or two) die (or dice), then

(i) Sample space of a die $= \{1, 2, 3, 4, 5, 6\}$

(ii) Sample space of two dice

$$= \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{Bmatrix}$$

To solve such type of problems, we can understand easily with the help of following examples

Example 5. A die is tossed once. Find the probability of getting

(i) a number 4. (ii) a number greater than 4.

(iii) a number less than 4. (iv) an even number.

(v) a number greater than 6.

Sol. When a die is tossed once, then sample space,

$$S = \{1, 2, 3, 4, 5, 6\}.$$

So, the total number of possible outcomes = 6

(i) Let E_1 = Event of getting a number 4.

Then, the number of outcomes favourable to $E_1 = 1$

$\therefore P$ (getting a number 4) $= P(E_1)$

$$= \frac{\text{Number of outcomes favourable to } E_1}{\text{Total number of possible outcomes}} = \frac{1}{6}$$

(ii) Let E_2 = Event of getting a number greater than 4, i.e. 5 and 6.

Then, the number of outcomes favourable to $E_2 = 2$

$\therefore P$ (getting a number greater than 4) $= P(E_2) = \frac{2}{6} = \frac{1}{3}$

(iii) Let E_3 = Event of getting a number less than 4, i.e. 1, 2 and 3.

Then, the number of outcomes favourable to $E_3 = 3$

$\therefore P$ (getting a number less than 4) $= P(E_3) = \frac{3}{6} = \frac{1}{2}$

(iv) Let E_4 = Event of getting an even number, i.e. 2, 4 and 6.

Then, the number of outcomes favourable to $E_4 = 3$

$\therefore P$(getting an even number) $= P(E_4) = \frac{3}{6} = \frac{1}{2}$

(v) Let E_5 = Event of getting a number greater than 6.

Then, the number of outcomes favourable to $E_5 = 0$

$\therefore P$ (getting a number greater than 6) $= P(E_5) = \frac{0}{6} = 0$

Note In this experiment, getting a number greater than 6 is an impossible event.

Example 6. Two dice are thrown simultaneously. Find the probability of getting an even number on one die and a multiple of 3 on the other die.

Sol. When two dice are thrown, so sample space,

$$S = \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{Bmatrix}$$

$\therefore$ Total number of possible outcomes = 36.

Let E be the event of getting an even number on one die and a multiple of 3 on the other die.

Here, even numbers are 2, 4 and 6 and multiples of 3 are 3 and 6.

So, outcomes favourable to E are {(2, 3), (4, 3), (6, 3), (2, 6), (4, 6), (6, 6), (3, 2), (3, 4), (3, 6), (6, 2) and (6, 4)}

$\therefore$ Number of outcomes favourable to $E = 11$.

Now, required probability,

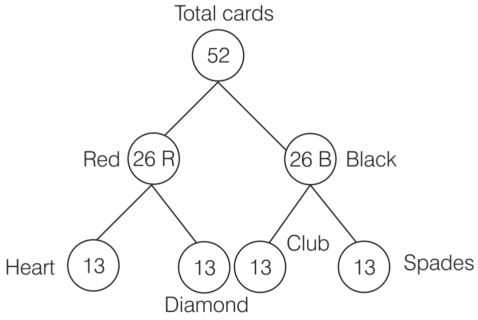
$$P(E) = \frac{\text{Number of outcomes favourable to } E}{\text{Total number of possible outcomes}} = \frac{11}{36}$$

Type III Problems Based on Playing Cards

A deck of playing cards consists of 52 cards out of which 26 are black cards and other 26 are red cards, where red cards consist of 13 cards of heart (♥), 13 cards of diamond (♦) and black cards consist of 13 cards of spades (♠) and 13 cards of club (♣).

Thus, 52 playing cards are divided into four parts (called suits) of 13 cards each, namely heart, diamond, spades and club.

Also, 13 cards in each suit are ace, king, queen, jack, 10, 9, 8, 7, 6, 5, 4, 3 and 2.



Tree diagram of playing cards

Face cards King, Queen and Jack are called face cards.
So, total face cards = 12

To solve such type of problem, we can understand easily with the help of following examples

Example 7. A card is drawn from a well-shuffled pack of 52 playing cards. Find the probability that the card drawn is

(i) a diamond. (ii) an ace or a queen.
(iii) Jack of heart.

Sol. The total number of cards in one pack of cards is 52.

$\therefore$ Total number of possible outcomes = 52.

(i) Let E_1 = Event of drawing a diamond card.

$\therefore$ Number of outcomes favourable to $E_1 = 13$

So, P (drawing a diamond card)

$$P(E_1) = \frac{\text{Number of outcomes favourable to } E_1}{\text{Total number of possible outcomes}}$$

$$\Rightarrow \quad P(E_1) = \frac{13}{52} = \frac{1}{4}$$

(ii) Let E_2 = Event of drawing an ace or a queen card.

$\therefore$ Number of outcomes favourable to $E_2 = 4 + 4 = 8$

So, P (drawing an ace or a queen) $= P(E_2) = \frac{8}{52} = \frac{2}{13}$

(iii) Let E_3 = Event of drawing a jack of heart.

$\therefore$ Number of outcomes favourable to $E_3 = 1$

So, P (drawing a jack of heart) $= P(E_3) = \frac{1}{52}$.

Example 8. One card is drawn from a well-shuffled deck of 52 cards. Find the probability of getting

(i) a king of red colour. (ii) a face card.
(iii) a red face card. (iv) the queen of diamonds.

Sol. Total number of cards in one deck of cards is 52.

$\therefore$ Total number of possible outcomes = 52

(i) Let E_1 = Event of getting a king of red colour.

$\therefore$ Number of outcomes favourable to $E_1 = 2$

[$\because$ there are four kings in a deck, in which two are red and two are black]

Hence, $P(E_1) = \frac{\text{Number of outcomes favourable to } E_1}{\text{Total number of possible outcomes}}$

$$= \frac{2}{52} = \frac{1}{26}$$

(ii) Let E_2 = Event of getting a face card.

Then, number of outcomes favourable to $E_2 = 12$

[$\because$ in a deck of cards, there are 12 face cards, namely, 4 kings, 4 jacks and 4 queens]

Hence, $P(E_2) = \frac{12}{52} = \frac{3}{13}$.

(iii) Let E_3 = Event of getting a red face card.

Then, number of outcomes favourable to $E_3 = 6$

[$\because$ in a deck of cards, there are 12 face cards in which 6 red and 6 black]

Hence, $P(E_3) = \frac{6}{52} = \frac{3}{26}$.

(iv) Let E_4 = Event of getting the queen of diamonds.

Then, number of outcomes favourable to $E_4 = 1$

[$\because$ in a deck of cards, there are 13 diamond cards, in which there is only one queen]

Hence, $P(E_4) = \frac{1}{52}$.

Type IV Problems Based on Selection of an Object from a Bag/Box

In these types of problems, we need to find the probability of selecting an item/object such as ball, marble, orange, bulb, etc., from a bag/box which have these items/objects in more than one colour/type.

This type of problem can easily understand with the help of following examples

Example 9. Three bags containing 10 orange, 10 green and 10 red balls, respectively are mixed together in one large bag. If one of the ball is taken out at random without looking into the bag, then what is the probability that it is (i) orange or green (ii) not orange?

Sol. Given, number of orange balls = 10

Number of green balls = 10

and number of red balls = 10

Since, all balls are mixed in one large bag.

So, total number of balls in large bag = 10 + 10 + 10 = 30

$\therefore$ Total number of possible outcomes = 30

(i) Let E be the event of selecting an orange or green ball. Since, there are 10 orange and 10 green balls, out of which one can be select in 20 ways.

Then, number of outcomes favourable to $E = 20$

$\therefore$ P (getting orange or green ball) $= P(E)$

$$= \frac{\text{Number of outcomes favourable to } E}{\text{Total number of possible outcomes}} = \frac{20}{30} = \frac{2}{3}$$

(ii) Let F be the event of selecting a ball which is not orange.

Since, there are 10 green and 10 red balls, so one ball which is not orange can be select in 20 ways.

Then, number of outcomes favourable to $F = 20$

$\therefore$ P (getting not orange ball) $= P(F) = \frac{20}{30} = \frac{2}{3}$

Example 10. A bag contains 5 white balls, 6 red balls and 9 green balls. A ball is drawn at random from the bag. Find the probability that the ball drawn is

(i) a green ball. (ii) a white or a red ball.

(iii) neither a green ball nor a white ball. ***[2015]***

Sol. Given, a bag contains 5 white balls, 6 red balls and 9 green balls.

$\therefore$ Total number of possible outcomes $= 5 + 6 + 9 = 20$

(i) Let G be the event of getting a green ball.

Then, number of outcomes favourable to $G = 9$

$\therefore$ Probability of drawing a green ball $= P(G) = \dfrac{9}{20}$

(ii) Let F be the event of getting a white or a red ball.

Then, number of outcomes favourable to $F = 5 + 6 = 11$

$\therefore$ Probability of drawing a white or red ball $= P(F) = \dfrac{11}{20}$

(iii) Let E be the event of getting neither a green ball nor a white ball, i.e. getting a red ball.

Then, number of outcomes favourable to $E = 6$

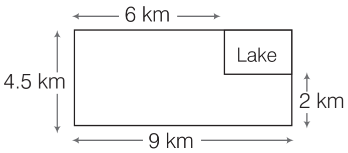
$\therefore$ Probability of drawing neither a green ball nor a white ball $= P(E) = \dfrac{6}{20} = \dfrac{3}{10}$

Type V Problems Based on Geometry

In this type of problems, a geometrical figure is given to us and we have to find the probability that, a given element lies inside the small part of the geometrical figure. For this, we first find the area/volume of both geometrical figures (small and big) separately and then find the required probability by taking area/volume of small part as number of favourable outcomes and area/volume of whole part as total number of possible outcomes.

To solve such type of problem, we can understand easily with the help of following example

Example 11. A missing helicopter is reported to have crashed somewhere in the rectangular region shown in figure. What is the probability that it crashed inside the lake shown in the figure?



Sol. Here, the helicopter is equally likely to crash anywhere in the region. The given geometrical figure is a rectangle and its small part, i.e. lake is also a rectangle.

For rectangular region,

Length = 9 km and Breadth = 4.5 km

$\therefore$ Area of entire rectangular region, where the helicopter can crash $= 4.5 \times 9 = 40.5 \text{ km}^2$

For rectangular lake,

Length $= 9 - 6 = 3$ km

Breadth $= 4.5 - 2 = 2.5$ km

$\therefore$ Area of the lake $= 2.5 \times 3 = 7.5 \text{ km}^2$

Hence, probability that helicopter crashed inside the lake

$$= \frac{\text{Area of the lake}}{\text{Area of entire rectangular region, where the helicopter can crash}}$$

$$= \frac{7.5}{40.5} = \frac{75}{405} = \frac{5}{27}$$

Type VI Miscellaneous Problems

Sometimes, the given problem is not of the type I to V which are discussed before. To solve these problems, first find the possible outcomes for these problems very carefully and then find the outcomes favourable to the given event for which we want to find the probability. Now, use the definition of probability, i.e. to calculate the required probability.

Probability

$$= \frac{\text{Number of outcomes favourable to the given event}}{\text{Total number of possible outcomes}}$$

Example 12. Two players Neha and Shivani play a tennis match. It is known that the probability of Neha winning the match is 0.62. What is the probability of Shivani winning the match?

Sol. Let E and F denote the events that Neha and Shivani win the match, respectively. It is clear that, if Neha wins the match, then Shivani losses the match and if Shivani wins the match, then Neha losses the match. Thus, E and F are complementary events.

$\therefore \quad P(E) + P(F) = 1$

Since, probability of Neha's winning the match,

i.e. $P(E) = 0.62$

$\therefore$ Probability of Shivani's winning the match,

$P(F) = P$ (Neha losses the match)

$= 1 - P(E) \quad [\because P(E) + P(F) = 1]$

$= 1 - 0.62 = 0.38$

Example 13. 12 defective pens are accidentally mixed with 132 good ones. It is not possible to just look at a pen and tell whether or not it is defective. One pen is taken out at random from this lot. Determine the probability that the pen taken out is a good one.

Sol. Total number of pens = 12 defective pens + 132 good pens

= 144 pens

$\therefore$ Total number of possible outcomes = 144

Let E be the event of selecting a good pen.

Then, number of outcomes favourable to $E = 132$

$\therefore$ Probability of selecting a good pen,

$$P(E) = \frac{132}{144} = \frac{11}{12}$$

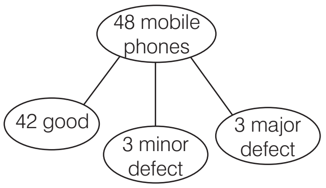
Example 14. A lot consists of 48 mobile phones of which 42 are good, 3 have only minor defect and 3 have major defect. Varnika will buy a phone, if it is good but the trader will only buy a mobile, if it has no major defect. One phone is selected at random from the lot. What is the probability that it is

(i) acceptable to Varnika?

(ii) acceptable to the trader?

Sol. Given, total number of mobile phones is 48.

So, number of possible outcomes = 48



(i) Let E_1 be the event that phone is acceptable to Varnika, i.e. E_1 be the event of getting a good phone.

$\therefore$ Number of outcomes favourable to $E_1 = 42$

Hence, probability that phone is acceptable to Varnika,

$$P(E_1) = \frac{42}{48} = \frac{7}{8}$$

(ii) Let E_2 be the event that phone is acceptable to the trader, i.e. E_2 be the event of getting a good phone or a phone with minor defect.

$\therefore$ Number of outcomes favourable to E_2

= Number of good phones

+ Number of phone with minor defect

$= 42 + 3 = 45$

Hence, probability that phone is acceptable to the trader,

$$P(E_2) = \frac{45}{48} = \frac{15}{16}$$

Example 15. Savita and Hamida are friends. What is the probability that both will have

(i) the same birthday?

(ii) different birthdays?

(ignoring a leap year)

Sol. There are 365 days in a year, so Savita's birthday can be at any day of 365 days in the year.

Similarly, Hamida's birthday can be at any day of 365 days in the year.

So, total number of possible outcomes $= 365 \times 365$

(*i*) If both have same birthday, then the number of outcomes favourable for their birthday = 365

$\therefore$ P (Savita and Hamida have same birthday)

$$= \frac{365}{365 \times 365} = \frac{1}{365}$$

(*ii*) P (Savita and Hamida have different birthdays)

$= 1 - P$ (Savita and Hamida have same birthday)

$[\because P(E) + P(\overline{E}) = 1]$

$$= 1 - \frac{1}{365} = \frac{364}{365}$$

Example 16. An integer is chosen between 0 and 100. What is the probability that it is

(i) divisible by 7?

(ii) not divisible by 7?

Sol. The total number of integers between 0 and 100 is 99.

$\therefore$ Total number of possible outcomes = 99

(i) Let E_1 = Event of choosing an integer which is divisible by 7

$= \{7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98\}$

Then, number of outcomes favourable to $E_1 = 14$

Hence, required probability

$$= P(E_1) = \frac{14}{99}$$

(ii) Let E_2 = Event of choosing an integer which is not divisible by 7.

Then, number of outcomes favourable to

$$E_2 = 99 - 14 = 85$$

$\therefore$ Required probability

$$= P(E_2) = \frac{85}{99}$$

Example 17. A game of numbers has cards marked with 11, 12, 13,..., 40. A card is drawn at random. Find the probability that the number on the card drawn is

(i) a perfect square.

(ii) divisible by 7. ***[2016]***

Sol. Given cards marked with number

11, 12, 13, 14, …, 40.

Total number of cards = 30

$\therefore$ Total number of possible outcomes = 30

(i) A perfect square number $= \{16, 25, 36\}$

$\therefore$ Number of outcomes favourable to event = 3

So, required probability $= \frac{3}{30} = \frac{1}{10}$

(ii) Number divisible by $7 = \{14, 21, 28, 35\}$

$\therefore$ Number of outcomes favourable to event = 4

So, required probability $= \frac{4}{30} = \frac{2}{15}$

Example 18. A box contains 17 cards numbered, 1, 2, 3, ... , 17 and are mixed thoroughly. A card is drawn at random from the box. Find the probability that the number on the card is

(i) odd. (ii) even.

(iii) prime.

(iv) divisible by 3.

(v) divisible by 3 and 2 both.

(vi) divisible by 3 or 2.

Sol. The cards are mixed thoroughly and a card is drawn at random from the box, means that all the outcomes are equally likely. As the box contains 17 cards, total number of possible outcomes = 17

(i) Let E be the event of getting 'the number on the card is odd'. Then, the outcomes favourable to E are 1, 3, 5, 7, 9, 11, 13, 15, 17.

$\therefore$ Number of outcomes favourable to $E = 9$

$\therefore$ P (drawing card is odd) $= P(E) = \frac{9}{17}$

(ii) Let F be the event of getting 'the number on the card is even'. Then, the outcomes favourable to F are 2, 4, 6, 8, 10, 12, 14, 16.

$\therefore$ Number of outcomes favourable to $F = 8$

$\therefore$ P (drawing card is even) $= P(F) = \frac{8}{17}$

(iii) Let G be the event of getting 'the number on the card is prime'. Then, the outcomes favourable to the event are 2, 3, 5, 7, 11, 13, 17.

$\therefore$ Number of outcomes favourable to $G = 7$

$\therefore$ P (drawing card is prime) $= P(G) = \frac{7}{17}$

(iv) Let H be the event of getting 'the number on the card is divisible by 3'. Then, the outcomes favourable to the event are 3, 6, 9, 12, 15.

$\therefore$ Number of outcomes favourable to $H = 5$

$\therefore$ P (drawing card is divisible by 3) $= P(H) = \frac{5}{17}$

(v) Let I be the event of getting 'the number on the card is divisible by 3 and 2 both'. Then, the outcomes favourable to the event are 6 and 12.

$\therefore$ Number of outcomes favourable to $I = 2$

$\therefore$ P (drawing card is divisible by 3 and 2 both)

$$= P(I) = \frac{2}{17}$$

(vi) Let J be the event of getting 'the number on the card is divisible by 3 or 2'. Then, the outcomes favourable to the event are 2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16.

$\therefore$ Number of outcomes favourable to $J = 11$

$\therefore$ P (drawing card is divisible by 3 or 2)

$$= P(J) = \frac{11}{17}$$

Example 19. Sixteen cards are labelled as $a, b, c, \ldots, m, n, o, p$. They are put in a box and shuffled. A boy is asked to draw a card from the box. What is the probability that the card drawn is

(i) a vowel?

(ii) a consonant?

(iii) none of the letters of the word 'median'? ***[2017]***

Sol. Total number of cards = 16

$\therefore$ Total number of possible outcomes = 16

Clearly, the cards labelled a, e, i, o are vowels and rest 12 are consonant.

(i) There are 4 ways of drawing a vowel card.

$\therefore$ Number of outcomes favourable to event = 4

So, probability of drawing a vowel card

$$= \frac{4}{16} = \frac{1}{4}$$

(ii) There are 12 ways of drawing a consonant card.

$\therefore$ Number of outcomes favourable to event = 12

So, probability of drawing a consonant card $= \frac{12}{16} = \frac{3}{4}$

(iii) Number of cards not labelled with letters of the word median $= 16 - 6 = 10$

Then, number of ways of drawing a card not labelled with letters of the word median = 10

$\therefore$ Number of outcomes favourable to event = 10

So, probability of drawing a card not labelled with letters of the median $= \frac{10}{16} = \frac{5}{8}$

Example 20. If 65% of the population have black eyes, 25% have brown eyes and the remaining have blue eyes, then what is the probability that a person selected at random has

(i) blue eyes? (ii) brown or black eyes?

(iii) neither blue nor brown eyes?

Sol. Given, 65% of the population have black eyes, 25% have brown eyes and the remaining $(100 - 65 - 25)\%$, i.e. 10% have blue eyes. Let the total number of people be 100, then 65 have black eyes, 25 have brown eyes and 10 have blue eyes. Therefore, the sample space of the experiment has 100 equally likely outcomes.

$\therefore$ Total number of possible outcomes = 100

(i) Let the event be 'have blue eyes'. As 10 people have blue eyes.

So, the number of outcomes favourable to the event 'have blue eyes' = 10.

$\therefore$ P (blue eyes) $= \frac{10}{100} = \frac{1}{10}$

(ii) Let the event be 'have brown or black eyes'.

Then, number of persons who have brown or black eyes

$$= 25 + 65 = 90$$

So, the number of outcomes favourable to the event 'have brown or black eyes' = 90.

$\therefore$ P(brown or black eyes) $= \frac{90}{100} = \frac{9}{10}$

(iii) The event 'have neither blue nor brown eyes' mean have black eyes.

Thus, number of persons who have neither blue nor brown eyes = Number of persons who have black eyes = 65

So, the number of outcomes favourable to the event 'have neither blue nor brown eyes' = 65

$\therefore$ P (neither blue nor brown eyes) $= \frac{65}{100} = \frac{13}{20}$

Example 21. A child's game has 8 triangles of which 3 are blue and rest are red, and 10 squares of which 6 are blue and rest are red. If one piece is lost at random, then find the probability that it is a

(i) triangle.

(ii) square.

(iii) square of blue colour.

(iv) triangle of red colour.

Sol. A child's game has 8 triangular pieces and 10 squared pieces. So, total number of pieces in the game $= 8 + 10 = 18$.

One piece is lost at random means that all pieces are equally likely to be lost. Therefore, the sample space of the experiment has 18 equally likely outcomes.

$\therefore$ Total number of possible outcomes = 18

(i) Number of triangular pieces = 8

$\therefore \quad P\text{ (triangle)} = \dfrac{8}{18} = \dfrac{4}{9}$

(ii) Number of squared pieces = 10

$\therefore \quad P\text{ (square)} = \dfrac{10}{18} = \dfrac{5}{9}$

(iii) Number of blue squared pieces = 6

$\therefore \quad P\text{ (square of blue colour)} = \dfrac{6}{18} = \dfrac{1}{3}$

(iv) Number of red coloured triangular pieces

= Total number of triangular pieces − Blue coloured triangular pieces

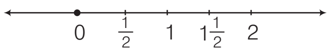
$= 8 - 3 = 5$

$\therefore \quad P\text{(triangle of red colour)} = \dfrac{5}{18}$

Example 22. In a musical chair game, the person playing the music has been advised to stop playing the music at any time within 2 min after she starts playing. What is the probability that the music will stop within the first half-minute after starting?

Sol. Since, the music is stopped at any time within 2 min. So, the possible outcomes are all real numbers (representing minute) between 0 and 2 which are represented on the number line.

Let E be the event that the music is stopped within the first half-minute. Then, outcomes favourable to E are points on the number line from 0 to $\dfrac{1}{2}$.



Here, all the outcomes are equally likely.

$\therefore$ Total number of possible outcomes

= Total distance from 0 to 2 = 2

$\because$ Number of outcomes favourable to E

= Distance from 0 to $\dfrac{1}{2} = \dfrac{1}{2}$

Now, required probability $= P(E) = \dfrac{1/2}{2} = \dfrac{1}{4}$.

Example 23. Three-digit numbers are made using the digits 4, 5, 9 (without repetition). If a number among them is selected at random, then what is the probability that the number will

(i) be a multiple of 5?

(ii) be a multiple of 9?

(iii) end with 9?

Sol. Here, three-digit numbers made from the digits 4, 5, 9 (without repetition) are 459, 495, 549, 594, 945, 954.

$\therefore$ Same space, $S = \{459, 495, 549, 594, 945, 954\}$

The sample space has 6 equally likely outcomes.

$\therefore$ Total number of possible outcomes = 6

(i) Let E be the event of getting a multiple of 5, then $E = \{495, 945\}$.

So, number of outcomes favourable to $E = 2$

$\therefore P(E) = P\text{ (a multiple of 5)} = \dfrac{2}{6} = \dfrac{1}{3}$

(ii) Let F be the event of getting a multiple of 9.

$\because$ All the six numbers are divisible by 9.

So, number of outcomes favourable to $F = 9$.

$\therefore P\text{ (a multiple of 9)} = \dfrac{6}{6} = 1$

(iii) Let G be the event of getting the number end with 9.

Since, the numbers that end with 9 are 459, 549.

So, number of outcomes favourable to $G = 2$

$\therefore P\text{ (end with 9)} = \dfrac{2}{6} = \dfrac{1}{3}$

CHAPTER EXERCISE

a 3 Marks Questions

1. In a simultaneous toss of two coins, find the probability of exactly one head.

2. A coin is tossed two times. Find the probability of getting atmost one head.

3. Two coins are tossed once. Find the probability of getting

(i) 2 heads. (ii) atleast 1 tail. ***[2012]***

4. In a game, the entry fee is ₹ 5. The game consists of tossing a coin 3 times. If one or two heads show, Sweta gets her entry fee back. If she tosses 3 heads, she receives double the entry fee. Otherwise, she will loss. For tossing a coin three times, find the probability that she

(i) losses the entry fee. (ii) gets double entry fee.
(iii) just gets her entry fee.

5. Three different coins are tossed together. Find the probability of getting

(i) exactly two heads. (ii) atleast two heads.
(iii) atleast two tails.

6. A die is thrown once. Find the probability of getting an even number less than 5.

7. If a die is thrown, what is the probability of getting a number less than 3 and greater than 2?

8. If two dice are thrown, then find the probability of getting an odd number on one and multiple of 3 on other die.

9. A die is thrown once. Find the probability of getting a number which is not a factor of 36.

10. Two dice are thrown at the same time. Find the probability that the sum of the two numbers appearing on the top of the dice is more than 9.

11. A child has a die whose six faces shows the letters as given below

$\boxed{A}\ \boxed{B}\ \boxed{C}\ \boxed{D}\ \boxed{E}\ \boxed{F}$

The die is thrown once. What is the probability of getting

(i) *A*? (ii) *D*?

12. A die is thrown once. Find the probability of getting

(i) a prime number.
(ii) a number lying between 2 and 6.
(iii) an odd number.

13. A die has its six faces marked 0, 1, 1, 1, 6, 6. Two such dice are thrown together and the total score is recorded.

(i) How many different scores are equal?
(ii) What is the probability of getting a total of 7?

14. Two dice are thrown at the same time and the product of numbers appearing on them is noted. Find the probability that the product is less than 9.

15. Two dice are thrown simultaneously. What is the probability that the sum of the numbers appearing on the dice, is

(i) 7?
(ii) not a prime number?
(iii) 1?

16. Two dice are thrown together. Find the probability that the product of the numbers on the top of the dice is

(i) 6 (ii) 12 (iii) 7

17. A pair of dice is thrown once. Find the probability of getting

(i) doublet of prime numbers.
(ii) a doublet of odd numbers.

18. From a pack of 52 playing cards, whose numbers are multiples of 3 are removed. A card is now drawn at random. What is the probability that the card drawn is

(i) a face card (king, jack or queen)?
(ii) an even numbered red card? ***[2011]***

19. A card is drawn from a well-shuffled pack of 52 cards. Find the probability that the card drawn is neither a red card nor a queen.

20. The king, queen and jack of clubs are removed from a deck of 52 playing cards and then well-shuffled. Now, one card is drawn at random from the remaining cards. Find the probability of getting a card of

(i) a heart. (ii) a king.

21. One card is drawn from a pack of 52 cards, each of the 52 cards being equally likely to be drawn. Find the probability that the card drawn is

(i) a red king.
(ii) '2' of spade.
(iii) '10' of a black suit.

22. Cards marked with numbers 5 to 75 are placed in a box and mixed thoroughly. One card is drawn from the box. Find the probability that the number on the card is odd.

23. A box contains some black balls and 30 white balls. If the probability of drawing a black ball is two-fifth of a white ball, then find the number of black balls in the box. ***[2013]***

24. A box contains 90 discs, which are numbered from 1 to 90. If one disc is drawn at random from the box, then find the probability that it bears

(i) a two-digit number.

(ii) a perfect square number.

(iii) a number divisible by 5.

25. A bag contains 2 green, 3 red and 4 black balls. A ball is taken out of the bag at random. Find the probability that the selected ball is

(i) not green. (ii) not black.

26. Box A contains 25 slips of which 19 are marked ₹ 1 and other are marked ₹ 5. Box B contains 50 slips of which 45 are marked ₹ 1 and others are marked ₹ 13. Slips of both boxes are put into a third box and reshuffled. A slip is drawn at random. What is the probability that it is marked other than ₹ 1?

27. In a box contain 40 electric rods in which 30 are goods, 4 have minor defect and 6 have major defect Anita will buy a electric rod, if it has no major defect. If one electric rod is selected at random from the lot, then what is the probability that it is

(i) acceptable to Anita?

(ii) acceptable to the trader?

28. There are 100 cards in a bag on which numbers from 1 to 100 are written. A card is taken out from the bag at random. Find the probability that the number on the selected card

(i) is divisible by 9 and is a perfect square.

(ii) is a prime number greater than 80.

29. A bag contains 24 balls of which x are red, $2x$ are white and $3x$ are blue. A ball is selected at random. What is the probability that it is

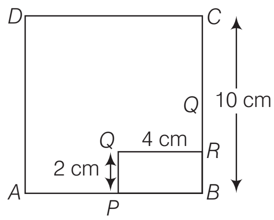
(i) not red? (ii) white?

30. A bag contains 18 balls out of which x balls are red.

(i) If one ball is drawn at random from the bag, what is the probability that it is red ball?

(ii) If 2 more red balls are put in the bag, the probability of drawing a red ball will be 9/8 times that of probability of red ball coming in part (i). Find the value of x.

31. A rectangle of dimension $2 \text{ cm} \times 4 \text{ cm}$ is drawn in one of the corner of square of side 10 cm and shaded as shown in the adjoining figure. A point is selected at random from the interior of square $ABCD$. What is the probability that the point will be chosen from the shaded part?



32. A number is chosen from 1 to 100. Find the probability that it is a prime number.

33. The probability of getting a rotten egg from a lot of 400 eggs is 0.035. Find the number of rotten eggs in the lot.

34. A single letter is selected at random from the word 'PROBABILITY'. Find the probability that it is a vowel.

35. A letter of English alphabet is chosen at random. Determine the probability that the letter is a consonant.

36. A letter is chosen at random from the English alphabet. What is the probability that it is a letter of the word 'RAMANUJAN'?

37. Find the probability of having 53 Sunday in

(i) a non-leap year. (ii) a leap year.

38. Find the probability that a non-leap year selected at random will contain 53 Friday.

39. Two friends were born in the year 2000. What is the probability that they have the same birthday?

40. A school has five houses A, B, C, D and E. A class has 23 students, 4 from house A, 8 from house B, 5 from house C, 2 from house D and rest from house E. A single student is selected at random to be the class monitor. Find the probability that the selected students is not from A, B and C.

41. The probability of guessing the correct answer to certain question is $p/12$. If the probability of not guessing the correct answer to same question is $3/4$, then find the value of p.

42. A girl calculates that the probability of her winning the first prize in a lottery is 0.08. If 6000 tickets are sold, then how many tickets has she bought?

43. Malini buys a fish from a shop for her aquarium. The shopkeeper takes out one fish at random from a tank containing 5 male fish and 8 female fish. What is the probability that the fish taken out is a male fish?

44. A book contain 85 pages. A page is chosen at random. What is the probability that the sum of the digits on the page is 8?

45. Cards marked with numbers 1, 2, 3, 4, …, 20 are well-shuffled and a card is drawn at random. What is the probability, that the number of the cards is
(i) a prime number? (ii) divisible by 3?
(iii) a perfect square? *[2010]*

46. A piggy bank contains hundred 50 paise coins, fifty ₹ 1 coins, twenty ₹ 2 coins and ten ₹ 5 coins. If it is equally likely that one of the coins will fall out when the bank is turned upside down, then what is the probability that the coin
(i) will be a 50 paise coin?
(ii) will not be a ₹ 5 coin?

47. From a group of 2 boys and 2 girls, two children are selected at random. List the possible outcomes. Find the probability that one boy and one girl are selected.

48. An integer is chosen between 0 and 100. What is the probability that it is
(i) divisible by 8? (ii) not divisible by 8?

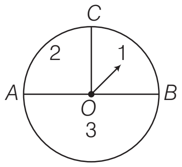
49. A carton of 24 bulbs contains 6 defective bulbs. One bulb is drawn at random. What is the probability that the bulb is not defective? If the bulb selected is defective and it is not replaced and a second bulb is selected at random from the rest, then what is the probability that the second bulb is defective?

50. A number is selected at random from first 50 natural numbers. Find the probability that it is a multiple of 3 and 4.

51. A number is selected from the numbers 2, 3, 3, 5, 5, 5, 7, 7, 7, 7, 9, 9, 9, 9, 9 at random. Find the probability that the number selected is
(i) their median. (ii) their mode.

52. A carton consists of 100 shirts of which 88 are good, 8 have minor defect and 4 have major defect. Jimmy, a trader, will only accept the shirts which are good but Sujata, another trader, will only reject the shirts which have major defect. One shirt is drawn at random from the carton. What is the probability that it is acceptable to
(i) Jimmy? (ii) Sujata?

53. A game of chance consists of an arrow which comes to rest pointing at one of the regions 1, 2 or 3. O is the centre of the circle, $OC \perp AB$.



Find the probability that
(i) arrow is resting on 3.
(ii) arrow is resting on 1.
(iii) arrow is not resting on 2.

54. Cards bearing numbers 2, 4, 6, 8, 10, 12, 14, 16, 18 and 20 are kept in a bag. A card is drawn at random from the bag. Find the probability of getting a card which is
(i) a prime number.
(ii) a number divisible by 4.
(iii) a number that is a multiple of 6.
(iv) an odd number. *[2018]*

b 4 Marks Questions

55. A die has six faces marked by the given numbers as shown below

$\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{-1}$ $\boxed{-2}$ $\boxed{-3}$

The die is thrown once. What is the probability of getting
(i) a positive integer?
(ii) an integer greater than −3?
(iii) the smallest integer? *[2014]*

56. Two dice are thrown simultaneously. Find the probability of getting
(i) an even number as the sum.
(ii) the sum as a prime number.
(iii) a total of atleast 10.
(iv) a doublet of even number.
(v) a multiple of 2 on one die and a multiple of 3 on the other.
(vi) same number on both dice i.e. a doublet.
(vii) a multiple of 3 as the sum.

57. The king, queen and jack of clubs are removed from a deck of 52 playing cards and then well-shuffled. One card is selected from the remaining cards. Find the probability of getting
(i) a heart. (ii) a king.
(iii) a club. (iv) the '10' of hearts.

58. All the three face cards of spades are removed from a well-shuffled pack of 52 cards. A card is drawn at random from the remaining pack. Find the probability of getting
(i) a black face card. (ii) a queen.
(iii) a black card. (iv) a spade.

59. A bag contains white, black and red balls only. A ball is drawn at random from the bag. The probability of getting a white ball is 3/10 and that of a black ball is 2/5. Find the probability of getting a red ball. If the bag contains 20 black balls, then find the total number of balls in the bag.

60. A traffic signal displays green light for two minutes to allow passage of traffic on a particular road. If the signal is currently displaying green light, then find the probability that it will turn red within the next half a minute.

61. There are 1000 sealed envelopes in a box, 10 of them contain a cash prize of ₹ 100 each, 100 of them contain a cash prize of ₹ 50 each and 200 of them contain a cash prize of ₹ 10 each and rest do not contain any cash prize. If they are well-shuffled and an envelope is picked up out, what is the probability that it contains no cash prize?

62. A number x is selected at random from the numbers 1, 4, 9, 16 and another number y is selected at random from the numbers 1, 2, 3, 4. Find the probability that the value of xy is more than 16.

Hints and Answers

1. Hint Here, number of possible outcomes = 4

Let E = Event of getting exactly one head.

So, E would consist of two outcomes, namely HT and TH.

$\therefore$ Number of outcomes favourable to $E = 2$. **Ans.** 1/2

2. Hint Let E = Event of getting atmost one head

$= \{(HT), (TH), (TT)\}$

Ans. 3/4

3. (i) **Hint** Favourable outcomes for getting 2 heads is HH.

$\therefore$ Number of outcomes favourable to given event = 1

Ans. $\frac{1}{4}$

(ii) **Hint** Favourable outcomes for getting atleast 1 tail are HT, TH and TT.

$\therefore$ Number of outcomes favourable to given event = 3

Ans. 3/4

4. Hint Total number of possible outcomes = 8

(i) She tosses tail three times, i.e. TTT. **Ans.** $\frac{1}{8}$

(ii) She tosses heads three times, i.e. HHH. **Ans.** $\frac{1}{8}$

(iii) Sweta gets heads one or two times, i.e. event of getting HTT, THT, TTH, HHT, HTH or THH.

Ans. $\frac{3}{4}$

5. Hint When three coins are tossed together, all possible outcomes are $HHH, HHT, HTH, THH, HTT, THT, TTH$ and TTT.

$\therefore$ Total number of possible outcomes = 8

(i) The favourable outcomes of getting exactly 2 heads are HHT, HTH and THH. **Ans.** $\frac{3}{8}$

(ii) The event of getting atleast two heads, i.e. HHT, HTH, THH or HHH. **Ans.** $\frac{1}{2}$

(iii) Do same as part (ii). **Ans.** $\frac{1}{2}$

6. Hint Total number of possible outcomes = 6

There are two even numbers, 2 and 4, which is less than 5.

$\therefore$ Number of outcomes favourable to given event = 2.

Ans. 1/3

7. Hint There is no such number lies on a die which is less than 3 and greater than 2. **Ans.** 0

8. Do same as Example 6. **Ans.** $\frac{11}{36}$

9. Hint Total number of possible outcomes = 6

The factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, 36.

Here, we see that 5 is not a factor of 36. **Ans.** 1/6

10. Hint Let E be the event of getting the sum of the numbers appearing on the top of the two dice is more than 9. Then, the favourable outcomes are (4, 6), (5, 5), (6, 4), (5, 6), (6, 5) and (6, 6).

$\therefore$ Number of favourable outcomes to $E = 6$ **Ans.** $\frac{1}{6}$

11. (i) **Hint** Let E_1 = Event of getting a letter A.

Number of outcomes favourable to $E_1 = 1$ **Ans.** $\frac{1}{6}$

(ii) **Hint** Let E_2 = Event of getting a letter D

Number of outcomes favourable to $E_2 = 1$ **Ans.** 1/6

12. (i) **Hint** Let E_1 = Event of getting a prime number.

Then, $E_1 = \{2, 3, 5\}$ **Ans.** 1/2

(ii) **Hint** Let E_2 = Event of getting a number lying between 2 and 6. Then, $E_2 = \{3, 4, 5\}$ **Ans.** 1/2

(iii) **Hint** Let E_3 = Event of getting an odd number.

Then, $E_3 = \{1, 3, 5\}$ **Ans.** 1/2

13. Hint The different scores are recorded as 0, 1, 2, 6, 7, 12.

Ans. (i) 6 (ii) $\frac{1}{6}$

14. Hint The favourable events are (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3), (2, 4), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (3, 2), (4, 2). **Ans.** $\frac{4}{9}$

15. Hint Total number of possible outcomes = 36

(i) The event of getting a sum of numbers is seven

$= \{(1, 6), (2, 5), (3, 4), (6, 1), (5, 2), (4, 3)\}$

Ans. 1/6

(ii) The event of getting a sum of number is not prime,

i.e. 6, 8, 9, 10, 12
$=\{(1,3),(3,1),(2,2),(1,5),(5,1),(2,4),(4,2),$
$(3,3),(2,6),(6,2),(3,5),(5,3),(4,4),(3,6),$
$(6,3),(4,5),(5,4),(4,6),(6,4),(5,5),(6,6)\}$

Ans. $\frac{7}{12}$

(iii) No such a pair exist, in which sum of numbers is 1.
Ans. 0

16. **Hint** Total number of possible outcomes $=36$
(i) Favourable outcomes are (1, 6), (2, 3), (3, 2) and (6, 1). **Ans.** $\frac{1}{9}$
(ii) Favourable outcomes are (2, 6), (3, 4), (4, 3) and (6, 2). **Ans.** $\frac{1}{9}$
(iii) Product of the numbers on the top of the dice cannot be 7. So, its probability is zero. **Ans.** 0

17. **Hint** Total number of possible outcomes $=6\times 6=36$
(i) Doublets of prime numbers are (2, 2), (3, 3) and (5, 5). They are three in numbers. **Ans.** $\frac{1}{12}$
(ii) Doublets of odd numbers are (1, 1), (3, 3) and (5, 5). They are three in number. **Ans.** $\frac{1}{12}$

18. **Hint** Total number of cards in a pack $=52$
Multiples of 3 in each suit are 3, 6 and 9.
We know that in a pack of cards, there are four suits.
Number of remove cards $=3\times 4=12$
Total number of cards left $=52-12=40$
(i) Number of favourable cards $=3\times 4=12$ **Ans.** $\frac{3}{10}$
(ii) Even numbered red cards
$=2\times$ even number in each red card i.e.
$\{2,4,8,10\}=2\times 4=8$
[since, there are two suits of red cards]
Number of favourable cards $=8$ **Ans.** $\frac{1}{5}$

19. **Hint** Total number of red cards $=26$ (including 2 queens)
Total number of queen cards is 4, out of which 2 are black queen cards and 2 are red queen cards.
Total number of red cards and queen cards
$=26+2=28$
Let E be the event of getting neither red nor a queen card, then number of outcomes favourable to E
$=52-28=24$ **Ans.** $\frac{6}{13}$

20. **Hint** Total number of left card in a pack $=49$
(i) Number of heart card in a pack $=13$
(ii) Number of king card in a pack $=3$
Ans. (i) 13/49 (ii) 3/49

21. **Hint** Out of 52 cards, one card can be drawn in 52 ways. So, total number of possible outcomes $=52$
(i) The number of red king in a pack of card is 2.
Ans. $\frac{1}{26}$
(ii) Only one card exist in spade which shows number 2.
Ans. $\frac{1}{52}$
(iii) There are two suits of black cards, namely spades and clubs. Each suit contains one card bearing number 10.
$\therefore$ Total number of outcomes favourable to the given event $=2$
Ans. 1/26

22. **Hint** Total number of possible outcomes $=71$
Let $E=$ Event of getting an odd number from 5 to 75, i.e. event of getting $=5,7,9,11,\ldots,75$
Ans. 36/71

23. **Hint** Let number of black balls be x.
So, total number of balls $=x+30$
Now, probability of drawing a white ball $=\frac{30}{x+30}$
and probability of drawing a black ball $=\frac{x}{x+30}$
According to the question,
Probability of drawing a black ball
$=\frac{2}{5}$ (Probability of drawing a white ball)
Ans. 12

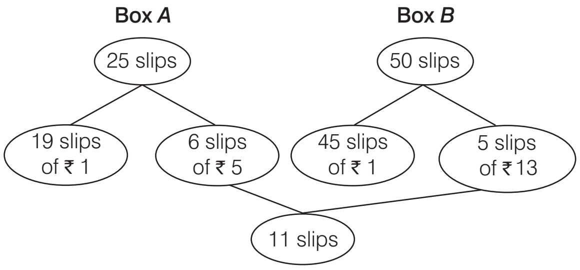
24. (i) **Hint** Let $E_1=$ Event of selecting a disc bearing a two-digit number.
Here, two-digit numbers are 10, 11, ... and 90.
$\therefore$ Number of outcomes favourable to $E_1=81$
Ans. 9/10
(ii) **Hint** Let $E_2=$ Event of selecting a disc bearing a perfect square number.
Then, $E_2=\{1,4,9,16,25,36,49,64,81\}$
Number of outcomes favourable to $E_2=9$
Ans. 1/10
(iii) **Hint** Let $E_3=$ Event of selecting a disc bearing a number divisible by 5.
Then, $E_3=\{5,10,15,20,25,30,35,40,45,$
$50,55,60,65,70,75,80,85,90\}$
Number of outcomes favourable to $E_3=18$
Ans. 1/5

25. Hint Total number of balls $= 9$

(i) Number of outcomes favourable to the given event $= 7$ **Ans.** 7/9

(ii) Number of outcomes favourable to the given event $= 5$ **Ans.** 5/9

26. Hint Total number of slips in third box $= 25 + 50 = 75$



From the chart, it is clear that there are 11 slips which are marked other than ₹ 1.

Ans. $\frac{11}{75}$

27. Do same as Example 14. **Ans.** (i) $\frac{3}{4}$ (ii) $\frac{17}{20}$

28. Hint Number of all possible outcomes $= 100$

(i) Let E_1 be the event of getting a number which is divisible by 9 and is a perfect square.

Then, outcomes favourable to E_1 are $3^2, 6^2, 9^2$.
Ans. 3/100

(ii) Let E_2 be the event of getting a prime number greater than 80.
Then, outcomes favourable to E_2 are 83, 89 and 97.

Ans. $\frac{3}{100}$

29. Hint Total number of balls in a bag $= 24$

Number of red balls $= x$

Number of white balls $= 2x$

Number of blue balls $= 3x$

$\therefore\ x + 2x + 3x = 24$ **Ans.** (i) 5/6 (ii) 1/3

30. Hint

(i) Total number of balls in the bag $= 18$

Let total number of red balls in the bag $= x$

Ans. $\frac{x}{18}$

(ii) Now, 2 red balls are added to the bag.

$\therefore$ Total balls in the bag $= 18 + 2 = 20$

Total number of red balls in the bag $= x + 2$

$\therefore$ Probability of drawing red ball $= \frac{x+2}{20}$

According to the question, we get

$$\frac{x+2}{20} = \frac{9}{8}\left(\frac{x}{18}\right)$$ **Ans.** 8

31. Hint P (point will be chosen from the shaded part)

$$= \frac{\text{Area of the rectangle } PQRB}{\text{Area of the square } ABCD}$$ **Ans.** 0.8

32. Hint Total number of possible outcomes $= 100$

Let $E =$ Event of getting a prime number, i.e. event of getting 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89 or 97

Ans. 1/4

33. Hint Number of rotten eggs

$= \text{Total number of eggs} \times P(E)$.

Ans. 14

34. Hint There are three vowels and one is repeated. So, number of outcomes favourable to the given event $= 4$. **Ans.** $\frac{4}{11}$

35. Hint We know that in English alphabet, there are 26 letters (5 vowels + 21 consonants).

So, total number of possible outcomes $= 26$

Let $E =$ Event of choosing a consonant, i.e. choosing $b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y$ or z

Ans. 21/26

36. Hint Number of possible outcomes $= 26$

Letters in the word 'RAMANUJAN' are R, A, M, N, U, J. **Ans.** 3/13

37. (i) **Hint** In non-leap year, there are 365 days, out of which 364 days make 52 weeks and in each week there is one Sunday.

Therefore, we have to find probability of having a Sunday on the remaining 1 day. Now, that 1 day can be Monday, Tuesday, Wednesday, Thursday, Friday, Saturday and Sunday. **Ans.** 1/7

(ii) **Hint** In a leap year, there are 366 days, out of which 364 days make 52 weeks and in each week there is one Sunday. Therefore, we have to find the probability of having a Sunday on one of the remaining 2 days.

Now, the 2 days can be (Sunday, Monday) or (Monday, Tuesday) or (Tuesday, Wednesday) or (Wednesday, Thursday) or (Thursday, Friday) or (Friday, Saturday) or (Saturday, Sunday). Note that Sunday occurs 2 times in these 7 pairs.

Ans. $\frac{2}{7}$

38. Hint Non-leap year contains 365 days, in which 52 weeks and one day extra. **Ans.** 1/7

39. Hint Total number of days in the year 2000 $= 366$
[$\because$ year 2000 is a leap year]

Total number of ways in which two friends may have their birthday $= 366 \times 366$

Number of ways in which two friends have same birthday $= 366$

Ans. $\dfrac{1}{366}$

40. Hint Total number of students $= 23$

$\therefore$ Total number of possible outcomes $= 23$

Number of students in houses A, B and C

$= 4 + 8 + 5 = 17$

$\therefore$ Remaining students $= 23 - 17 = 6$

Thus, number of outcomes favourable to the given event $= 6$ **Ans.** $\dfrac{6}{23}$

41. Hint We know that, $1 - P(E) = P(\overline{E}) \Rightarrow 1 - \dfrac{p}{12} = \dfrac{3}{4}$

Ans. $p = 3$

42. Hint Let she bought x tickets.
Then, probability of her winning the first prize

$\dfrac{x}{6000} = 0.08$ **Ans.** 480

43. Hint Let E be the event of getting a male fish. As there are 5 male fish in the tank, the number of outcomes favourable to $E = 5$ **Ans.** 5/13

44. Hint Let $E =$ Event of getting the sum of the digits on the page is $8 = \{17, 26, 35, 44, 53, 62, 71\}$

Then, the number of outcomes favourable to $E = 7$

Ans. 7/85

45. Hint Total number of cards $= 20$

$\therefore$ Total number of possible outcomes $= 20$

(i) Prime numbers are 2, 3, 5, 7, 11, 13, 17 and 19.

$\therefore$ Number of favourable cards $= 8$ **Ans.** $\dfrac{2}{5}$

(ii) Numbers divisible by 3 are 3, 6, 9, 12, 15 and 18.

$\therefore$ Number of favourable cards $= 6$ **Ans.** $\dfrac{3}{10}$

(iii) Perfect square numbers are 1, 4, 9 and 16.

$\therefore$ Number of favourable cards $= 4$ **Ans.** $\dfrac{1}{5}$

46. (i) **Hint** Total number of coins

$= 100 + 50 + 20 + 10 = 180$ **Ans.** 5/9

(ii) **Hint** Number of coins, which are not of ₹5

$= 180 - 10 = 170$ **Ans.** 17/18

47. Hint Let the 2 boys and 2 girls be denoted by B_1, B_2, G_1 and G_2, respectively.

$\therefore$ The total possible outcomes are

B_1B_2, B_1G_1, B_1G_2, B_2G_1, B_2G_2 and G_1G_2.

$\therefore$ Total number of possible outcomes $= 6$

If one boy and one girl are selected, then favourable outcomes are B_1G_1, B_1G_2, B_2G_1 and B_2G_2.

Ans. 2/3

48. Hint The numbers divisible by 8 are 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96.

Ans. (i) $\dfrac{4}{33}$ (ii) $\dfrac{29}{33}$

49. Hint Number of not defective bulb $= 18$

$\therefore$ Probability of not defective bulb $= \dfrac{18}{24}$

If in first selection, defective ball is not replaced, then in the second selection, total number of balls left 24 and number of defective balls left 5.

Ans. 3/4 and 5/23

50. Hint The multiples of 3 and 4 are 12, 24, 36, 48.

Ans. $\dfrac{2}{25}$

51. (i) **Hint** Median of the numbers is 7.

$\therefore$ Number of outcomes favourable to the given event $= 4$ **Ans.** $\dfrac{4}{15}$

(ii) **Hint** Mode $= 9$

$\therefore$ Number of outcomes favourable to the given event $= 5$ **Ans.** $\dfrac{1}{3}$

52. (i) **Hint** Favourable outcomes for Jimmy acceptable $= 88$

(ii) **Hint** Favourable outcomes for Sujata acceptable

$= 88 + 8 = 96$

Ans. (i) 0.88 (ii) 0.96

53. Hint Total angle made by the circle at O is $360°$.

Angle subtended by region 1 at $O = 90°$

Angle subtended by region 2 at $O = 90°$

Angle subtended by region 3 at $O = 180°$

Ans. (i) 1/2 (ii) 1/4 (iii) 3/4

54. Hint Total number of outcomes $= 10$

(i) Number of favourable outcomes (prime numbers) $= 1$

Ans. $\dfrac{1}{10}$

(ii) Number of favourable outcomes (numbers divisible by 4) $= 5$ **Ans.** $\dfrac{1}{2}$

(iii) Number of favourable outcomes (numbers which is a multiple of 6) $= 3$ **Ans.** $\dfrac{3}{10}$

(iv) Number of favourable outcomes (odd numbers) $= 0$

Ans. 0

55. Hint Total number of possible outcomes $=6$

(i) Positive integers are 1, 2 and 3.

$\therefore$ Number of outcomes favourable to the given event $=3$ **Ans.** $\frac{1}{2}$

(ii) Integers greater than -3 are 1, 2, 3, -1 and -2.

$\therefore$ Number of outcomes favourable to the given event $=$ Number of integers greater than $-3=5$ **Ans.** $\frac{5}{6}$

(iii) Smallest integer is -3 only.

$\therefore$ Number of outcomes favourable to the given event $=1$ **Ans.** $\frac{1}{6}$

56. Hint Total number of possible outcomes $=36$

(i) The event of getting an even number as the sum are 2, 4, 6, 8, 10 and 12.

$\therefore$ Number of outcomes favourable to the given event $=18$ **Ans.** $\frac{1}{2}$

(ii) The event of getting the sum as a prime number are 2, 3, 5, 7 and 11.

$\therefore$ Number of outcomes favourable to the given event $=15$ **Ans.** $\frac{5}{12}$

(iii) The event of getting a total of atleast 10 are 10, 11 and 12.

$\therefore$ Number of outcomes favourable to the given event $=6$

Ans. $\frac{1}{6}$

(iv) $\therefore$ Number of outcomes favourable to the given event, i.e. event of getting a doublet of even number $=3$

Ans. $\frac{1}{12}$

(v) $\therefore$ Number of outcomes favourable to the given event, i.e. event of getting a multiple of 2 on one die and a multiple of 3 on the other $=11$

Ans. $\frac{11}{36}$

(vi) Let A_6 be the event of getting the same number on both dice.

$\therefore$ Number of outcomes favourable to the given event $=6$. **Ans.** $\frac{1}{6}$

(vii) Number of outcomes favourable to the given event, i.e. events of getting a multiple of 3 as the sum, i.e. event of getting the sum 3, 6, 9 and 12 $=12$

Ans. $\frac{1}{3}$

57. Hint After removing king, queen and jack of clubs from a deck of 52 playing cards, there are 49 cards left in the deck.

$\therefore$ Total number of possible outcomes $=49$

(i) Here, number of outcomes favourable to the given event $=13$ **Ans.** 13/49

(ii) Here, number of outcomes favourable to the given event $=3$ **Ans.** 3/49

(iii) After removing king, queen and jack of clubs only 10 clubs cards are left in the deck.

Here, number of outcomes favourable to the given event $=10$ **Ans.** 10/49

(iv) Number of outcomes favourable to the given event $=1$ **Ans.** $\frac{1}{49}$

58. Hint Total number of left card in a pack $=49$

(i) Number of black face cards in a pack $=3$

Ans. $\frac{3}{49}$

(ii) Number of queen cards in a pack $=3$ **Ans.** 3/49

(iii) Number of black cards in a pack $=23$ **Ans.** 23/49

(iv) Number of spade cards in a pack $=10$ **Ans.** $\frac{10}{49}$

59. Hint Probability of getting a red ball

$=1-[P(\text{white ball})+P(\text{black ball})]=1-\frac{3}{10}-\frac{2}{5}=\frac{3}{10}$

Now, let the number of balls in the bag be x.

and probability of getting a black ball $=\frac{20}{x} \Rightarrow \frac{2}{5}=\frac{20}{x}$

Ans. 50

60. Hint Total number of minutes $=2$

Outcomes favourable to the given event $=\frac{1}{2}$ **Ans.** $\frac{1}{4}$

61. Hint Number of envelopes containing cash prize

$=10+100+200=310$

Number of envelopes containing no cash prize

$=1000-310=690$ **Ans.** 0.69

62. Hint Two numbers can be selected in 16 ways as listed below

(1, 1), (1, 2), (1, 3), (1, 4), (4, 1), (4, 2), (4, 3), (4, 4), (9, 1), (9, 2), (9, 3), (9, 4), (16, 1), (16, 2), (16, 3) and (16, 4)

$\therefore$ Total number of possible outcomes $=16$

Now, the product of xy will be more than 16, if x and y are chosen in one of the following ways

(9, 2), (9, 3), (9, 4), (16, 2), (16, 3) and (16, 4)

$\therefore$ Number of outcomes favourable to the given event $=6$

Ans. $\frac{3}{8}$

ARCHIVES* *(Last 8 Years)*

Collection of Questions Asked in Last 8 Years' (2018-2011) ICSE Class 10th Examinations

2018

1 Cards bearing numbers 2, 4, 6, 8, 10, 12, 14, 16, 18 and 20 are kept in a bag. A card is drawn at random from the bag. Find the probability of getting a card which is

(i) a prime number. (ii) a number divisible by 4.

(iii) a number that is a multiple of 6. (iv) an odd number.

2017

2 Sixteen cards are labelled as $a, b, c, \ldots, m, n, o, p$. They are put in a box and shuffled. A boy is asked to draw a card from the box. What is the probability that the card drawn is

(i) a vowel? (ii) a consonant?

(iii) none of the letters of the word 'median'?

2016

3 A game of numbers has cards marked with 11, 12, 13, …, 40. A card is drawn at random. Find the probability that the number on the card drawn is

(i) a perfect square. (ii) divisible by 7.

2015

4 A bag contains 5 white balls, 6 red balls and 9 green balls. A ball is drawn at random from the bag. Find the probability that the ball drawn is

(i) a green ball. (ii) a white or a red ball.

(iii) neither a green ball nor a white ball.

2014

5 A die has six faces marked by the given numbers as shown below

$\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{-1}$ $\boxed{-2}$ $\boxed{-3}$

The die is thrown once. What is the probability of getting

(i) a positive integer? (ii) an integer greater than –3? (iii) the smallest integer?

2013

6 A box contains some black balls and 30 white balls. If the probability of drawing a black ball is two-fifth of a white ball, then find the number of black balls in the box.

2012

7 Two coins are tossed once. Find the probability of getting

(i) 2 heads. (ii) atleast 1 tail.

2011

8 From a pack of 52 playing cards, whose numbers are multiples of 3 are removed. A card is now drawn at random. What is the probability that the card drawn is

(i) a face card (king, jack or queen)? (ii) an even numbered red card?

* All these questions are completely covered in chapter either as solved examples or in chapter exercise.

CHALLENGERS*

A Set of Brain Teasing Questions for Exercise of Your Mind

1. The probability that the minute hand lies from 5 to 15 min in the wall clock, is

(a) $\frac{1}{6}$ (b) $\frac{5}{6}$ (c) $\frac{1}{5}$ (d) $\frac{1}{10}$

Directions (Q.Nos. 2-4) The diameters of circles (in mm) drawn on a table are given below

Diameters	14–20	21–27	28–34	35–41	42–48
Number of circles	3	5	8	11	7

2. If a circle is chosen at random, then the probability that chosen circle has diameter less than 28.

(a) $\frac{4}{17}$ (b) $\frac{17}{4}$ (c) $\frac{15}{2}$ (d) $\frac{13}{5}$

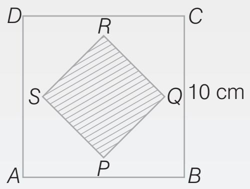
3. If a circle is chosen at random, then find the probability that chosen circle has radius lying between 14 to 17.

(a) $\frac{15}{2}$ (b) $\frac{4}{17}$ (c) $\frac{17}{4}$ (d) $\frac{19}{4}$

4. If a circle is chosen at random, then find the probability that chosen circle has diameter above 50.

(a) $\frac{17}{4}$ (b) $\frac{4}{17}$ (c) 0 (d) $\frac{15}{2}$

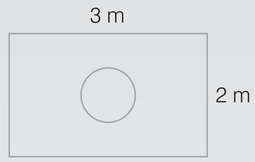
5. A square of side 5 cm is drawn in the interior of another square of side 10 cm and shaded as shown in the figure below



A point is selected at random from the interior of square *ABCD*. The probability that the point will be chosen from the shaded part is

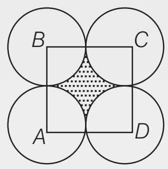
(a) 0.05 (b) 0.10 (c) 0.25 (d) 0.30

6. Suppose you drop a die at random on the rectangular region shown in the figure. The probability that it will land inside the circle of diameter 1m, is



(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{24}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

7 In the given figure, points A, B, C and D are the centres of four circles that each have a radius of length one unit. If a point is selected at random from the interior of square *ABCD*, then the probability that the point will be chosen from the shaded region, is



(a) $\frac{4-\pi}{2}$ (b) $\frac{4-\pi}{4}$

(c) $\frac{4+\pi}{2}$ (d) None of these

* These questions may or may not be asked in the examination, have been given just for additional practice required for olympiads Scholarship Exams etc. For detailed explanations refer Page No. 420.

Explanations to Challengers

Chapter 2 Banking

1. *(a)* Given, number of instalments (n) = 15 months,

amount of each instalment (P) = ₹ 500

and interest (I) = ₹ 300

Now, as $I = \dfrac{Prn(n+1)}{2400}$

$\therefore \quad r = \dfrac{2400\, I}{Pn(n+1)}$

$= \dfrac{2400 \times 300}{500 \times 15 \times 16} = 6\%$

2. *(b)* Let the time period be T yr.

Then, T yr = 12 T months

$\because$ Interest received by him = ₹ 1687.5

$\therefore \quad \dfrac{900 \times 7.5 \times 12T \times (12T+1)}{2400} = 1687.5 \quad \left[\because I = \dfrac{Prn(n+1)}{2400}\right]$

$\Rightarrow \quad \dfrac{9(T)(12T+1)(7.5)}{2} = 1687.5$

$\Rightarrow \quad T(12T+1) = \dfrac{1687.5 \times 2}{9 \times 7.5} = 50$

$\Rightarrow \quad 12T^2 + T - 50 = 0$

$\Rightarrow \quad 12T^2 - 24T + 25T - 50 = 0$

$\Rightarrow \quad 12T(T-2) + 25(T-2) = 0$

$\Rightarrow \quad (12T+25)(T-2) = 0$

$\Rightarrow \quad T = \dfrac{-25}{12}$ or 2

[$\because$ time cannot be negative]

$\therefore \quad T = 2$ yr

3. *(a)* We have, MV $= Pn\left[1 + \dfrac{r(n+1)}{2400}\right]$

$\therefore \quad 12440 = 1200\, n\left[1 + \dfrac{8(n+1)}{2400}\right]$

$\Rightarrow \quad 12440 = 1200n\left(\dfrac{2400 + 8n + 8}{2400}\right)$

$\Rightarrow \quad 24880 = 8n^2 + 2408n$

$\Rightarrow \quad n^2 + 301n - 3110 = 0$

$\Rightarrow \quad n = -311$ or 10

$\Rightarrow \quad n = 10$ months

4. *(c)* We know that, MV $= Pn\left[1 + \dfrac{r(n+1)}{2400}\right]$

$\therefore \quad 9390 = P \times 12\left(1 + \dfrac{8 \times 13}{2400}\right)$

$\Rightarrow \quad 9390 = P \times 12\left(\dfrac{2400 + 104}{2400}\right)$

$\Rightarrow \quad 9390 = P \times \dfrac{2504}{200}$

$\Rightarrow \quad \dfrac{9390 \times 200}{2504} = P$

$\Rightarrow \quad P = ₹\, 750$

5. *(b)* We know that, $I = \dfrac{Prn(n+1)}{2400}$

$\therefore$ Interest earned by Ravi

$= \dfrac{400 \times 9 \times 24 \times 25}{2400} = ₹\, 900$

and interest earned by Sonu

$= \dfrac{400 \times 12 \times 30 \times 31}{2400} = ₹\, 1860$

Hence, option (b) is correct.

Chapter 03 Shares and Dividends

1. *(a)* (i) Percentage return $= \left(\dfrac{\text{Dividend}}{\text{Investment}} \times 100\right)\%$

$= \left(\dfrac{10}{120} \times 100\right)\% = 8\dfrac{1}{3}\%$

(ii) Percentage return $= \left(\dfrac{\text{Dividend}}{\text{Investment}} \times 100\right)\%$

$= \left(\dfrac{9}{110} \times 100\right)\% = 8\dfrac{18}{99}\% = 8\dfrac{2}{11}\%$

Since, $8\dfrac{1}{3}\% > 8\dfrac{2}{11}\%$, therefore part (i) is better.

2. *(d)* Number of shares $= \dfrac{\text{Total investment}}{\text{Market value of one share}}$

$= \dfrac{125X}{X + \dfrac{25}{100}X} = \dfrac{125X}{\left(\dfrac{5X}{4}\right)}$

$= 125X \times \dfrac{4}{5X} = 100$

3. *(b)* $\because$ Dividend percentage $\times$ Nominal value

= Percentage return $\times$ Market value

$\therefore \quad \dfrac{8}{100} \times 150 = \dfrac{10}{100} \times$ Market value

$\Rightarrow \quad$ Market value $= \dfrac{8 \times 150}{10} = ₹\, 120$

4. *(a)* From the given data, 400 shares can be purchased with an investment of ₹ 25000.

$\therefore$ Market value of each share $= ₹\left(\dfrac{25000}{400}\right) = ₹\, 62.50$

5. *(d)* $\because$ Dividend $\times$ Face value = Market value $\times$ Rate of return

$\therefore \quad \dfrac{x}{100} \times 110 = 100 \times \dfrac{y}{100}$

$\Rightarrow \quad \dfrac{x}{y} = \dfrac{100}{110} = \dfrac{10}{11}$

$\Rightarrow \quad x : y = 10 : 11$

6. *(a)* Number of shares bought by Ms. Neelam

$= \dfrac{27000}{90} = 300$

$\Rightarrow$ Dividend earned on 300 shares

$$= 300 \times \frac{10}{100} \times 100 = ₹\,3000$$

Money received by selling these 300 shares

$$= 300 \times 110 = ₹\,33000$$

$\Rightarrow$ Earning of Ms. Neelam

$$= 33000 + 3000 - 27000 = ₹\,9000$$

Number of shares bought by Mr. Ram

$$= \frac{\text{Total investment}}{\text{Market value}}$$

$$= \frac{30000}{120} = 250$$

$\Rightarrow$ Dividend earned on 250 shares

$$= 250 \times \frac{8}{100} \times 150 = ₹\,3000$$

Money received by selling these 250 shares

$$= 250 \times 110 = ₹\,27500$$

$\therefore$ Earning of Mr. Ram $= 27500 + 3000 - 30000 = ₹\,500$

7. *(c)* **Jindal Electronics**

Investment = ₹54000, Dividend = 18%, Face value = ₹200, Percentage return = 16%

$\because$ Market price $\times$ Percentage return = Face value $\times$ Dividend

$\Rightarrow$ Market price $\times 16 = 200 \times 18$

$\therefore$ Market price $= \frac{200 \times 18}{16} = ₹\,225$

Number of shares $= \frac{\text{Investment}}{\text{Market price}} = \frac{54000}{225} = 240$

Annual income = Number of shares

$$\times \text{Nominal value} \times \frac{\text{Rate of dividend}}{100}$$

$$= 240 \times 200 \times \frac{18}{100} = ₹\,8640$$

Number of shares = 240, Market price = ₹275

Sale proceeds = 240×275 = ₹66000

Golmal Entertainment

Investment = Amount received = ₹66000

Dividend = 24%

Face value = ₹100

Annual income = Number of shares

$$\times \text{Nominal value} \times \frac{\text{Rate of dividend}}{100}$$

$\Rightarrow$ $10560 = \text{Number of shares} \times 100 \times \frac{24}{100}$

[$\because$ Annual income = 8640 + 1920 = ₹ 10560]

Number of shares $= 10560 \times \frac{1}{100} \times \frac{100}{24} = 440$

Market price of one share

$$= \frac{\text{Investment}}{\text{Number of shares}}$$

$$= \frac{66000}{440} = ₹\,150$$

8. *(d)* Let the man invests ₹ x in each company.

For company X

Nominal value of each share = ₹ 100

Market value of each share

$$= ₹\,100 + 20\% \text{ of } ₹\,100 = ₹\,120$$

$\therefore$ Number of shares bought $= \frac{x}{120}$

and dividend on each share = 5% of ₹ 100 = ₹ 5 [$\because$ rate = 5%]

$\Rightarrow$ Total dividend $= 5 \times \frac{x}{120} = ₹\,\frac{x}{24}$

For company Y

Nominal value of each share = ₹ 100

Market value of each share = ₹ 100 − 10% of ₹ 100

$$= ₹\,90$$

$\therefore$ Number of shares bought $= \frac{x}{90}$

and dividend on each share = 7% of ₹ 100 = ₹ 7 [$\because$ rate = 7%]

$\Rightarrow$ Total dividend $= 7 \times \frac{x}{90} = ₹\,\frac{7x}{90}$

Given, sum of dividends (returns) from both the companies is ₹ 1290.

$$\therefore \quad \frac{x}{24} + \frac{7x}{90} = 1290$$

On solving, we get $x = 10800$

Since, the man invests ₹ 10800 in each of the two companies.

$\therefore$ Man invests in all companies

$$= 2 \times 10800 = ₹\,21600$$

Chapter 4 Linear Inequations

1. *(c)* We have, $(x+1)^2 - (x-1)^2 < 6$

$$\Rightarrow \quad (x^2 + 1 + 2x) - (x^2 + 1 - 2x) < 6$$

$[\because (a \pm b)^2 = a^2 + b^2 \pm 2ab]$

$$\Rightarrow \quad 4x < 6$$

$$\Rightarrow \quad x < \frac{6}{4}$$

$$\Rightarrow \quad x < \frac{3}{2}$$

$$\Rightarrow \quad x \in \left(-\infty, \frac{3}{2}\right)$$

2. *(d)* Since, $|x|$ is always non-negative.

$$\therefore \quad a - |x| \le a$$

So, the maximum value of $23 - |2x + 3|$ is 23.

3. *(d)* We have, $\frac{1}{5 + 3x} \le 0$.

This is possible, when $5 + 3x < 0$

$$\Rightarrow \quad 3x < -5 \quad \Rightarrow \quad x < -\frac{5}{3}$$

$$\Rightarrow \quad x \in \left(-\infty, -\frac{5}{3}\right)$$

4. *(c)* Given inequation is $\left|\frac{2}{x-4}\right| > 1$, $x \ne 4$

$\Rightarrow$ $1 < \frac{2}{x-4}$ or $\frac{2}{x-4} < -1$ [splitting the inequation]

Consider $\frac{2}{x-4} > 1$. If $x - 4 > 0$.

Then, $2 > x - 4$ [multiplying by $(x-4)$ on both sides]

$$\Rightarrow \quad x < 6$$

So, $\quad 4 < x < 6$

If $x - 4 < 0$. Then, $2 < x - 4$

[multiplying by $(x-4)$ both sides]

$\Rightarrow \quad x > 6$

But, $x < 4$ and $x > 6$ is not possible simultaneously.

Consider $\dfrac{2}{x-4} < -1$

If $x - 4 < 0$. Then, $2 > -(x-4)$

[multiplying by $(x-4)$ both sides]

$\Rightarrow \quad 2 > -x + 4 \Rightarrow -x < -2 \Rightarrow x > 2$

So, $2 < x < 4$.

If $x - 4 > 0$. Then, $2 < -(x-4)$

[multiplying by $(x-4)$ on both sides]

$\Rightarrow \quad 2 < -x + 4 \Rightarrow -x > -2 \Rightarrow x < 2$

But, $x < 2$ and $x > 4$ is not possible simultaneously. The required solution set is $\{x : x \in (2, 4) \cup (4, 6), x \in R\}$.

5. *(a)* Given inequation is $|x + 1| + |x| > 3$

Put $x + 1 = 0 \Rightarrow x = -1$ and $x = 0$.

So, we will consider three intervals $(-\infty, -1)$, $(-1, 0)$ and $(0, \infty)$.

***Case* I** When $-\infty < x < -1$, then $|x + 1| = -(x + 1)$ and $|x| = -x$.

$\therefore \quad |x+1| + |x| > 3 \Rightarrow -(x+1) - x > 3$

$\Rightarrow \quad -x - 1 - x > 3 \Rightarrow -2x - 1 > 3$

$\Rightarrow \quad -2x - 1 + 1 > 3 + 1$ [adding 1 both sides]

$\Rightarrow \quad -2x > 4 \Rightarrow \dfrac{-2x}{-2} < \dfrac{4}{-2}$ [dividing both sides by -2]

$\Rightarrow \quad x < -2$

***Case* II** When $-1 \le x < 0$, then $|x + 1| = x + 1$ and $|x| = -x$.

$\therefore \quad |x+1| + |x| > 3$

$\Rightarrow \quad x + 1 - x > 3$

$\Rightarrow \quad 1 > 3$, which is not possible.

***Case* III** When $0 \le x < \infty$, then $|x + 1| = x + 1$ and $|x| = x$.

$\therefore \quad |x+1| + |x| > 3 \Rightarrow x + 1 + x > 3$

$\Rightarrow \quad 2x + 1 > 3 \Rightarrow 2x + 1 - 1 > 3 - 1$

[subtracting 1 from both sides]

$\Rightarrow \quad 2x > 2 \Rightarrow \dfrac{2x}{2} > \dfrac{2}{2}$ [dividing both sides by 2]

$\Rightarrow \quad x > 1$

On combining results of the above cases, we get

$x < -2 \quad$ or $\quad x > 1$

$\therefore \quad x \in (-\infty, -2) \cup (1, \infty)$

6. *(d)* Given, $-2\dfrac{2}{3} \le x + \dfrac{1}{3} \le 3\dfrac{1}{3}$

$\Rightarrow \quad \dfrac{-8}{3} \le x + \dfrac{1}{3} \le \dfrac{10}{3}$

$\dfrac{-8}{3} \times 3 \le \left(x + \dfrac{1}{3}\right)3 \le \dfrac{10}{3} \times 3$ [multiplying by 3 in each term]

$\Rightarrow \quad -8 \le 3x + 1 \le 10$

$\Rightarrow -8 - 1 \le 3x + 1 - 1 \le 10 - 1$

[subtracting 1 from each term]

$\Rightarrow \quad -9 \le 3x \le 9$

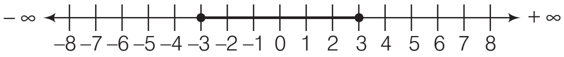
$\Rightarrow \quad \dfrac{-9}{3} \le \dfrac{3x}{3} \le \dfrac{9}{3}$ [dividing by 3 each term]

$\Rightarrow \quad -3 \le x \le 3$

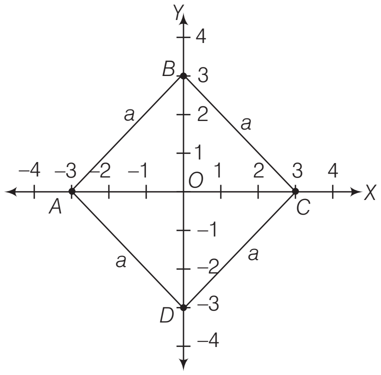
Since, $x \in R$.

$\therefore$ Range of x is $[-3, 3]$.

Representation of range of x on the number line is given as



Now, consider the following figure



Here, $AC = 6$ units, which is a diagonal of square.

Let side of a square $ABCD$ be a.

In right angled ΔABC,

$$AC^2 = AB^2 + BC^2$$

$\Rightarrow \quad 6^2 = a^2 + a^2$

$\Rightarrow \quad 36 = 2a^2$

$\Rightarrow \quad a^2 = 18$

Now, area of a square $ABCD = (\text{Side})^2 = a^2 = 18$ sq units

7. *(b)* Given system of linear inequations is $-2 < 2x - 6$ or $-2x + 5 \ge 13$.

Consider $\quad -2 < 2x - 6$

$\Rightarrow \quad -2 + 6 < 2x - 6 + 6$ [adding 6 both sides]

$\Rightarrow \quad 4 < 2x$

$\Rightarrow \quad 2 < x$ [dividing both sides by 2] ...(i)

or $\quad -2x + 5 \ge 13$

$\Rightarrow -2x + 5 + 2x \ge 13 + 2x$ [adding $2x$ on both sides]

$\Rightarrow \quad 5 \ge 13 + 2x \Rightarrow 5 - 13 \ge 13 + 2x - 13$

[subtracting 13 from both sides]

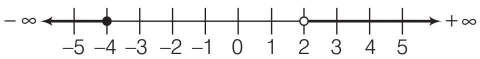
$\Rightarrow \quad -8 \ge 2x$

$\Rightarrow \quad \dfrac{-8}{2} \ge \dfrac{2x}{2}$ [dividing both sides by 2]

$\Rightarrow \quad -4 \ge x$ ...(ii)

From Eqs. (i) and (ii), we get the solution set is $\{x : x > 2$ or $x \le -4, x \in R\}$.

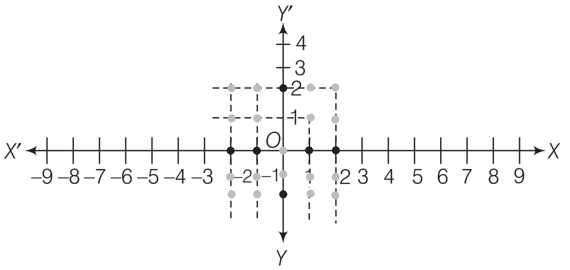
The representation of solution set on the number line is



When we shift the origin at the position 1, then the solution set will be $\{x : x + 1 > 2$ or $x + 1 \le -4, x \in R\}$

or $\quad \{x : x > 1$ or $x \le -5, x \in R\}$.

8. *(a)* The number line P on Y-axis and number line Q on X-axis is shown below



In the graph, the dotted points show all the pairs of coordinate axes.

Thus, the coordinates are (0, 0), (1, 0), (2, 0), (1, 1), (2, 1), (1, 2), (2, 2), (0, 1), (0, 2), (–1, 0), (–2, 0), (–1, 1), (–2, 1), (–1, 2), (–2, 2), (–1, –1), (–2, –1), (–1, –2), (–2, –2) (0, –1), (0, –2), (1, –1), (2, –1), (1, –2), (2, –2).

Hence, the number of all pairs of coordinates of the solution set is 25.

Chapter 5 Quadratic Equations in One Variable

1. *(c)* We have, $x^2 + kx + 1 = 0$

On comparing it with $ax^2 + bx + c = 0$, we get

$$a = 1, b = k \text{ and } c = 1$$

For the linear factors, $D \geq 0$

$\Rightarrow \quad b^2 - 4ac \geq 0$

$\Rightarrow \quad k^2 - 4 \times 1 \times 1 \geq 0$

$\Rightarrow \quad (k^2 - 2^2) \geq 0$

$\Rightarrow \quad (k - 2)(k + 2) \geq 0$

$\therefore \quad k \geq 2 \text{ or } k \leq -2$

2. *(b)* Given equation is

$$2\sin\theta + \frac{3}{\sin\theta} + 7 = 0$$

$\Rightarrow \quad 2\sin^2\theta + 3 + 7\sin\theta = 0$

$\Rightarrow \quad 2\sin^2\theta + 7\sin\theta + 3 = 0$

$\Rightarrow \quad 2\sin^2\theta + 6\sin\theta + \sin\theta + 3 = 0$ [splitting the middle term]

$\Rightarrow 2\sin\theta(\sin\theta + 3) + 1(\sin\theta + 3) = 0$

$\Rightarrow \quad (2\sin\theta + 1)(\sin\theta + 3) = 0$

$\Rightarrow \quad 2\sin\theta + 1 = 0 \text{ and } \sin\theta + 3 = 0$

But we know that, $\sin\theta$ lies between –1 and 1.

$\therefore \quad \sin\theta = -\frac{1}{2}$

Hence, the root of given equation is $-\frac{1}{2}$.

3. *(a)* Given equation is

$$x^2(a^2 + b^2) + 2x(ac + bd) + (c^2 + d^2) = 0$$

On comparing it with $Ax^2 + Bx + C = 0$, we get

$$A = (a^2 + b^2), B = 2(ac + bd)$$

and $\quad C = c^2 + d^2$

Now, discriminant, $D = B^2 - 4AC$

$= 4(ac + bd)^2 - 4(a^2 + b^2)(c^2 + d^2)$

$= 4[(ac + bd)^2 - (a^2 + b^2)(c^2 + d^2)]$

$= 4(a^2c^2 + b^2d^2 + 2acbd - a^2c^2 - a^2d^2 - b^2c^2 - b^2d^2)$

$= 4(2acbd - a^2d^2 - b^2c^2)$

$= -4(a^2d^2 + b^2c^2 - 2adbc) = -4(ad - bc)^2$

Clearly, when $ad \neq bc$, then $(ad - bc)^2 > 0$

$\Rightarrow \quad -4(ad - bc)^2 < 0 \Rightarrow \quad D < 0$

Hence, the given equation has no real roots.

4. *(d)* Let P be the initial production (2 yr ago) and the increase in production every year be x%. Then, production at the end of first year $= P + \frac{Px}{100} = P\left(1 + \frac{x}{100}\right)$

Production at the end of second year

$$= P\left(1 + \frac{x}{100}\right) + \frac{Px}{100}\left[\left(1 + \frac{x}{100}\right)\right]$$

$$= P\left(1 + \frac{x}{100}\right)\left(1 + \frac{x}{100}\right) = P\left(1 + \frac{x}{100}\right)^2$$

Since, the production is doubled in last two years.

$\therefore \quad P\left(1 + \frac{x}{100}\right)^2 = 2P \Rightarrow \left(1 + \frac{x}{100}\right)^2 = 2$

$\Rightarrow \quad (100 + x)^2 = 2 \times (100)^2 \Rightarrow 10000 + x^2 + 200x = 20000$

$\Rightarrow \quad x^2 + 200x - 10000 = 0$

On comparing it with $ax^2 + bx + c = 0$, we get

$$a = 1, b = 200 \text{ and } c = -10000$$

By quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore \quad x = \frac{-200 \pm \sqrt{(200)^2 + 40000}}{2}$

$= -100 \pm 100\sqrt{2} = 100(-1 \pm \sqrt{2})$

$= 100(-1 + 1.414)$

[$\because$ percentage cannot be negative]

$= 100(0.414) = 41.4$

Hence, the required percentage is 41.4%.

5. *(a)* As the CP of x articles is ₹ 1200.

$\therefore$ CP of one article = ₹ $\frac{1200}{x}$

As the selling price of each article is ₹ 2 more than its CP.

$\therefore$ SP of each article = ₹$\left(\frac{1200}{x} + 2\right)$

Since, 10 articles were damaged, therefore

Number of articles left for selling $= x - 10$

$\therefore$ SP of all articles (worth selling) = ₹$(x - 10)\left(\frac{1200}{x} + 2\right)$

As the trader earns a net profit of ₹ 60.

$\therefore \quad (x - 10)\left(\frac{1200}{x} + 2\right) = 1200 + 60$

$\Rightarrow \quad (x - 10)\left(\frac{1200 + 2x}{x}\right) = 1260$

$\Rightarrow \quad (x - 10)(1200 + 2x) = 1260x$

$\Rightarrow 1200x + 2x^2 - 12000 - 20x - 1260x = 0$

$\Rightarrow \quad 2x^2 - 80x - 12000 = 0$

$\Rightarrow \quad x^2 - 40x - 6000 = 0$ [dividing both sides by 2]

$\Rightarrow \quad x^2 - 100x + 60x - 6000 = 0$

[splitting the middle term]

$\Rightarrow \quad x(x-100)+60(x-100)=0$

$\Rightarrow \quad (x-100)(x+60)=0$

$\Rightarrow \quad x=100$ or $x=-60$

$\therefore \quad x=100$

[$\because$ number of articles cannot be negative]

6. *(b)* Given equations are $x^2+2cx+ab=0$ ...(i)

and $x^2-2(a+b)x+a^2+b^2+2c^2=0$ ...(ii)

On comparing above equations with $A_1x^2+B_1x+C_1=0$ and $A_2x^2+B_2x+C_2=0$ respectively, we get

$A_1=1, B_1=2c, C_1=ab$

and $A_2=1, B_2=-2(a+b), C_2=a^2+b^2+2c^2$

Let D_1 and D_2 be the discriminants of Eqs. (i) and (ii), respectively.

Then, $D_1=(2c)^2-4\times1\times ab$ [$\because D=B^2-4AC$]

$=4c^2-4ab=4(c^2-ab)$

and $D_2=[-2(a+b)]^2-4\times1\times(a^2+b^2+2c^2)$

$=4(a+b)^2-4(a^2+b^2+2c^2)$

$=4(a^2+b^2+2ab-a^2-b^2-2c^2)$

$=4(2ab-2c^2)=-8(c^2-ab)$

Since, the roots of Eq. (i) are real and unequal.

$\therefore \quad D_1>0 \Rightarrow 4(c^2-ab)>0$

$\Rightarrow \quad c^2-ab>0 \Rightarrow -8(c^2-ab)<0$ [$\because A>0 \Rightarrow -A<0$]

$\Rightarrow \quad D_2<0$

Hence, roots of Eq. (ii) are not real.

7. *(a)* We have, $p^2x^2+(p^2-q^2)x-q^2=0$

On comparing it with $ax^2+bx+c=0$, we get

$a=p^2, b=p^2-q^2$ and $c=-q^2$

Now,discriminant, $D=b^2-4ac$

$=(p^2-q^2)^2-4\times p^2\times(-q^2)$

$=(p^2-q^2)^2+4p^2q^2$

$=p^4+q^4-2p^2q^2+4p^2q^2$

$=p^4+q^4+2p^2q^2=(p^2+q^2)^2>0$

So, the given equation has real roots, which are given by

$$\alpha=\frac{-b+\sqrt{D}}{2a}=\frac{-(p^2-q^2)+\sqrt{(p^2+q^2)^2}}{2p^2}$$

$$=\frac{-p^2+q^2+p^2+q^2}{2p^2}=\frac{2q^2}{2p^2}=\frac{q^2}{p^2}$$

and $$\beta=\frac{-b-\sqrt{D}}{2a}=\frac{-(p^2-q^2)-\sqrt{(p^2+q^2)^2}}{2p^2}$$

$$=\frac{-p^2+q^2-(p^2+q^2)}{2p^2}=\frac{-p^2+q^2-p^2-q^2}{2p^2}=\frac{-2p^2}{2p^2}=-1$$

Hence, the required roots are $\frac{q^2}{p^2}$ and -1.

8. *(d)* Let the total number of Saras birds be x.

Then, number of Saras birds moving in lotus plants $=\frac{x}{4}$

Number of Saras birds moving on a hill $=\frac{x}{9}+\frac{x}{4}+7\sqrt{x}$

Number of Saras birds sitting on the Bakula trees $=56$

According to the question,

$$\frac{x}{4}+\frac{x}{9}+\frac{x}{4}+7\sqrt{x}+56=x$$

$\Rightarrow \quad 7\sqrt{x}=x-\frac{x}{4}-\frac{x}{9}-\frac{x}{4}-56$

$\Rightarrow \quad 7\sqrt{x}=\frac{36x-9x-4x-9x}{36}-56$

$\Rightarrow \quad 7\sqrt{x}=\frac{7x}{18}-56 \Rightarrow \sqrt{x}=\frac{x}{18}-8$

$\Rightarrow \quad x=\frac{x^2}{324}+64-\frac{8x}{9}$ [squaring on both sides]

$\Rightarrow \quad x=\frac{x^2+20736-288x}{324}$

$\Rightarrow \quad 324x=x^2+20736-288x$

$\Rightarrow \quad x^2-612x+20736=0$

$\Rightarrow \quad x^2-36x-576x+20736=0$ [splitting the middle term]

$\Rightarrow \quad x(x-36)-576(x-36)=0$

$\Rightarrow \quad (x-36)(x-576)=0$

$\Rightarrow \quad x-36=0$ or $x-576=0$

$\Rightarrow \quad x=576$ or $x=36$

Here, $x=36$ is not possible, because if there are only 36 birds, then 56 cannot be on the trees.

Thus, total number of Saras birds is 576.

Chapter 6 Ratio and Proportion

1. *(d)* In order to compare the price of coffee with that of tea, we must find the cost of the same quantity of each of them. Let us find the cost of 1 kg of each of the two items.

We have, cost of 100 g of coffee = ₹ 24

Cost of 1 g of coffee = ₹ $\left(\frac{24}{100}\right)$

$\Rightarrow$ Cost of 1000 g of coffee = ₹ $\left(\frac{24}{100}\times1000\right)$ = ₹ 240

$\therefore$ Cost of 1 kg of coffee = ₹ 240 [$\because$ 1 kg = 1000 g]

It is given that, the cost of 1 kg of tea is ₹ 80.

$\therefore$ Ratio of price of coffee to that of tea

= Cost of 1 kg of coffee : Cost of 1 kg of tea

= ₹ 240 : ₹ 80 = 240 : 80 = 3 : 1

2. *(a)* Given, $(x^2+y^2):(x^2+y^2+xy)$

According to the question,

$$\frac{x^2+y^2-2xy}{x^2+y^2+xy-2xy}=\frac{x^2-y^2}{1}$$

$\Rightarrow \quad \frac{(x-y)^2}{x^2+y^2-xy}=\frac{(x-y)(x+y)}{1}$

[$\because (a-b)^2=a^2+b^2-2ab$]

$\Rightarrow \quad \frac{(x-y)}{x^2+y^2-xy}=(x+y)$

$\Rightarrow \quad x-y=(x+y)(x^2+y^2-xy)$

$\Rightarrow \quad x-y=x^3+y^3$

$\Rightarrow \quad x-x^3=y^3+y$

$\Rightarrow \quad x(1-x^2)=y(y^2+1)$

$\Rightarrow \quad \frac{x}{y}=\frac{1+y^2}{1-x^2}$

$\therefore \quad x:y=(1+y^2):(1-x^2)$

3. *(d)* We have, $15 \times 42 = 630$ and $18 \times 35 = 630$

$\therefore \quad 15 \times 42 = 18 \times 35 \quad \ldots(i)$

$\Rightarrow$ First term = 15 , fourth term = 42, second term = 18 and third term = 35

Hence, $\quad 15 : 18 :: 35 : 42$

We can also write Eq. (i) as $15 \times 42 = 35 \times 18$

$\Rightarrow$ First term = 15, fourth term = 42, second term = 35 and third term = 18

Hence, $\quad 15 : 35 :: 18 : 42$

Again, Eq. (i) can also be written as

$$42 \times 15 = 18 \times 35$$

$\Rightarrow$ First term = 42, fourth term = 15, second term = 18 and third term = 35

Hence, $\quad 42 : 18 :: 35 : 15$

Finally, Eq. (i) can be written as

$$42 \times 15 = 35 \times 18$$

$\Rightarrow$ First term = 42, fourth term = 15, second term = 35 and third term = 18

Hence, $\quad 42 : 35 :: 18 : 15$

Thus, the required proportions are

$15 : 18 :: 35 : 42$, $15 : 35 :: 18 : 42$,

$42 : 18 :: 35 : 15$ and $42 : 35 :: 18 : 15$.

Hence, required number of proportions are 4.

4. *(b)* Given, $\dfrac{x+y}{a^3-b^3} = \dfrac{y+z}{b^3-c^3} = \dfrac{z+x}{c^3-a^3} = k$ (let)

$\therefore \quad x + y = k(a^3 - b^3) \quad \ldots(i)$

$y + z = k(b^3 - c^3) \quad \ldots(ii)$

and $\quad z + x = k(c^3 - a^3) \quad \ldots(iii)$

On adding Eqs. (i), (ii) and (iii), we get

$x + y + y + z + z + x = k(a^3 - b^3) + k(b^3 - c^3) + k(c^3 - a^3)$

$\Rightarrow \quad 2x + 2y + 2z = k(a^3 - b^3 + b^3 - c^3 + c^3 - a^3)$

$\Rightarrow \quad 2(x + y + z) = k \times 0$

$\Rightarrow \quad 2(x + y + z) = 0$

$\therefore \quad x + y + z = 0 \quad [\because 2 \neq 0]$

5. *(a)* Given, $\dfrac{3k+4l+6m+7n}{3k+4l-6m-7n} = \dfrac{3k-4l+6m-7n}{3k-4l-6m+7n}$

$$\Rightarrow \frac{(3k+4l)+(6m+7n)}{(3k+4l)-(6m+7n)} = \frac{(3k-4l)+(6m-7n)}{(3k-4l)-(6m-7n)}$$

$$\Rightarrow \frac{2(3k+4l)}{2(6m+7n)} = \frac{2(3k-4l)}{2(6m-7n)}$$

[applying componendo and dividendo]

$$\Rightarrow \frac{3k+4l}{3k-4l} = \frac{6m+7n}{6m-7n}$$

$$\Rightarrow \frac{2(3k)}{2(4l)} = \frac{2(6m)}{2(7n)}$$

[applying componendo and dividendo]

$$\Rightarrow \frac{k}{4l} = \frac{2m}{7n}$$

$$\Rightarrow \frac{k}{2m} = \frac{4l}{7n}$$

6. *(b)* Given, a, b, c and d are in proportion.

$$\therefore \quad \frac{a}{b} = \frac{c}{d}$$

On applying componendo and dividendo, we get

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

On cubing both sides, we get

$$\left(\frac{a+b}{a-b}\right)^3 = \left(\frac{c+d}{c-d}\right)^3$$

$$\Rightarrow \frac{a^3+3a^2b+3ab^2+b^3}{a^3-3a^2b+3ab^2-b^3} = \frac{c^3+3c^2d+3cd^2+d^3}{c^3-3c^2d+3cd^2-d^3}$$

$$\Rightarrow \frac{(a^3+3ab^2)+(3a^2b+b^3)}{(a^3+3ab^2)-(3a^2b+b^3)} = \frac{(c^3+3cd^2)+(3c^2d+d^3)}{(c^3+3cd^2)-(3c^2d+d^3)}$$

$$\Rightarrow \frac{a^3+3ab^2}{3a^2b+b^3} = \frac{c^3+3cd^2}{3c^2d+d^3}$$

[applying componendo and dividendo property]

7. *(c)* Given, $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \dfrac{121}{125}$

$$\Rightarrow \frac{3a}{3b} = \frac{4c}{4d} = \frac{9e}{9f} = \frac{121}{125}$$

We know that,

$$\text{Each ratio} = \frac{\text{Sum of antecedents}}{\text{Sum of consequents}} = \frac{3a+4c+9e}{3b+4d+9f}$$

$$\Rightarrow \frac{121}{125} = \frac{3a+4c+9e}{3b+4d+9f} \quad \left[\because \text{each ratio} = \frac{121}{125}\right]$$

8. *(b)* Given equation is $\dfrac{x}{1} = \dfrac{\sqrt[3]{m+1}+\sqrt[3]{m-1}}{\sqrt[3]{m+1}-\sqrt[3]{m-1}}$

On applying componendo and dividendo, we get

$$\frac{x+1}{x-1} = \frac{(\sqrt[3]{m+1}+\sqrt[3]{m-1})+(\sqrt[3]{m+1})-\sqrt[3]{m-1}}{(\sqrt[3]{m+1}+\sqrt[3]{m-1})-(\sqrt[3]{m+1})-\sqrt[3]{m-1}}$$

$$\Rightarrow \frac{x+1}{x-1} = \frac{2(\sqrt[3]{m+1})}{2(\sqrt[3]{m-1})} \Rightarrow \frac{x+1}{x-1} = \frac{\sqrt[3]{m+1}}{\sqrt[3]{m-1}}$$

On cubing both sides, we get

$$\frac{(x+1)^3}{(x-1)^3} = \frac{m+1}{m-1}$$

Again, applying componendo and dividendo, we get

$$\frac{(x+1)^3+(x-1)^3}{(x+1)^3-(x-1)^3} = \frac{m+1+m-1}{m+1-(m-1)}$$

$$\Rightarrow \frac{x^3+1^3+3x^2\times1+3x\times1^2+(x^3-1^3-3x^2\times1+3x\times1^2)}{x^3+1^3+3x^2\times1+3x\times1^2-(x^3-1^3-3x^2\times1+3x\times1^2)} = \frac{2m}{2}$$

$$\Rightarrow \frac{2x^3+6x}{6x^2+2} = \frac{m}{1}$$

$$\Rightarrow \frac{2(x^3+3x)}{2(3x^2+1)} = \frac{m}{1}$$

$$\Rightarrow x^3+3x = 3mx^2+m$$

$$\Rightarrow x^3-3mx^2+3x = m$$

Chapter 8 Matrices

1. *(a)* Given quadratic equation is $x^2 + x - 6 = 0$

$\Rightarrow \quad x^2 + 3x - 2x - 6 = 0$ [by splitting the middle term]

$\Rightarrow \quad x(x+3) - 2(x+3) = 0$

$\Rightarrow \quad (x+3)(x-2) = 0$

$\Rightarrow \quad x = -3, 2$

Also, given $\beta > \alpha$.

$\therefore$ We take $\beta = 2$ and $\alpha = -3$

Now, $\begin{bmatrix} 0 & \alpha \\ \alpha & \beta \end{bmatrix}\begin{bmatrix} \beta+1 & 0 \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} 0-\alpha\beta & 0+\alpha^2 \\ \alpha\beta+\alpha-\beta^2 & 0+\alpha\beta \end{bmatrix}$

$$= \begin{bmatrix} -(-3)(2) & (-3)^2 \\ (-3)(2)-3-(2)^2 & (-3)(2) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 9 \\ -6-3-4 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 9 \\ -13 & -6 \end{bmatrix}$$

2. *(a)* From option (a), we have

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Then, $A \neq O, B \neq O$

Now, $\quad AB = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 2\times1+2(-1) & 2\times1+2(-1) \\ 2\times1+2(-1) & 2\times1+2(-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

3. *(a)* Given, $\quad A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

Now, $\quad A^2 = A\cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$\therefore A^4 = A^2\cdot A^2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 64-25 & 40+15 \\ -40-15 & -25+9 \end{bmatrix}$

$$= \begin{bmatrix} 39 & 55 \\ -55 & -16 \end{bmatrix}$$

Now, $\quad A^5 = A^4\cdot A = \begin{bmatrix} 39 & 55 \\ -55 & -16 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 117-55 & 39+110 \\ -165+16 & -55-32 \end{bmatrix} = \begin{bmatrix} 62 & 149 \\ -149 & -87 \end{bmatrix}$$

4. *(b)* Since, $A+B$ is defined, therefore both A and B are of the same type.

Suppose that both A and B are of order $m \times n$.

Also, AB is defined.

Thus, the number of columns in the pre-factor A must be equal to the number of rows in the post-factor B, i.e. $n = m$

Hence, both A and B are of order $n \times n$, i.e. A and B are square matrices of the same type.

5. *(c)* We have, $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 3I$

$\therefore \quad A^n = (3I)^n = 3^n I^n = 3^n I$

[$\because I^n = I$, for all natural numbers n]

$$= 3^n\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

6. *(b)* We have, $\begin{bmatrix} a^x \\ a^{-x} \end{bmatrix}[1 \;\; 2] = \begin{bmatrix} p & a^{-2} \\ q & \log_2 2 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} a^x & 2a^x \\ a^{-x} & 2a^{-x} \end{bmatrix} = \begin{bmatrix} p & a^{-2} \\ q & 1 \end{bmatrix} \quad \left[\because \log_2 2 = \frac{\log 2}{\log 2} = 1\right]$

On comparing the corresponding elements both sides, we get

$\Rightarrow \quad a^x = p \quad$ ...(i)

$\Rightarrow \quad 2a^x = a^{-2} \quad$ ...(ii)

$\Rightarrow \quad a^{-x} = q \quad$ ...(iii)

and $\quad 2a^{-x} = 1 \quad$ ...(iv)

On multiplying Eqs. (ii) and (iv), we get

$$4a^{x-x} = a^{-2}$$

$\Rightarrow \quad 4a^0 = a^{-2} \Rightarrow 4 = a^{-2} \Rightarrow 4 = \frac{1}{a^2}$

$\Rightarrow \quad a^2 = \frac{1}{4} \Rightarrow a = \frac{1}{2} \quad [\because a > 0]$

Now, $\quad a^{p-q} = a^{a^x - a^{-x}}$ [from Eqs. (i) and (iii)]

$= a^{\frac{1}{2}a^{-2}-\frac{1}{2}} = a^{2-\frac{1}{2}}$ [from Eqs. (ii) and (iv)]

$= a^{\frac{1}{2}\cdot 4-\frac{1}{2}}$ [$\because a^{-2} = 4$]

$= \left(\frac{1}{2}\right)^{3/2} = 2^{-3/2}$

7. *(a)* We have, $\quad A = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 25 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5^2 & 5^2 \\ 0 & 0 \end{bmatrix}$

$\therefore \quad A^2 = A\cdot A = \begin{bmatrix} 5^2 & 5^2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 5^2 & 5^2 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 5^4 & 5^4 \\ 0 & 0 \end{bmatrix}$$

$\vdots \qquad \vdots$

$$A^n = \begin{bmatrix} 5^{2n} & 5^{2n} \\ 0 & 0 \end{bmatrix}$$

Thus, $\begin{bmatrix} 5^{2n} & 5^{2n} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5^{200} & 5^{200} \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \quad 5^{2n} = 5^{200}$

$\Rightarrow \quad 2n = 200$

$\Rightarrow \quad n = 100$

8. *(c)* Let $A = \begin{bmatrix} 7 & -6 \\ 8 & -7 \end{bmatrix}$.

Then, $A^2 = \begin{bmatrix} 7 & -6 \\ 8 & -7 \end{bmatrix}\begin{bmatrix} 7 & -6 \\ 8 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore \quad \begin{bmatrix} 7 & -6 \\ 8 & -7 \end{bmatrix}^{2017} = A^{2017} = A^{2016}\cdot A$

$= (A^2)^{1008}\cdot A = I^{2008}\cdot A = I\cdot A \quad [\because I^{1008} = I]$

$$= A = \begin{bmatrix} 7 & -6 \\ 8 & -7 \end{bmatrix}$$

Chapter 9 Arithmetic and Geometric Progression

1. *(c)* Let the five integers be $a-2d, a-d, a, a+d, a+2d$.

Then, we have

$(a-2d)+(a-d)+a+(a+d)+(a+2d)=10020$

$\Rightarrow \quad 5a = 10020$

$\Rightarrow \quad a = 2004$

Now, as smallest possible value of d is 1.

$\therefore$ Smallest possible value of $a+2d = 2004+2 = 2006$

2. *(b)* Let $S = 1+(1+2)+(1+2+3)+(1+2+3+4) + \ldots + (1+2+3+\ldots+20)$

$= 1+3+6+10+15+21+28+36+45+55+66+78 + 91+105+120+136+153+171+190+210$

$= 1540 \quad \left[\because 1+2+3+\ldots+n = \dfrac{n(n+1)}{2}\right]$

3. *(a)* Given, $S_{11} = 33$

$\Rightarrow \quad \dfrac{11}{2}(2a+10d) = 33 \quad \left[\because S_n = \dfrac{n}{2}[2a+(n-1)d]\right]$

$\Rightarrow \quad a+5d = 3$

i.e. $a_6 = 3 \Rightarrow a_4 = 2$

[$\because$ alternate terms are integers and the given sum is possible]

4. *(b)* Let $S = (45^2-43^2)+(44^2-42^2)+(43^2-41^2) + (42^2-40^2)+\ldots$upto 15 terms

$= (45+43)(45-43)+(44+42)(44-42) + (43+41)(43-41)+(42+40)(42-40) + \ldots$ upto 15 terms $\quad [\because a^2-b^2 = (a-b)(a+b)]$

$= (45+43)2+(44+42)2+(43+41)2 + (42+40)2+\ldots$upto 15 terms

$= 2[\{45+44+43+\ldots$upto 15 terms$\} + \{43+42+41+\ldots$upto 15 terms$\}]$

$= 2\times\left[\dfrac{15}{2}\{2\times 45+(15-1)(-1)\} + \dfrac{15}{2}\{2\times 43+(15-1)(-1)\}\right] \quad \left[\because S_n = \dfrac{n}{2}\{2a+(n-1)d\}\right]$

$= 2\left[\dfrac{15}{2}(76)+\dfrac{15}{2}(72)\right]$

$= 15(76+72)$

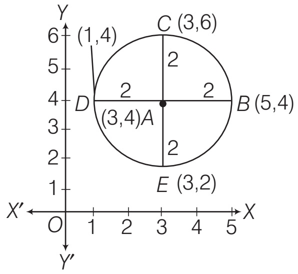
$= 15\times 148 = 2220$

Hence, the sum of the given series is 2220.

5. *(c)* Given point is $A(3, 4)$. The x-coordinate of A is 3. Since, the common difference is 2.

$\therefore$ x-coordinates of the point before A is $(3-2)$, i.e. 1 and after A is $(3+2)$, i.e. 5.

$\Rightarrow D(1, 4)$ and $B(5, 4)$ are the required points.



The y-coordinate of A is 4.

Since, the common difference is 2.

$\therefore$ y-coordinates of the point above A is $(4+2)$, i.e. 6 and below A is $(4-2)$, i.e. 2.

$\Rightarrow C(3, 6)$ and $E(3, 2)$ are the required points.

A circle shown in the figure having centre $A(3,4)$ and passing through the points B, C, D and E.

Clearly, radius of circle is, $AB = 5-3 = 2$ units

$\therefore$ Area of circle $= \pi r^2 = 3.14\times(2)^2 = 3.14\times 4$

$= 12.56$ sq units

6. *(b)* Given, a, b, c and d are in GP.

$\therefore \quad \dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c} = r$ (say)

$\Rightarrow \quad b = ar, c = br, d = cr$

$\Rightarrow \quad b = ar, c = (ar)r, d = (br)r$

$\Rightarrow \quad b = ar, c = ar^2, d = br^2$

$\Rightarrow \quad b = ar, c = ar^2, d = (ar)r^2 = ar^3 \quad \ldots$(i)

Now, consider

$(a^2+b^2+c^2)(b^2+c^2+d^2)$

$= (a^2+a^2r^2+a^2r^4)(a^2r^2+a^2r^4+a^2r^6)$

$= a^2(1+r^2+r^4)a^2r^2(1+r^2+r^4)$

$= a^4r^2(1+r^2+r^4)^2 = [a^2r(1+r^2+r^4)]^2$

$= (a^2r+a^2r^3+a^2r^5)^2$

$= (a\cdot ar+ar\cdot ar^2+ar^2\cdot ar^3)^2$

$= (ab+bc+cd)^2 \quad$ [from Eq. (i)]

7. *(d)* Let A be the first term and R be the common ratio of the given GP.

Then, $\quad a = p$th term $\Rightarrow a = AR^{p-1}$

$\Rightarrow \quad \log a = \log A + (p-1)\log R \quad \ldots$(i)

$b = q$th term $\Rightarrow b = AR^{q-1}$

$\Rightarrow \quad \log b = \log A + (q-1)\log R \quad \ldots$(ii)

$c = r$th term $\Rightarrow c = AR^{r-1}$

$\Rightarrow \quad \log c = \log A + (r-1)\log R \quad \ldots$(iii)

Now, consider $(q-r)\log a + (r-p)\log b + (p-q)\log c$

$= (q-r)\{\log A + (p-1)\log R\} + (r-p)\{\log A + (q-1)\log R\} + (p-q)\{\log A + (r-1)\log R\}$

[from Eqs. (i), (ii) and (iii)]

$= \log A\{(q-r)+(r-p)+(p-q)\} + \log R\{(p-1)(q-r) + (q-1)(r-p)+(r-1)(p-q)\}$

$= (\log A)\,0 + \{p(q-r)+q(r-p)+r(p-q)-(q-r) - (r-p)-(p-q)\}\log R$

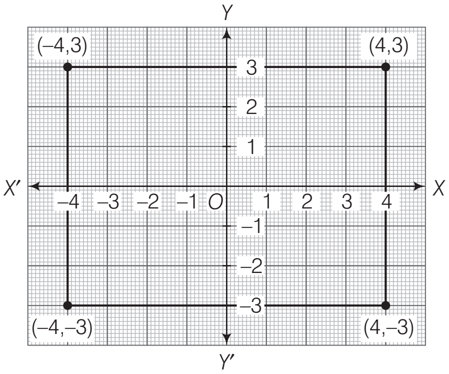
$= (\log A)\,0 + (\log R)\,0 = 0$

Chapter 10 Reflection

1. *(b)* Clearly, point P will be the reflection of $(-3, 2)$ in the X-axis.

Thus, $(-3, -2)$ is the required point.

2. *(a)* Since, the image of point $(4, 3)$ under X-axis is $(4, -3)$ and the image of point $(4, 3)$ under Y-axis is $(-4, 3)$.



$\therefore$ Other two vertices of the rectangle are $(4, -3)$ and $(-4, 3)$.

3. *(c)* Since, the image of any point (x, y) under X-axis is $(x, -y)$.

$\therefore$ Coordinate of $M \equiv (4, 5)$

Since, the image of any point (x, y) under Y-axis is $(-x, y)$.

$\therefore$ Coordinate of $M'' \equiv (-4, 5)$

Since, the image of any point (x, y) under origin is $(-x, -y)$.

$\therefore$ Coordinate of $M''' \equiv (4, -5)$

4. *(a)* $\because$ Image of $(-8, 5)$ under origin $\equiv (8, -5)$

and image of $(8, -5)$ under X-axis $\equiv (8, 5)$

$\therefore$ $h = 8$ and $k = 5$

5. *(b)* We know that a point is invariant when the line of reflection passing through it.

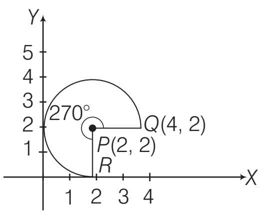
$\because$ $(3, -2)$ lie on $y = -2$.

$\therefore$ $(3, -2)$ is the required point.

Chapter 11 Section and Mid-Point Formulae

1. *(c)* When we rotate the line PQ in anti-clockwise direction at an angle of 270°, then the new coordinates of point Q will be at R, which touches the X-axis at $(2, 0)$.

Hence, the coordinates of R are $(2, 0)$.



Now, $PQ = \sqrt{(4-2)^2 + (2-2)^2} = \sqrt{2^2 + 0} = 2$ units

$\left[\because \text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right]$

Area of the figure $= \frac{3}{4}\pi r^2 = \frac{3}{4} \times 3.14 \times 4 = 9.42$ sq units

2. *(d)* Let the vertices of a triangle be $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$.

Again, let coordinates of P be (h, k).

Since, G is the centroid of a ΔABC.

$\therefore$ Coordinates of $G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

Let $P(h, k)$ be a point in the plane.

Now, $PA^2 = (x_1 - h)^2 + (y_1 - k)^2$

$\left[\because \text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right]$

$= x_1^2 + h^2 - 2x_1h + y_1^2 + k^2 - 2y_1k$

$PB^2 = (x_2 - h)^2 + (y_2 - k)^2$

$= x_2^2 + h^2 - 2x_2h + y_2^2 + k^2 - 2y_2k$

and $PC^2 = (x_3 - h)^2 + (y_3 - k)^2$

$= x_3^2 + h^2 - 2x_3h + y_3^2 + k^2 - 2y_3k$

Now, $GA^2 = \left(x_1 - \frac{x_1 + x_2 + x_3}{3}\right)^2 + \left(y_1 - \frac{y_1 + y_2 + y_3}{3}\right)^2$

$= \left(\frac{2x_1 - x_2 - x_3}{3}\right)^2 + \left(\frac{2y_1 - y_2 - y_3}{3}\right)^2$

$= \frac{1}{9}\begin{bmatrix} 4x_1^2 + x_2^2 + x_3^2 - 4x_1x_2 + 2x_2x_3 - 4x_1x_3 \\ + 4y_1^2 + y_2^2 + y_3^2 - 4y_1y_2 + 2y_2y_3 - 4y_1y_3 \end{bmatrix}$

$[\because (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca]$

Similarly,

$GB^2 = \frac{1}{9}\begin{bmatrix} x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3 \\ + y_1^2 + 4y_2^2 + y_3^2 - 4y_1y_2 + 2y_1y_3 - 4y_2y_3 \end{bmatrix}$

and $GC^2 = \frac{1}{9}\begin{bmatrix} x_1^2 + x_2^2 + 4x_3^2 + 2x_1x_2 - 4x_1x_3 \\ - 4x_2x_3 + y_1^2 + y_2^2 + 4y_3^2 \\ + 2y_1y_2 - 4y_2y_3 - 4y_1y_3 \end{bmatrix}$

Also, $GP^2 = \left(h - \frac{x_1 + x_2 + x_3}{3}\right)^2 + \left(k - \frac{y_1 + y_2 + y_3}{3}\right)^2$

$= h^2 + \left(\frac{x_1 + x_2 + x_3}{3}\right)^2 - 2h\left(\frac{x_1 + x_2 + x_3}{3}\right)$

$+ k^2 + \left(\frac{y_1 + y_2 + y_3}{3}\right)^2 - 2k\left(\frac{y_1 + y_2 + y_3}{3}\right)$

$\Rightarrow$ $3GP^2 = 3h^2 + \frac{1}{3}(x_1 + x_2 + x_3)^2 - 2h(x_1 + x_2 + x_3)$

$+ 3k^2 + \frac{1}{3}(y_1 + y_2 + y_3)^2 - 2k(y_1 + y_2 + y_3)$

Now, $PA^2 + PB^2 + PC^2 = 3(x_1^2 + x_2^2 + x_3^2)$

$+ 3(y_1^2 + y_2^2 + y_3^2) - 2h(x_1 + x_2 + x_3)$

$+ 3h^2 + 3k^2 - 2k(y_1 + y_2 + y_3)$

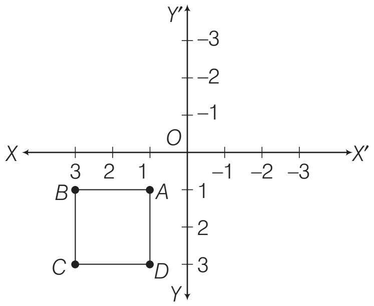
Similarly, $GA^2 + GB^2 + GC^2 + 3GP^2$

$= 3(x_1^2 + x_2^2 + x_3^2) + 3(y_1^2 + y_2^2 + y_3^2)$

$+ 3h^2 + 3k^2 - 2k(y_1 + y_2 + y_3)$

$\therefore$ $PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3GP^2$

3. *(b)* When we rotate the given graph at an angle of 180°, then the new graph is shown below



Thus, the new coordinates will be remain same, i.e. $A(1, 1)$, $B(3, 1)$, $C(3, 3)$ and $D(1, 3)$.

We know that in a square, the diagonals bisect each other.

$\therefore$ Mid-point of $BD = \left(\dfrac{3+1}{2}, \dfrac{1+3}{2}\right)$

$= \left(\dfrac{4}{2}, \dfrac{4}{2}\right) = (2, 2)$

4. *(c)* Let the third vertex of an equilateral triangle be (x, y).

Then, vertices of triangles are $A(-4, 3)$, $B(4, 3)$ and $C(x, y)$.

We know that in equilateral triangle, the angle between two adjacent sides is 60° and all three sides are equal.

$\therefore \quad AB = BC = CA$

$\Rightarrow \quad AB^2 = BC^2 = CA^2 \quad$ ...(i)

Now, taking first two terms, $AB^2 = BC^2$

$\Rightarrow (4+4)^2 + (3-3)^2 = (x-4)^2 + (y-3)^2$

$\left[\because \text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right]$

$\Rightarrow \quad 64 + 0 = x^2 + 16 - 8x + y^2 + 9 - 6y$

$[\because (a-b)^2 = a^2 + b^2 - 2ab]$

$\Rightarrow x^2 + y^2 - 8x - 6y = 39 \quad$ ...(ii)

Now, taking first and third terms, $AB^2 = CA^2$

$\Rightarrow \quad (4+4)^2 + (3-3)^2 = (-4-x)^2 + (3-y)^2$

$\Rightarrow \quad 64 + 0 = 16 + x^2 + 8x + 9 + y^2 - 6y$

$\Rightarrow \quad x^2 + y^2 + 8x - 6y = 39 \quad$ ...(iii)

On subtracting Eq. (ii) from Eq. (iii), we get

$$\begin{array}{r} x^2 + y^2 + 8x - 6y = 39 \\ x^2 + y^2 - 8x - 6y = 39 \\ - \quad - \quad + \quad + \quad - \\ \hline 16x = 0 \\ \hline \end{array}$$

$\Rightarrow \quad x = 0$

Now, putting the value of x in Eq. (ii), we get

$0 + y^2 - 0 - 6y = 39$

$\Rightarrow \quad y^2 - 6y - 39 = 0$

$\therefore \quad y = \dfrac{6 \pm \sqrt{(-6)^2 - 4(1)(-39)}}{2 \times 1}$

$\left[\text{by quadratic formula, } x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$

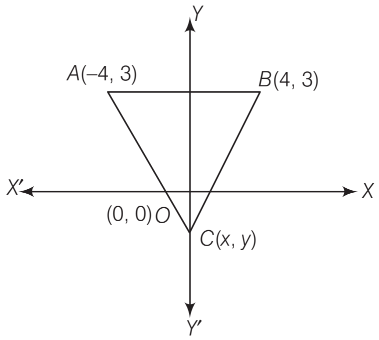
$= \dfrac{6 \pm \sqrt{36 + 156}}{2} = \dfrac{6 \pm \sqrt{192}}{2}$

$= \dfrac{6 \pm 2\sqrt{48}}{2} = 3 \pm \sqrt{48}$

$= 3 \pm 4\sqrt{3} = 3 + 4\sqrt{3} \quad \text{or} \quad 3 - 4\sqrt{3}$

So, the points of third vertex are $(0, 3 + 4\sqrt{3})$ or $(0, 3 - 4\sqrt{3})$.

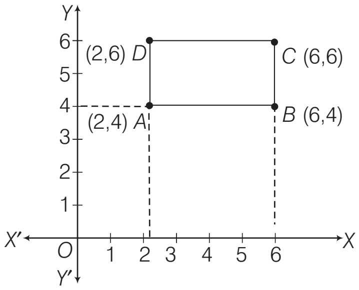
But given that, the origin lies in the interior of the ΔABC and the x-coordinate of third vertex is zero. Then, y-coordinate of third vertex should be negative.



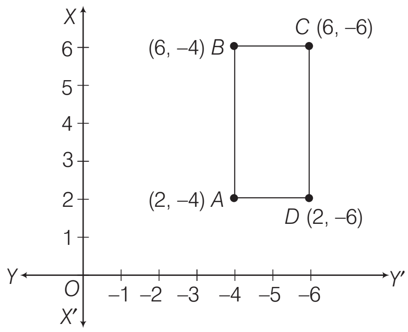
Hence, the required coordinates of third vertex are

$C(0, 3 - 4\sqrt{3}) \qquad [\because C \not\equiv (0, 3 + 4\sqrt{3})]$

5. *(b)* Given points $A(2, 4)$, $B(6, 4)$, $C(6, 6)$ and $D(2, 6)$ plotting on a graph paper, is shown below



When we rotate only the axes at an angle of 90° in anti-clockwise direction, the new axes are shown below



Here, we see that, in first quadrant, y-coordinates will be negative.

$\therefore$ The new coordinates of A, B, C and D are respectively

$A(2, -4)$, $B(6, -4)$, $C(6, -6)$ and $D(2, -6)$.

Now, $\quad AB = \sqrt{(6-2)^2 + (-4+4)^2}$

$$= \sqrt{4^2 + 0^2} = 4 \text{ units}$$

$$\left[\because \text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right]$$

$$BC = \sqrt{(6-6)^2 + (-6+4)^2}$$

$$= \sqrt{(0)^2 + (-2)^2} = 2 \text{ units}$$

$$CD = \sqrt{(2-6)^2 + (-6+6)^2}$$

$$= \sqrt{(-4)^2 + 0^2} = 4 \text{ units}$$

and $$DA = \sqrt{(2-2)^2 + (-6+4)^2}$$

$$= \sqrt{0^2 + (-2)^2} = 2 \text{ units}$$

$\therefore \quad AB = CD$ and $BC = DA$

Now, the diagonals are

$$AC = \sqrt{(6-2)^2 + (-6+4)^2}$$

$$= \sqrt{4^2 + (-2)^2} = \sqrt{16+4} = \sqrt{20}$$

$$= 2\sqrt{5} \text{ units}$$

and $$BD = \sqrt{(2-6)^2 + (-6+4)^2}$$

$$= \sqrt{(-4)^2 + (-2)^2} = \sqrt{16+4} = \sqrt{20}$$

$$= 2\sqrt{5} \text{ units}$$

$\therefore \quad AC = BD$

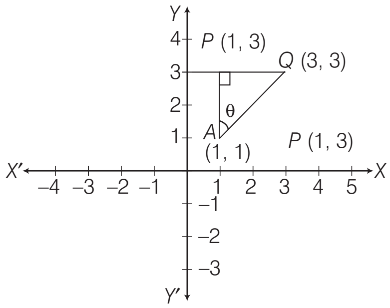
Hence, the adjacent points A, B, C and D from a rectangle.

6. *(a)* Given, coordinates of pole be $P(1, 3)$ and $Q(3, 3)$ and $A(1, 1)$ be the position of man.

(i) Now, $AP = \sqrt{(1-1)^2 + (3-1)^2}$

$$\left[\because \text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right]$$

$$= \sqrt{0^2 + 2^2} = 2 \text{ units}$$



and $PQ = \sqrt{(3-1)^2 + (3-3)^2} = \sqrt{2^2 + 0^2} = 2$ units

Now, in ΔAPQ, we have

$$\tan\theta = \frac{PQ}{AP} \Rightarrow \tan\theta = \frac{2}{2} = 1$$

$\Rightarrow \quad \theta = 45° \quad [\because \tan 45° = 1]$

(ii) When we shift the origin at (1, 1), then the angle will remain same, i.e. $\theta = 45°$.

7. *(d)* Let $P(x, y)$ be the moving point.

Let given two fixed points be $A(a, 0)$ and $B(-a, 0)$.

According to the given condition,

$$PA^2 + PB^2 = 2b^2$$

$\Rightarrow \quad (x-a)^2 + (y-0)^2 + (x+a)^2 + (y-0)^2 = 2b^2$ [by distance formula]

$\Rightarrow \quad x^2 - 2ax + a^2 + y^2 + x^2 + 2ax + a^2 + y^2 = 2b^2$

$\Rightarrow \quad 2x^2 + 2y^2 + 2a^2 = 2b^2$

$\Rightarrow \quad x^2 + y^2 + a^2 = b^2$ [dividing both sides by 2]

8. *(c)* Let $P(x, y)$ be the moving point such that the sum of its distances from $A(ae, 0)$ and $B(-ae, 0)$ is $2a$.

Then, $PA + PB = 2a$

$\Rightarrow \quad \sqrt{(x-ae)^2 + (y-0)^2} + \sqrt{(x+ae)^2 + (y-0)^2} = 2a$

$$\left[\because \text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right]$$

$\Rightarrow \quad \sqrt{(x-ae)^2 + y^2} = 2a - \sqrt{(x+ae)^2 + y^2}$

$\Rightarrow \quad (x-ae)^2 + y^2 = 4a^2 + (x+ae)^2 + y^2 - 4a\sqrt{(x+ae)^2 + y^2}$ [squaring on both sides]

$\Rightarrow \quad x^2 + a^2e^2 - 2xae = 4a^2 + x^2 + a^2e^2 + 2xae - 4a\sqrt{(x+ae)^2 + y^2}$

$\Rightarrow \quad -4aex - 4a^2 = -4a\sqrt{(x+ae)^2 + y^2}$

$\Rightarrow \quad (ex + a) = \sqrt{(x+ae)^2 + y^2}$

$\Rightarrow \quad (ex + a)^2 = (x+ae)^2 + y^2$ [again, squaring on both sides]

$\Rightarrow \quad e^2x^2 + 2aex + a^2 = x^2 + a^2e^2 + 2aex + y^2$

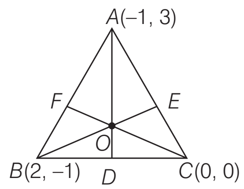
$\Rightarrow \quad x^2(1-e^2) + y^2 = a^2(1-e^2)$

$\Rightarrow \quad \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2(1-e^2)} = 1$

$\therefore \quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b^2 = a^2(1-e^2)$

Chapter 12 Equation of a Straight Line

1. *(b)* Let the vertices of ΔABC be $A(-1, 3)$, $B(2, -1)$ and $C(0, 0)$.



Let AD, BE and CF be the altitudes and $O(h, k)$ be the orthocentre of ΔABC.

$\because \quad AO \perp BC$

$\therefore$ Slope of line $AO \times$ Slope of line $BC = -1$

$\Rightarrow \quad \dfrac{k-3}{h+1} \times \left(\dfrac{0+1}{0-2}\right) = -1 \quad [\because m_1 \times m_2 = -1]$

$\Rightarrow \quad \dfrac{k-3}{h+1} = 2 \Rightarrow 2h - k + 5 = 0 \quad \ldots\text{(i)}$

Also, $BO \perp AC$

$\therefore$ Slope of line $BO \times$ Slope of line $AC = -1$

$$\frac{k+1}{h-2} \times \frac{3-0}{-1-0} = -1 \quad [\because m_1 \times m_2 = -1]$$

$$\Rightarrow \quad \frac{k+1}{h-2} \times (-3) = -1$$

$$\Rightarrow \quad h - 2 = 3k + 3$$

$$\Rightarrow \quad h - 3k - 5 = 0 \qquad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

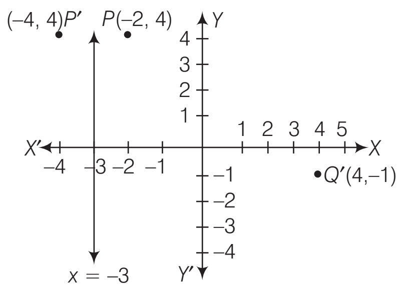
$$\frac{h}{5+15} = \frac{k}{5+10} = \frac{1}{-6+1}$$

$$\Rightarrow \quad \frac{h}{20} = \frac{k}{15} = \frac{1}{-5}$$

$$\therefore \quad h = -4 \text{ and } k = -3$$

Hence, the orthocentre is $(-4, -3)$.

2. *(a)* The equation of the line parallel to Y-axis at a distance 3 units to the left of Y-axis, is $x = -3$.



We know that reflection of the point (x, y) in the line $x = a$ is $(-x + 2a, y)$. Therefore, the reflection of the point $P(-2, 4)$ in the line $x = -3$ is the point $P'(-(-2) + 2 \times (-3), 4)$, i.e. $P'(-4, 4)$.

Since, $Q'(4, -1)$ is the image of the point Q in the origin.

Therefore, the coordinates of the point Q are $(-4, 1)$.

Now, the equation of a line $P'Q$ passing through $P'(-4, 4)$ and $Q(-4, 1)$ is

$$y - 4 = \frac{1-4}{-4+4}(x+4)$$

$$\Rightarrow \quad y - 4 = \frac{-3}{0}(x+4)$$

$$\Rightarrow \quad 0 = -3(x+4)$$

$$\Rightarrow \quad x + 4 = 0$$

3. *(d)* Equation of the line passing through $(3, 2)$ having slope $\frac{3}{4}$ is given by

$$y - 2 = \frac{3}{4}(x - 3) \qquad [\because y - y_1 = m(x - x_1)]$$

$$\Rightarrow \quad 4y - 3x + 1 = 0 \qquad \text{...(i)}$$

Let (h, k) be the required point on the line, such that distance between (h, k) and $(3, 2)$ is 5.

Then, $$(h-3)^2 + (k-2)^2 = 25 \qquad \text{...(ii)}$$

Also, point (h, k) lies on the line.

$$\therefore \quad 4k - 3h + 1 = 0 \qquad \text{[from Eq. (i)]...(iii)}$$

$$\Rightarrow \quad k = \frac{3h-1}{4} \qquad \text{...(iv)}$$

On putting the value of k in Eq. (ii) and simplifying, we get

$$25h^2 - 150h - 175 = 0$$

$$\Rightarrow \quad h^2 - 6h - 7 = 0$$

$$\Rightarrow \quad (h+1)(h-7) = 0$$

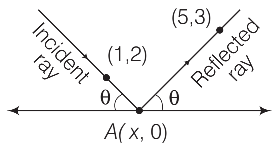
$$\Rightarrow \quad h = -1, 7$$

On putting these values of h in Eq. (iv), we get

$$k = -1 \text{ and } k = 5$$

Therefore, the coordinates of the required points are either $(-1, -1)$ or $(7, 5)$.

4. *(c)* Let the incident ray strike X-axis at the point A, whose coordinates be $(x, 0)$. From the figure, the slope of the reflected ray is given by



$$\tan\theta = \frac{3}{5-x} \qquad \text{...(i)}$$

Again, the slope of the incident ray is given by

$$\tan(\pi - \theta) = \frac{-2}{x-1} \Rightarrow -\tan\theta = \frac{-2}{x-1}$$

$$\Rightarrow \quad \tan\theta = \frac{2}{x-1} \qquad [\because \tan(\pi - \theta) = -\tan\theta] \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$\frac{3}{5-x} = \frac{2}{x-1}$$

$$\Rightarrow \quad 3x - 3 = 10 - 2x \Rightarrow x = \frac{13}{5}$$

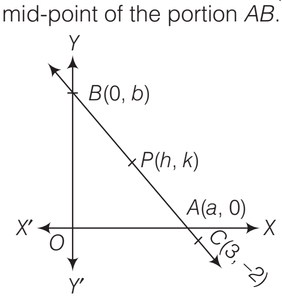
Therefore, the required coordinates of the point A are $\left(\frac{13}{5}, 0\right)$.

5. *(d)* Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$. ...(i)

Since, it passes through the point $(3, -2)$.

$$\therefore \quad \frac{3}{a} - \frac{2}{b} = 1 \qquad \text{...(ii)}$$

The line (i) cuts the coordinate axes at $A(a, 0)$ and $B(0, b)$. Let $P(h, k)$ be the mid-point of the portion AB.



Then,

$$h = \frac{a+0}{2} \text{ and } k = \frac{0+b}{2}$$

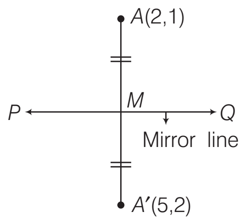
$$\Rightarrow \quad a = 2h \text{ and } b = 2k$$

On substituting the values of a and b in Eq. (ii), we get

$$\frac{3}{2h} - \frac{2}{2k} = 1$$

Hence, the locus of $P(h, k)$ is $\frac{3}{2x} - \frac{1}{y} = 1$ or $3y - 2x = 2xy$.

6. *(a)* Let PQ be the mirror line. Given, image of point $A(2, 1)$ with respect to the line PQ is A' (5, 2). Since, image line is the perpendicular bisector of the mirror line. So, mid-point (M) of AA' lies on PQ.



Now, coordinates of mid-point of

$$AA' = M\left(\frac{2+5}{2}, \frac{1+2}{2}\right) = M\left(\frac{7}{2}, \frac{3}{2}\right)$$

Now, slope of line AA' is $m_1 = \frac{2-1}{5-2} = \frac{1}{3}$ $\left[\because \text{slope} = \frac{y_2 - y_1}{x_2 - x_1}\right]$

Let the slope of a mirror line PQ be m_2.

Since, PQ is perpendicular to AA'.

$\therefore \quad m_1 \times m_2 = -1$

$\Rightarrow \quad \frac{1}{3} \times m_2 = -1 \Rightarrow m_2 = -3$

Now, equation of the mirror line PQ, which is passing through $M\left(\frac{7}{2}, \frac{3}{2}\right)$ and having slope $m_2 = -3$, is

$$y - \frac{3}{2} = -3\left(x - \frac{7}{2}\right)$$

$\Rightarrow \quad \left(\frac{2y-3}{2}\right) = -3\left(\frac{2x-7}{2}\right)$

$\Rightarrow \quad 2y - 3 = -3(2x - 7)$

$\Rightarrow \quad 2y - 3 = -6x + 21$

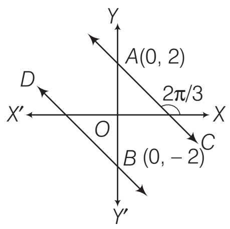
$\Rightarrow \quad 6x + 2y - 24 = 0$

$\Rightarrow \quad 3x + y - 12 = 0$ [dividing both sides by 2]

7. *(b)* Since, $m = \tan\theta = \tan\frac{2\pi}{3}$

$$= \tan\left(\pi - \frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}$$

[given $\tan(\pi - \theta) = -\tan\theta$]



Hence, the equation of line AC passing through the point $A(0, 2)$ is $y - y_1 = m(x - x_1)$

$\Rightarrow \quad y - 2 = -\sqrt{3}(x - 0)$ $[\because x_1 = 0, y_1 = 2]$

$\Rightarrow \quad y - 2 = -\sqrt{3}x$

$\Rightarrow \quad \sqrt{3}x + y - 2 = 0$

Also, the line BD intersects Y-axis below origin. It means that the y-coordinate of intersection point is negative and x-coordinate is 0, i.e. point is (0, –2).

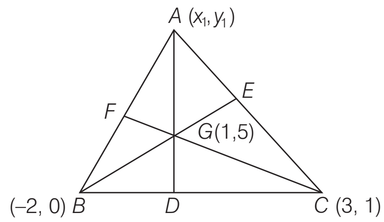
Hence, equation of line passing through the point $B(0, -2)$ and parallel to AC is

$y + 2 = -\sqrt{3}(x - 0)$ $[\because y - y_1 = m(x - x_1)]$

$\Rightarrow \quad \sqrt{3}x + y + 2 = 0$

8. *(a)* Let third vertex of a ΔABC be $A(x_1, y_1)$.

Given vertices and centroid of a ΔABC are $B(-2, 0)$, $C(3, 1)$ and $G(1, 5)$.



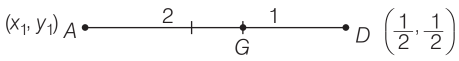
We know that centroid G divides the median line AD into 2 : 1,

i.e. $\quad \frac{AG}{GD} = \frac{2}{1}$

Since, AD is a median line, so D is the mid-point of BC.

Now, the coordinates of mid-point of BC are

$$D\left(\frac{-2+3}{2}, \frac{0+1}{2}\right), \text{ i.e. } D\left(\frac{1}{2}, \frac{1}{2}\right).$$



Now, using the internal division formula,

$$\text{Coordinates of } G = \left(\frac{2 \times \frac{1}{2} + 1 \times x_1}{2+1}, \frac{2 \times \frac{1}{2} + 1 \times y_1}{2+1}\right)$$

$\Rightarrow \quad (1, 5) = \left(\frac{1+x_1}{3}, \frac{1+y_1}{3}\right)$

On equating the x and y-coordinates of both sides, we get

$$1 = \frac{1+x_1}{3} \text{ and } 5 = \frac{1+y_1}{3}$$

$\Rightarrow \quad 3 = 1 + x_1$ and $15 = 1 + y_1$

$\therefore \quad x_1 = 2$ and $y_1 = 14$

Thus, the coordinates of A are (2, 14).

Now, the equation of side AB, which is passing through $A(2, 14)$ and $B(-2, 0)$, is

$$y - 14 = \frac{0-14}{-2-2}(x - 2)$$

$\Rightarrow \quad y - 14 = \frac{-14}{-4}(x - 2)$

$\Rightarrow \quad 2y - 28 = 7x - 14$

$\Rightarrow \quad 2y - 7x = 28 - 14$

$\Rightarrow \quad 2y - 7x = 14 \Rightarrow 7x - 2y + 14 = 0$

Now, the equation of side AC, which is passing through $A(2, 14)$ and $C(3, 1)$, is

$$y - 14 = \frac{1-14}{3-2}(x - 2)$$

$\Rightarrow \quad y - 14 = \frac{-13}{1}(x - 2)$

$\Rightarrow \quad y - 14 = -13x + 26 \Rightarrow 13x + y - 40 = 0$

Chapter 13 Similarity

1. (*b*) Given, height of boy (BC) = 90 cm = 0.9 m

Height of lamp-post (PQ) = 3.6 m

Speed of boy = 1.2 m/s

Time taken = 6 s

$\therefore$ Distance moved by boy (CQ) = Speed × Time

$= 1.2 \times 6 = 7.2$ m

Let length of shadow be $AC = x$ cm

In ΔABC and ΔAPQ,

$\angle ACB = \angle AQP$ [each 90°]

and $\angle BAC = \angle PAQ$ [common angle]

$\therefore \Delta ABC \sim \Delta APQ$ [by AA similarity criterion]

Then, $\dfrac{AC}{AQ} = \dfrac{BC}{PQ}$

$\Rightarrow \dfrac{x}{x+7.2} = \dfrac{0.9}{3.6}$ [$\because AQ = AC + CQ = x + 7.2$]

$\Rightarrow \dfrac{x}{x+7.2} = \dfrac{1}{4}$

$\Rightarrow 4x = x + 7.2 \Rightarrow 4x - x = 7.2$

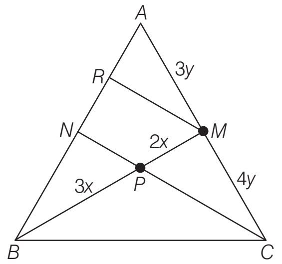
$\Rightarrow 3x = 7.2$

$\therefore x = \dfrac{7.2}{3} = 2.4$

2. (*c*) Given $AM : MC = 3 : 4$

$BP : PM = 3 : 2$ and $BN = 12$ cm

Draw MR parallel to CN which meets AB at the point R.



Consider ΔBMR. In this, we have

$PN \parallel MR$ [by construction]

$\therefore$ By BPT, $\dfrac{BN}{NR} = \dfrac{BP}{PM} \Rightarrow \dfrac{12}{NR} = \dfrac{3}{2}$

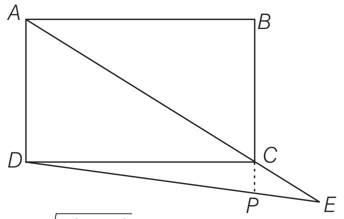
$\Rightarrow NR = 8$ cm

Now, consider ΔANC, $RM \parallel NC$ [by construction]

$\therefore$ By BPT, $\dfrac{AR}{RN} = \dfrac{AM}{MC} \Rightarrow \dfrac{AR}{8} = \dfrac{3}{4} \Rightarrow AR = 6$ cm

Thus, $AN = AR + RN = 6 + 18 = 14$ cm

3. (*c*) Given $AB = 8$ cm and $BC = 6$ cm



$\therefore AC = \sqrt{8^2 + 6^2} = 10$ cm

Also, given $AC : CE = 2 : 1$

Now, produce BC to meet DE at the point P as CP is parallel to AD,

$\Delta ECP \sim \Delta EAD$ ...(i)

$\Rightarrow \dfrac{CP}{AD} = \dfrac{CE}{AE} \Rightarrow \dfrac{CP}{6} = \dfrac{1}{3}$ ...(ii)

$\Rightarrow CP = 2$ cm

Also, ΔCPD is a right triangle.

$\therefore DP = \sqrt{CD^2 + CP^2}$

$= \sqrt{68} = 2\sqrt{17}$ cm

But $DP = PE = 2:1$ [from Eq.(i)]

$\therefore PE = \sqrt{17}$ cm

Thus, $DE = DP + PE = 2\sqrt{17} + \sqrt{17} = 3\sqrt{17}$ cm

4. (*a*) In ΔABC, we have $DE \parallel BC$

Now, in ΔADE and ΔABC, we have

$\angle A = \angle A$ [common angle]

$\angle ADE = \angle ABC$ [corresponding angles]

and $\angle AED = \angle ACB$ [corresponding angles]

$\therefore \Delta ADE \sim \Delta ABC$ [by AAA similarity criterion]

$\Rightarrow \dfrac{AD}{AB} = \dfrac{DE}{BC}$ ...(i)

We have, $\dfrac{AD}{DB} = \dfrac{5}{4}$ [$\because$ if two triangles are similar, then their corresponding sides are proportional]

$\Rightarrow \dfrac{DB}{AD} = \dfrac{4}{5}$ [given]

$\Rightarrow \dfrac{DB}{AD} + 1 = \dfrac{4}{5} + 1$ [adding 1 both sides]

$\Rightarrow \dfrac{DB + AD}{AD} = \dfrac{9}{5}$

$\Rightarrow \dfrac{AB}{AD} = \dfrac{9}{5} \Rightarrow \dfrac{AD}{AB} = \dfrac{5}{9}$

$\therefore \dfrac{DE}{BC} = \dfrac{5}{9}$ [from Eq. (i)]

In ΔDFE and ΔCFB, we have

$\angle 1 = \angle 3$ [alternate interior angles]

$\angle 2 = \angle 4$ [vertically opposite angles]

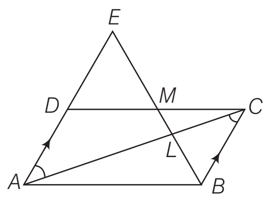
Therefore, by AA similarity criterion, we have

$\Delta DFE \sim \Delta CFB$

$\Rightarrow \dfrac{\text{ar}(\Delta DFE)}{\text{ar}(\Delta CFB)} = \dfrac{DE^2}{BC^2} = \left(\dfrac{5}{9}\right)^2 = \dfrac{25}{81}$

[by areas of similar triangles theorem]

5. (*c*) In ΔBMC and ΔEMD, we have



$\angle BMC = \angle EMD$ [vertically opposite angles]

$\Rightarrow MC = MD$ [$\because M$ is the mid-point of CD]

$\Rightarrow \quad \angle MCB = \angle MDE$ [alternate angles]

So, by AAS congruence criterion, we have

$$\Delta BMC \cong \Delta EMD$$

$\Rightarrow \quad BC = ED$

[$\because$ corresponding parts of congruent triangles are equal]

In ΔAEL and ΔCBL, we have

$\angle ALE = \angle CLB$ [vertically opposite angles]

and $\angle EAL = \angle BCL$ [alternate angles]

So, by AA criterion of similarity, we have

$$\Delta AEL \sim \Delta CBL$$

$$\Rightarrow \quad \frac{AE}{BC} = \frac{EL}{BL} = \frac{AL}{CL}$$

[$\because$ if two triangles are similar, then their corresponding sides are proportional]

On taking first two terms, we get

$$\frac{EL}{BL} = \frac{AE}{BC} = \frac{AD + DE}{BC}$$

$$= \frac{BC + BC}{BC} = \frac{2BC}{BC} = 2$$

[$\because AD = BC$ as sides opposite to parallelogram and $DE = BC$, proved above]

$\Rightarrow \quad EL = 2BL$ ...(i)

Now, $\dfrac{\text{ar}(\Delta AEL)}{\text{ar}(\Delta CBL)} = \left(\dfrac{EL}{BL}\right)^2$

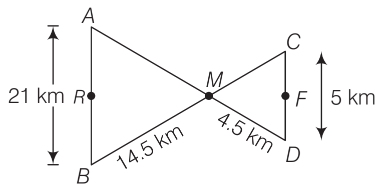
[$\because$ ratio of areas of two similar triangles is equal to the square of the ratio of their corresponding sides]

$$= \left(\frac{2BL}{BL}\right)^2 = (2)^2 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \frac{\text{ar}(\Delta AEL)}{\text{ar}(\Delta CBL)} = 4$$

$\Rightarrow \quad \text{ar}(\Delta AEL) = 4\ \text{ar}(\Delta CBL)$

6. *(a)* Given figure can be redrawn as



In ΔBAM and ΔCDM,

$\angle BAM = \angle CDM$ [alternate interior angles as $AB || CD$]

$\angle ABM = \angle DCM$ [alternate interior angles]

and $\angle AMB = \angle DMC$ [vertically opposite angles]

$\therefore \quad \Delta BAM \sim \Delta CDM$ [by AAA similarity criterion]

So, $\dfrac{AM}{DM} = \dfrac{BA}{CD} = \dfrac{BM}{CM}$

[since, corresponding sides of similar triangles are proportional]

$$\therefore \quad \frac{AM}{4.5} = \frac{21}{5} = \frac{14.5}{CM}$$

Now, consider $\dfrac{AM}{4.5} = \dfrac{21}{5}$

$$\Rightarrow \quad AM = \frac{21 \times 4.5}{5} = 21 \times 0.9 = 18.9 \text{ km}$$

and consider $\dfrac{21}{5} = \dfrac{14.5}{CM}$

$$\Rightarrow \quad CM = \frac{5 \times 14.5}{21} = 3.45 \text{ km}$$

There are four routes which Radhika may follow to reach friend's house, are given below

1st route $R \to A \to M \to C \to F$

Total distance $= RA + AM + MC + CF$

$= 10.5 + 18.9 + 3.45 + 2.5$

$= 35.35$ km

2nd route $R \to B \to M \to D \to F$

Total distance $= RB + BM + MD + DF$

$= 10.5 + 14.5 + 4.5 + 2.5 = 32$ km

3rd route $R \to A \to M \to D \to F$

Total distance $= RA + AM + MD + DF$

$= 10.5 + 18.9 + 4.5 + 2.5 = 36.4$ km

4th route $R \to B \to M \to C \to F$

Total distance $= RB + BM + MC + CF$

$= 10.5 + 14.5 + 3.45 + 2.5 = 30.95$ km

Hence, the shortest distance which Radhika has to travel is 30.95 km.

7. *(b)* Given PA, QB and RC are perpendicular to AC.

In ΔPAC and ΔQBC, we have

$\angle PCA = \angle QCB$ [common angle]

$\angle PAC = \angle QBC$ [each 90°]

$\therefore \quad \Delta PAC \sim \Delta QBC$ [by AA similarity criterion]

Now, $\dfrac{\text{ar}(\Delta PAC)}{\text{ar}(\Delta QBC)} = \left(\dfrac{PA}{QB}\right)^2$

[$\because$ ratio of areas of two similar triangles is equal to square of the ratio of their corresponding sides]

$$\Rightarrow \quad \frac{\text{ar}(\Delta PAC)}{\text{ar}(\Delta QBC)} = \frac{x^2}{y^2} \quad \text{...(i)}$$

Similarly, in ΔRCA and ΔQBA, we have

$\angle RCA = \angle QBA$ [each 90°]

$\angle RAC = \angle QAB$ [common angle]

$\therefore \quad \Delta RCA \sim \Delta QBA$ [by AA similarity criterion]

Now, $\dfrac{\text{ar}(\Delta RCA)}{\text{ar}(\Delta QBA)} = \left(\dfrac{RC}{QB}\right)^2$

[$\because$ ratio of areas of two similar triangles is equal to the square of the ratio of their corresponding sides]

$$\Rightarrow \quad \frac{\text{ar}(\Delta RCA)}{\text{ar}(\Delta QBA)} = \frac{z^2}{y^2} \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we have

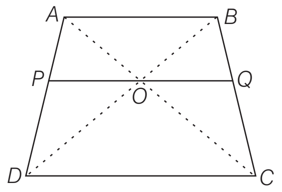
$$\frac{\text{ar}(\Delta PAC)}{\text{ar}(\Delta QBC)} + \frac{\text{ar}(\Delta RCA)}{\text{ar}(\Delta QBA)} = \frac{x^2}{y^2} + \frac{z^2}{y^2}$$

$$\Rightarrow \quad \frac{\text{ar}(\Delta PAC)\ \text{ar}(\Delta QBA) + \text{ar}(\Delta RCA)\ \text{ar}(\Delta QBC)}{\text{ar}(\Delta QBC)\ \text{ar}(\Delta QBA)} = \frac{x^2 + z^2}{y^2}$$

8. *(a)* **Given** $ABCD$ is a trapezium. Diagonals AC and BD intersect at O.

$\therefore$ $PQ \parallel AB \parallel DC$

To prove $PO = QO$



Proof In ΔABD and ΔPOD,

$PO \parallel AB$ [$\because PQ \parallel AB$]

$\angle ADB = \angle PDO$ [common angle]

$\angle ABD = \angle POD$ [corresponding angles]

$\therefore$ $\Delta ABD \sim \Delta POD$ [by AA similarity criterion]

Then, $\dfrac{OP}{AB} = \dfrac{PD}{AD}$ ...(i)

In ΔABC and ΔOQC, $OQ \parallel AB$ [$\because PQ \parallel AB$]

$\angle ACB = \angle OCQ$ [common angle]

and $\angle BAC = \angle QOC$ [corresponding angles]

$\therefore$ $\Delta ABC \sim \Delta OQC$ [by AA similarity criterion]

Then, $\dfrac{OQ}{AB} = \dfrac{QC}{BC}$ ...(ii)

Now, in ΔADC, $OP \parallel DC$

$\therefore$ $\dfrac{AP}{PD} = \dfrac{OA}{OC}$ ...(iii)

[by basic proportionality theorem]

In ΔABC, $OQ \parallel AB$

$\therefore$ $\dfrac{BQ}{QC} = \dfrac{OA}{OC}$ ...(iv)

[by basic proportionality theorem]

From Eqs. (iii) and (iv),

$$\frac{AP}{PD} = \frac{BQ}{QC}$$

On adding 1 to both sides, we get

$$\frac{AP}{PD} + 1 = \frac{BQ}{QC} + 1$$

$\Rightarrow$ $\dfrac{AP + PD}{PD} = \dfrac{BQ + QC}{QC}$

$\Rightarrow$ $\dfrac{AD}{PD} = \dfrac{BC}{QC}$

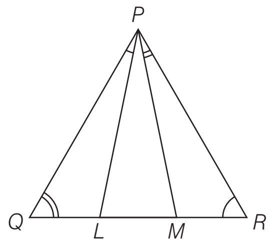
$\Rightarrow$ $\dfrac{PD}{AD} = \dfrac{QC}{BC}$ [reciprocal the terms]

$\Rightarrow$ $\dfrac{OP}{AB} = \dfrac{OQ}{AB}$ [from Eqs. (i) and (ii)]

$\Rightarrow$ $OP = OQ$

9. *(d)* **Given** $\angle LPQ = \angle QRP$

and $\angle RPM = \angle RQP$



In ΔPQL and ΔRPM, $\angle LPQ = \angle MRP$ [$\because \angle LPQ = \angle QRP$, given]

and $\angle LQP = \angle RPM$ [$\because \angle RQP = \angle RPM$, given]

$\therefore$ $\Delta PQL \sim \Delta RPM$ [by AA similarity criterion]

Since, $\Delta PQL \sim \Delta RPM$

$\therefore$ $\dfrac{QL}{PM} = \dfrac{PL}{RM} \Rightarrow QL \times RM = PL \times PM$

In ΔPQL and ΔRQP,

$\angle PQL = \angle RQP$ [common angle]

and $\angle QPL = \angle QRP$ [given]

$\therefore$ $\Delta PQL \sim \Delta RQP$ [by AA similarity criterion]

Then, $\dfrac{PQ}{QR} = \dfrac{QL}{PQ}$

$\Rightarrow$ $PQ^2 = QR \times QL$

Chapter 14 Locus

1. *(a)* **Given** A quadrilateral $ABCD$ in which $AB = BC$ and PE and QE are right bisectors of AD and CD respectively, such that they meet at E.

To prove BE bisects $\angle ABC$.

Construction Join AE, DE and CE.

Proof Since, PE is the right bisector of AD and E lies on it.

$\therefore$ $AE = ED$ ...(i)

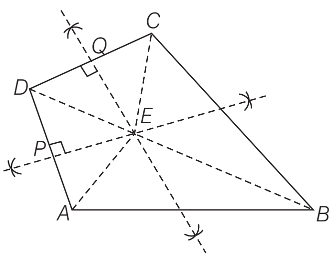
[$\because$ points on the right bisector of line segment are equidistant from the ends of the segment]

Also, QE is the right bisector of CD and E lies on it.

$\therefore$ $ED = EC$ ...(ii)

From Eqs. (i) and (ii), we get

$AE = EC$ ...(iii)



Now, in ΔABE and ΔCBE, we have

$AB = BC$ [given]

$BE = BE$ [common]

and $AE = EC$ [from Eq. (iii)]

So, by SSS criterion of congruence, we have

$\Delta ABE \cong \Delta ACE$

$\Rightarrow$ $\angle ABE = \angle CBE$

$\Rightarrow$ BE bisects $\angle ABC$

$\therefore \quad \angle ABE = \angle CBE \quad$ ...(iv)

Now, $\quad \angle ABC = \angle ABE + \angle CBE$

$= \angle ABE + \angle ABE \quad$ [from Eq. (iv)]

$= 2\angle ABE$

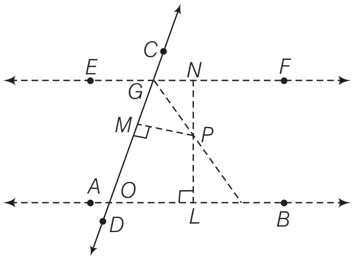
2. *(d)* **Given** Two lines AB and CD intersecting at O and P is a moving point such that the sum of its distances from AB and CD is constant, i.e. it always remains same.

Construction Draw $PL \perp AB$ and $PM \perp CD$.

Proof We have to find the locus of P, when it is given that $PL + PM = k$ constant.

Draw $EF \parallel AB$ at a distance k from AB. Suppose EF meets CD at G. Join GP and produce LP to meet EF at N.

Now, $EF \parallel AB$ at a distance k from it.



$\therefore$ LN = Distance between AB and EF.

$\Rightarrow \quad LN = k$

$\therefore \quad PL + PM = LN \quad [\because PL + PM = k \text{ (given)}]$

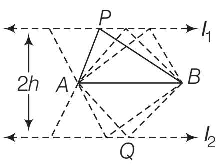
$\Rightarrow \quad PL + PM = PL + PN$

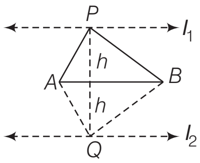
$\Rightarrow \quad PM = PN$

$\Rightarrow$ P is equidistant from CD and EF.

$\Rightarrow$ P lies on the bisector of $\angle FGO$.

3. *(a)* Given a ΔPAB, in which AB is the base and h is the height of the vertex P of the triangle. Take a point Q on the other side of AB at a distance of h from AB. Join PB and BQ.





Now, area of $\Delta APB = \frac{1}{2} \times AB \times h \quad$ ...(i)

$\left[\because \text{area of triangle} = \frac{1}{2} \times \text{base} \times \text{altitude}\right]$

Area of $\Delta AQB = \frac{1}{2} \times AB \times h \quad$ ...(ii)

From Eqs. (i) and (ii), we get

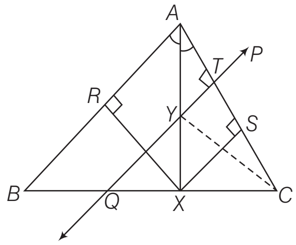
Area of ΔAPB = Area of ΔAQB

The areas of the two triangles remain the same for the same base AB and altitude h.

Hence, the lines l_1 and l_2 drawn from P and Q parallel to AB at a height of h, are the loci of the vertex of triangle.

4. *(c)* Given a ΔABC, in which AX is a bisector of $\angle BAC$ meeting BC at X. Also, PQ is a perpendicular bisector of AC, which meets AX at Y and AC at T.

Draw $XR \perp AB$ and $XS \perp AC$. Join YC also.



I. In ΔARX and ΔASX,

$AX = AX \quad$ [common side]

$\angle XAR = \angle XAS \quad$ [given]

$\angle XRA = \angle XSA \quad$ [each 90°]

$\therefore \quad \Delta ARX \cong \Delta ASX \quad$ [SAA congruency]

$\therefore \quad XR = XS \quad$ [by CPCT]

II. In ΔYTA and ΔYTC,

$TA = TC \quad$ [given]

$\angle YTA = \angle YTC \quad$ [each 90°]

$YT = YT \quad$ [common side]

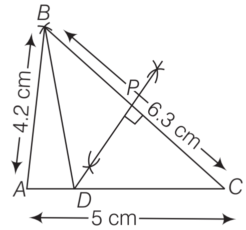
$\therefore \quad \Delta YTA \cong \Delta YTC \quad$ [SAS congruency]

$\therefore \quad YA = YC \quad$ [by CPCT]

Thus, Y is equidistant from A and C.

5. *(b)* **Steps of construction**

1. Draw $AC = 5$ cm.
2. From A, draw an arc of radius 4.2 cm and from C, draw an arc of radius 6.3 cm, so that both arcs intersect at B.
3. Join AB and BC. Thus, ABC is the required triangle.
4. Draw perpendicular bisector PD of BC, which meets AC at D.



5. Join DB. Measure DB and DC.

$\therefore$ $DB = DC = 4.3$ cm

Thus, D is equidistant from B and C.

Now, we have to prove this result.

In ΔDPB and ΔDPC,

$\angle DPB = \angle DPC \quad$ [each 90°]

$BP = PC \quad [\because DP$ is perpendicular bisector of $BC]$

$DP = DP \quad$ [common side]

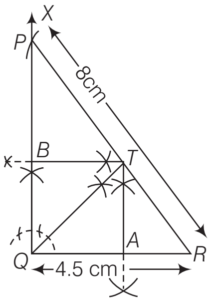
$\therefore \quad \Delta DPB \cong \Delta DPC \quad$ [SAS congruency]

$\therefore \quad DB = DC \quad$ [by CPCT]

Thus, D is equidistant from B and C.

6. *(a)* **Steps of construction**

1. Draw a line segment $QR = 4.5$ cm.
2. Draw $QX \perp QR$.
3. From R, draw an arc of radius 8 cm, which intersects QX at P.
4. Now, join PR.



5. Draw bisector QT of $\angle Q$, which meets PR at T.
6. From T, draw $TA \perp QR$ and $TB \perp QP$.
7. Measure TA and TB.

Thus, $TA = TB = 2.8$ cm

Now, we have to prove $TA = TB$.

In ΔTAQ and ΔTBQ,

$TQ = TQ$ [common side]

$\angle TQA = \angle TQB$ [$\because QT$ is bisector of $\angle AQB$]

$\angle TAQ = \angle TBQ$ [each 90°]

$\therefore \Delta TAQ \cong \Delta TBQ$ [SAA congruency]

$\therefore TA = TB$ [by CPCT]

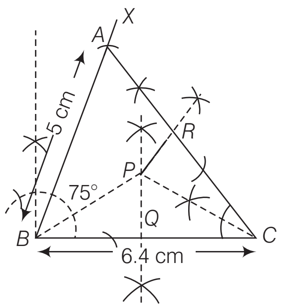
Thus, T is equidistant from PQ and QR.

7. *(c)* **Steps of construction**

1. Draw a line segment $BC = 6.4$ cm.
2. At B, draw $\angle XBC = 75°$ and cut $AB = 5$ cm from BX.
3. Join AC.
4. Draw perpendicular bisector of BC; and also bisector of $\angle ACB$, so that both lines intersect at P.
5. Measure PB, PC, PQ and PR.

$\therefore$ $PB = PC = 3.4$ cm

and $PQ = PR = 1.2$ cm



In ΔPBQ and ΔPCQ,

$BQ = CQ$ [$\because PQ$ is perpendicular bisector of BC]

$\angle BQP = \angle CQP$ [each 90°]

$PQ = PQ$ [common side]

$\therefore \Delta PBQ \cong \Delta PCQ$ [SAS congruency]

$\therefore PB = PC$ [by CPCT] ...(i)

In ΔPQC and ΔPRC,

$PC = PC$ [common side]

$\angle PCQ = \angle PCR$ [$\because PC$ is bisector of $\angle ACB$]

$\angle PQC = \angle PRC$ [each 90°]

$\Delta PQC \cong \Delta PRC$ [SAA congruency]

$\therefore PQ = PR$ [by CPCT] ...(ii)

From Eqs. (i) and (ii), it is clear that P is equidistant from B and C; and also equidistant from AC and BC.

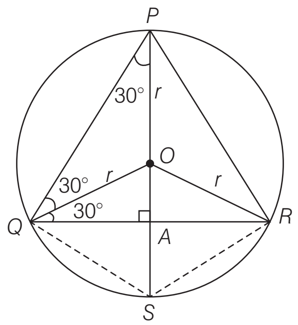
Chapter 15 Circles

1. *(c)* As PQR is an equilateral triangle, hence PS will be perpendicular to QR and will divide it into 2 equal parts.

Since, $\angle P$ and $\angle S$ will be supplementary, so

$$\angle S = 120° \text{ and } \angle QSA = \angle RSA = 60°.$$

Now, $PA = PQ \cos 30°$ and $OA = OQ \sin 30° = \dfrac{r}{2}$



$$\Rightarrow \quad AS = OA = \frac{r}{2} \text{ and } PA = PO + OA = r + \frac{r}{2}$$

Hence, $PQ = \dfrac{PA}{\cos 30°} = \dfrac{r + \dfrac{r}{2}}{\dfrac{\sqrt{3}}{2}} = \sqrt{3}r$

In ΔQAS, $AS = QS \cos 60° \Rightarrow QS = \dfrac{r/2}{1/2} = r$

Since, $AQ = AR$, AS is common and $\angle QAS = \angle RAS = 90°$

So, $QS = RS$.

$\therefore$ Perimeter of $PQSR = 2(PQ + QS) = 2(\sqrt{3} + 1)r$

2. *(c)* Join AE, AB and CD.

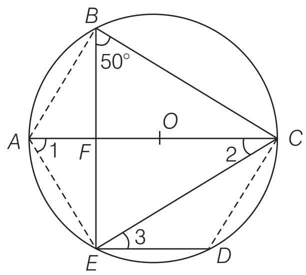
Then, $\angle CBE = \angle CAE$

[$\because$ angles in the same segment are equal]

$\Rightarrow \angle CAE = \angle 1 = 50°$ [$\because \angle CBE = 50°$] ... (i)

and $\angle AEC = 90°$ ...(ii)

[$\because$ angle in a semi-circle]



Now, in ΔAEC,

$\angle 1 + \angle AEC + \angle 2 = 180°$

[by angle sum property of a triangle]

$\therefore \quad 50° + 90° + \angle 2 = 180°$

$\Rightarrow \quad \angle 2 = 40° \quad$ ...(iii)

Also, $\quad ED \parallel AC \quad$ [given]

$\therefore \quad \angle 2 = \angle 3 \quad$ [alternate angles]

$\therefore \quad 40° = \angle 3$, i.e. $\angle 3 = 40°$

Hence, $\angle CED = 40°$.

3. *(d)* AS and AT are two tangents from A, so $AT = AS$.

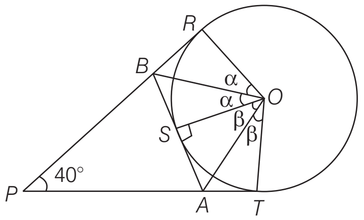
Join O and S.

Since, AB is tangent at S and OS is radius.

$\Rightarrow \quad OS \perp AB$ and $OS = OT$

BS and BR are two tangents from B. So, $BS = BR$. Since, OS and OR are radius of the same circle.

$\Rightarrow \quad OS = OR$



Therefore, OA and OB bisect $\angle SOT$ and $\angle SOR$.

OR and OT are radii and PR and PT are tangent at R and T, respectively.

So, $\angle ORP = \angle OTP = 90°$ and $\angle APB = 40° = \angle TPR$

In quadrilateral $PROT$,

$\angle ORP + \angle TPR + \angle OTP + \angle ROT = 360°$

[by angle sum property of a quadrilateral]

$\Rightarrow 90° + 40° + 90° + \angle ROT = 360° \Rightarrow \angle ROT = 140°$

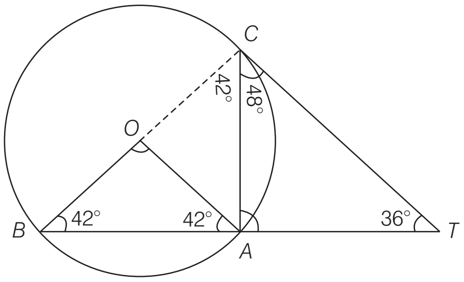
Let $\angle SOR = 2\alpha$ and $\angle SOT = 2\beta$ and OB and OA are respective bisectors, so $\angle ROB = \angle BOS = \alpha$ and $\angle SOA = \angle AOT = \beta$.

Then, $\angle AOB = \angle BOS + \angle SOA = \alpha + \beta$

$\Rightarrow \angle ROT = \angle SOR + \angle SOT = 2\alpha + 2\beta = 2(\alpha + \beta) = 140°$

$\Rightarrow \quad \alpha + \beta = 70° = \angle AOB$. So, $\angle AOB = 70°$.

4. *(b)* Let O be centre of the circle. Join OC.



Since, the angle between the radius and the tangent is 90°, i.e. $\angle OCT = 90°$.

$\therefore \quad \angle OCA = \angle OCT - \angle ACT$

$= 90° - 48° = 42°$

and $\quad OA = OC = OB \quad$ [radii of the same circle]

$\Rightarrow \quad \angle OAC = \angle OCA = 42°$

[$\because$ angle opposite to equal sides are equal]

In ΔACT, exterior $\angle CAB = \angle ACT + \angle CTA$

$\Rightarrow \quad \angle CAB = 48° + 36° = 84°$

$\therefore \quad \angle OAB = \angle CAB - \angle OAC$

$\Rightarrow \quad \angle OAB = 84° - 42° = 42°$

$\therefore \quad \angle OBA = \angle OAB = 42°$

In ΔAOB,

$\angle OBA + \angle OAB + \angle AOB = 180°$

[by angle sum property of a triangle]

$\Rightarrow \quad 42° + 42° + \angle AOB = 180°$

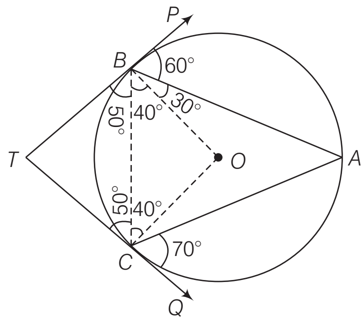
$\Rightarrow \quad \angle AOB = 180° - 84° = 96°$

Hence, the angle subtended by AB at centre is 96°.

5. *(b)* Join BC, BO and CO.

Clearly, $\angle OBP = 90°$

[$\because$ tangent to a circle is perpendicular to the radius through the point of contact]



$\therefore \quad \angle OBA = 90° - 60° = 30°$

Now, $\quad \angle ACQ = \angle CBA \quad$ [angles in the alternate segment]

$\therefore \quad \angle CBA = 70° \quad$ [alternate segments theorem]

$\therefore \quad \angle OBC = \angle CBA - \angle OBA = 70° - 30° = 40°$

In ΔOBC, $\quad OB = OC$

$\therefore \quad \angle OBC = \angle OCB \quad$ [$\because \angle OBC = 40°$]

$\therefore \quad \angle BOC = 180° - (40° + 40°) = 100°$

Again, $\quad \angle BAC = \frac{1}{2} \angle BOC$

[$\because$ angle subtended by an arc at the centre is double the angle subtended by the same arc at the circumference]

$\Rightarrow \quad \angle BAC = \frac{1}{2} \times 100° = 50°$

Again, $\angle OBT = 90°$

[$\because$ tangent to a circle is perpendicular to the radius through the point of contact]

$\therefore \quad \angle CBT = \angle OBT - \angle OBC = 90° - 40° = 50°$

Similarly, $\quad \angle BCT = 50°$

In ΔBCT, $\quad \angle BTC = 180° - (\angle CBT + \angle BCT)$

$= 180° - (50° + 50°) = 80°$

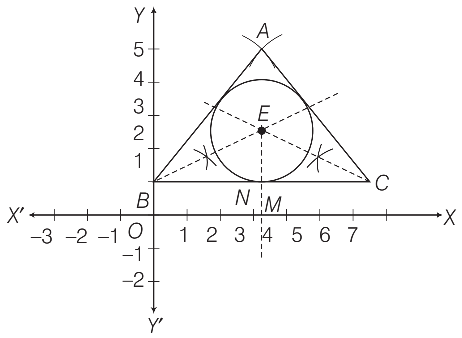
Hence, $\quad \angle BAC = 50°$ and $\angle BTC = 80°$.

Chapter 16 Constructions

Solutions (Q.Nos. 1-4)

1. ***Steps of construction***

1. Now, draw the internal bisectors of $\angle B$ and $\angle C$. Let these bisectors meet at point E.
2. From point E, draw a perpendicular on side BC, which cuts BC at N.



3. Taking E as centre and radius equal to EN, draw a circle, which touches all the sides of the ΔABC. Thus, it is the required incircle.

2. *(a)* Now, measure of radius, $r = EN = 1.5$ cm.

3. *(d)* Then, x-coordinate of E is $OM = 3.5$

and y-coordinate of E is $ME = MN + NE$

$$= 1 + 1.5 = 2.5$$

Hence, coordinates of E are (3.5, 2.5).

4. *(b)* Here, coordinates of B (0, 1)

Required distance, $BE = \sqrt{(3.5-0)^2 + (2.5-1)^2}$

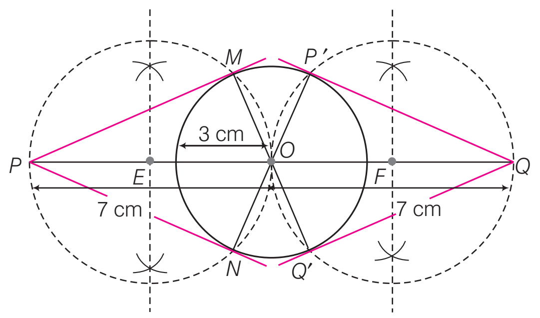
$$= \sqrt{12.25 + 2.25} = \sqrt{14.5} = 3.80 \text{ units}$$

5. Given, two points P and Q on the extended diameter of a circle with radius 3 cm, such that its distance from the centre equal to 7 cm.

We have to construct the tangents to the circle from the given points P and Q.

Steps of construction

1. Draw a circle of radius 3 cm with centre O.
2. Produce its diameter on both sides and take points P and Q on this diameter, such that $OP = OQ = 7$ cm.
3. Bisect OP and OQ. Let E and F be the mid-points of OP and OQ, respectively.



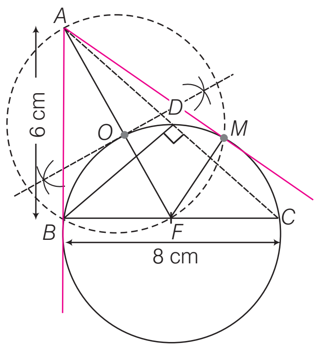
4. Now, taking E as centre and OE as radius, draw a circle, which intersects the given circle at two points M and N.

   Again, taking F as centre and OF as radius, draw a circle, which intersects the given circle at two points P' and Q'.
5. Join PM, PN, QP' and QQ'. These are the required tangents from P and Q to the given circle.

6. Given, ABC is a right angled triangle, in which $AB = 6$ cm, $BC = 8$ cm, $\angle B = 90°$ and BD is perpendicular to AC. Then, $\angle ADB = \angle CDB = 90°$.

Steps of construction

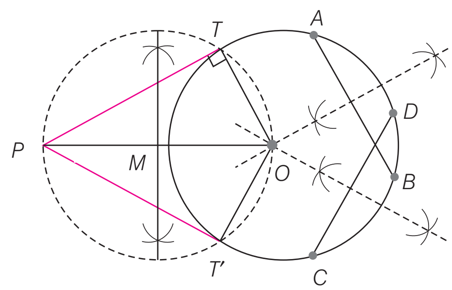
1. Draw the line segments $AB = 6$ cm and $BC = 8$ cm, such that they perpendicular to each other. Join AC. Thus, ΔABC is the given right angled triangle.
2. Draw $BD \perp AC$.
3. Taking the mid-point F of BC as centre and draw a circle with radius 4 cm, passing through B, C and D.
4. Now, join AF and bisect it. Let mid-point of AF be O.



5. Taking O as centre and OA as radius and draw a circle, which intersects the given circle (which passes through B,C and D) at B and M.
6. Now, join AB and AM, which are the required tangents.

7. ***Steps of construction***

1. First, draw a circle with the help of given circular solid ring and then draw two non-parallel chords AB and CD.
2. Draw perpendicular bisectors of AB and CD, which intersect each other at point O. Then, O is centre of the circle.
3. Now, take a point P outside the circle and join OP.
4. Draw bisector of OP. Let its mid-point be M.



5. Taking M as centre and MP as radius, draw a dotted circle which intersect the given circle at T and T'.
6. Join PT and PT'.

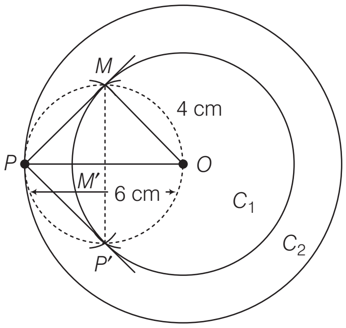
   Then, PT and PT' are the required pair of tangents drawn to the circle from P.

8. Given, two concentric circles of radii 4 cm and 6 cm with common centre O.

Here, we have to draw two tangents to inner circle C_1 from a point on the outer circle C_2.

Steps of construction

1. Draw two concentric circles C_1 and C_2 with common centre O and radii 4 cm and 6 cm, respectively.
2. Take any point P on outer circle C_2 and join OP.
3. Draw the bisector of OP, which bisect OP at M'.
4. Taking M' as centre and OM' as radius, draw a dotted circle which cuts the inner circle C_1 at two points M and P'.



5. Join PM and PP'. Thus, PM and PP' are required tangents. On measuring PM and PP', we get

$$PM = PP' = 4.47 \text{ cm}$$

Chapter 17 Surface Area and Volume

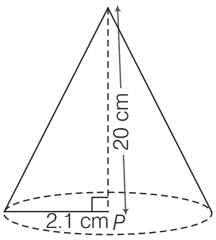
1. *(c)* Given, radius of the broadest end, $r = 2.1$ cm

and length, $h = 20$ cm.

Let l be the slant height of the conical corn cob.

Then, $l = \sqrt{r^2 + h^2} = \sqrt{(2.1)^2 + (20)^2}$

$= \sqrt{4.41 + 400} = \sqrt{404.41} = 20.11$ cm



Now, curved surface area of the corn cob $= \pi r l$

$= \frac{22}{7} \times 2.1 \times 20.11$

$= 132.726 \text{ cm}^2 = 132.73 \text{ cm}^2$

Since, the grains of corn are found on the curved surface of the corn cob.

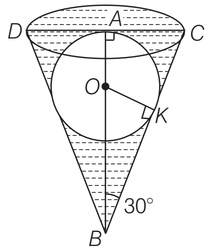
So, total number of grains on the corn cob = Curved surface area of the corn cob × Number of grains of corn on 1 cm^2

$= 132.73 \times 4 = 530.92$

So, there would be approximately 531 grains of corn on the entire cob.

2. *(a)* Let radius of sphere be a, i.e. $OK = OA = a$.

Then, the centre O of a sphere will be centroid of the ΔBCD.



$\therefore \quad OA = \frac{1}{3} AB \Rightarrow AB = 3\,(OA)$

In right angled ΔOKB,

$$\sin 30° = \frac{OK}{OB} = \frac{a}{OB}$$

$\Rightarrow \quad \frac{1}{2} = \frac{a}{OB}$

$\Rightarrow \quad OB = 2a$

Now, $\quad AB = OA + OB = a + 2a = 3a$

Now, in right angled ΔBAC,

$$\frac{AC}{AB} = \tan 30° \Rightarrow \frac{AC}{AB} = \frac{1}{\sqrt{3}}$$

$\Rightarrow \quad AC = \frac{AB}{\sqrt{3}} = \frac{3a}{\sqrt{3}}$

$\therefore \quad AC = \sqrt{3}a$ units

Now, volume of a cone $BCD = \frac{1}{3}\pi (AC)^2 \times AB$

$= \frac{1}{3}\pi (a\sqrt{3})^2 \times 3a = 3\pi a^3$

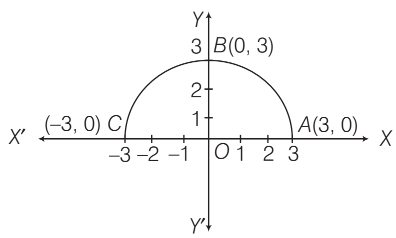
and volume of a sphere $= \frac{4}{3}\pi a^3$

$\therefore$ Volume of water remaining in the cone

= Volume of the cone BCD – Volume of a sphere

$= 3\pi a^3 - \frac{4}{3}\pi a^3 = \frac{5\pi}{3} a^3$ cu units

Solutions (Q. Nos. 3-4) A semi-circle passing through $A(3, 0)$, $B(0, 3)$ and $C(-3, 0)$ as shown in the figure.



3. *(b)* When we rotate the semi-circle along the side AC, a sphere is formed having radius, $r = OA = 3$ cm

$\therefore$ Space occupied by the sphere = Volume of the sphere

$= \frac{4}{3}\pi r^3 = \frac{4}{3} \times 3.14 \times (3)^3 = 113.04 \text{ cm}^3$

4. *(a)* When we rotate the semi-circle along the side BO, a hemisphere is formed having radius

$r = OB = 3$ cm

$\therefore$ Space occupied by the hemisphere

= Volume of the hemisphere

$= \frac{2}{3}\pi r^3 = \frac{2}{3} \times 3.14 \times (3)^3$

$= 56.52 \text{ cm}^3$

5. *(a)* Given, length of cuboid, $l = 15$ cm,

breadth of cuboid, $b = 10$ cm

and height of cuboid, $h = 3.5$ cm

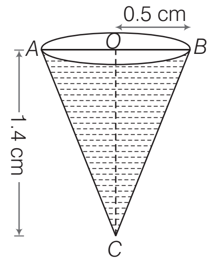
$\therefore$ Volume of the cuboid $= l \times b \times h$

$= 15 \times 10 \times 3.5 = 525 \text{ cm}^3$

Also, radius of a conical depression, $r = 0.5$ cm

and height of a conical depression, $h = 1.4$ cm

$\therefore$ Volume of one conical depression $= \frac{1}{3}\pi r^2 h$



$= \frac{1}{3} \times \frac{22}{7} \times 0.5 \times 0.5 \times 1.4$

$= \frac{22}{3} \times \frac{1}{2} \times \frac{1}{2} \times \frac{2}{10} = \frac{11}{30} \text{ cm}^3$

Now, volume of 4 conical depressions

$= 4 \times$ Volume of one conical depression $= 4 \times \frac{11}{30} = \frac{22}{15} \text{ cm}^3$

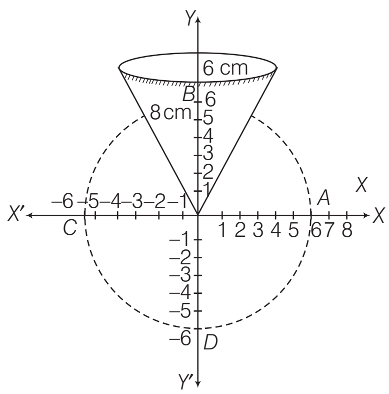
$\therefore$ Volume of wood in the pen-stand

= Volume of the cuboid – Volume of 4 conical depressions

$= 525 - \frac{22}{15} = 525 - 1.46$

$= 523.54 \text{ cm}^3$

6. *(b)* According to the given information, a shape of figure is shown below



When the hanging pipes touches the surface paper, a circular shape *ABCD* is formed on the graph paper. The size of circle *ABCD* is equal to the size of circular base of the cone.

$\therefore$ Radius of the circle *ABCD* is 6 cm.

Hence, the coordinates of *A, B, C* and *D* are (6, 0), (0, 6), (–6, 0) and (0, –6), respectively.

The figure formed in the given information is cylindrical in outer surface and conical in the inner surface.

Now, total surface area of the figure

= Curved surface area of the cylinder + Curved surface area of the cone

$= 2\pi rh + \pi rl = \pi r(2h + l)$

$= \pi r(2h + \sqrt{r^2 + h^2})$

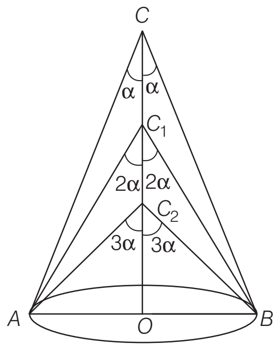
$= 3.14 \times 6\,(2 \times 8 + \sqrt{6^2 + 8^2})$

$= 18.84(16 + \sqrt{36 + 64})$

$= 18.84(16 + \sqrt{100}) = 18.84\,(16 + 10)$

$= 18.84 \times 26 = 489.84 \text{ cm}^2$

7. *(b)* Let V_1, V_2 and V_3 be the volumes of the cones CAB, C_1AB and C_2AB, respectively.



For cone *CAB*, we have

$CO = 3h$

In ΔCOA, $\tan\alpha = \frac{OA}{OC} = \frac{OA}{3h}$

$\Rightarrow \quad OA = 3h\tan\alpha$

$\therefore$ Volume, $V_1 = \frac{1}{3}\pi(3h\tan\alpha)^2 \times 3h$ [$\because$ radius $= 3h\tan\alpha$]

$\Rightarrow \quad V_1 = \frac{27}{3}\pi h^3 \tan^2\alpha$

For cone C_1AB, we have

$C_1O = 2h$

In ΔC_1AO, $\tan 2\alpha = \frac{OA}{OC_1} = \frac{OA}{2h}$

$\Rightarrow \quad OA = 2h\tan 2\alpha$

$\therefore$ Volume, $V_2 = \frac{1}{3}\pi(2h\tan 2\alpha)^2 \times 2h$ [$\because$ radius $= 2h\tan 2\alpha$]

$\Rightarrow \quad V_2 = \frac{8}{3}\pi h^3 \tan^2 2\alpha$

For cone C_2AB, we have $C_2O = h$

In ΔC_2OA, $\tan 3\alpha = \frac{OA}{OC_2} = \frac{OA}{h}$

$\Rightarrow \quad OA = h\tan 3\alpha$

$\therefore$ Volume, $V_3 = \frac{1}{3}\pi(h\tan 3\alpha)^2 \times h$ [$\because$ radius $= h\tan 3\alpha$]

$\Rightarrow \quad V_3 = \frac{1}{3}\pi h^3 \tan^2 3\alpha$

Now, $V_1 - V_2 = \frac{27}{3}\pi h^3 \tan^2\alpha - \frac{8}{3}\pi h^3 \tan^2 2\alpha$

$$= \frac{\pi h^3}{3}(27\tan^2\alpha - 8\tan^2 2\alpha)$$

and $V_2 - V_3 = \frac{8}{3}\pi h^3 \tan^2 2\alpha - \frac{1}{3}\pi h^3 \tan^2 3\alpha$

$$= \frac{\pi h^3}{3}(8\tan^2 2\alpha - \tan^2 3\alpha)$$

$\therefore$ Required ratio $= (V_1 - V_2) : (V_2 - V_3)$

$$= (27\tan^2\alpha - 8\tan^2 2\alpha) : (8\tan^2 2\alpha - \tan^2 3\alpha)$$

8. *(a)* When we rotate a right triangle about the origin O, a cylinder is formed.

To determine the radius, first draw a perpendicular line from origin to the side AB, which bisects it. So, C is the mid-point of AB. Thus, the coordinate of C is $\left(\frac{0+4}{2}, \frac{4+0}{2}\right)$, i.e. (2, 2)

Now, $OC = \sqrt{(2-0)^2 + (2-0)^2}$

[$\because$ distance between two points $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$]

$$= \sqrt{2^2 + 2^2} = 2\sqrt{2} \text{ units}$$

Here, radius of the cylinder, $r = OC = 2\sqrt{2}$ units

and height of the cylinder, $h = AB$

$$= \sqrt{(4-0)^2 + (0-4)^2} = \sqrt{4^2 + 4^2}$$

$$= 4\sqrt{1+1} = 4\sqrt{2} \text{ units}$$

Now, curved surface area of the cylinder

$$= 2\pi rh = 2 \times 3.14 \times (2\sqrt{2})^2 \times 4\sqrt{2}$$

$$= 6.28 \times 8 \times 4\sqrt{2} = 6.28 \times 32 \times 1.414$$

$$= 284.15 \text{ sq units}$$

Chapter 18 Trigonometric Identities

1. *(c)* Since, $\sin\alpha$ and $\cos\alpha$ are the roots of the equation

$$px^2 + qx + r = 0$$

$\therefore$ $\sin\alpha + \cos\alpha = -\frac{q}{p}$

and $p\sin^2\alpha + q\sin\alpha + r = 0$ ...(i)

or $p\cos^2\alpha + q\cos\alpha + r = 0$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$p(\sin^2\alpha + \cos^2\alpha) + q(\sin\alpha + \cos\alpha) + 2r = 0$$

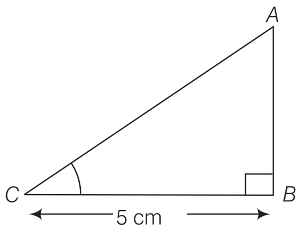
$\Rightarrow$ $p + q\left(-\frac{q}{p}\right) + 2r = 0$ [$\because \sin^2 A + \cos^2 A = 1$]

$\Rightarrow$ $p^2 - q^2 + 2rp = 0$

2. *(c)* In right angled ΔABC,

$AC^2 = BC^2 + AB^2$ [by using Pythagoras theorem]

$\Rightarrow AC^2 - AB^2 = BC^2$



$\Rightarrow$ $(AC + AB)(AC - AB) = (5)^2$

$\Rightarrow$ $(AC + AB)(1) = 25$

[$\because AC - AB = 1$ cm, given]

$\Rightarrow$ $AC + AB = 25$ and $AC - AB = 1$

On solving above equations, we get

$$AC = 13 \text{ and } AB = 12$$

Now, in ΔABC, $\sin C = \frac{AB}{AC} = \frac{12}{13}$

$\therefore$ $\frac{\cos^2 C}{1 - \sin C} = \frac{1 - \sin^2 C}{1 - \sin C} = \frac{(1 - \sin C)(1 + \sin C)}{1 - \sin C}$

$$= 1 + \sin C = 1 + \frac{12}{13} = \frac{13 + 12}{13} = \frac{25}{13}$$

3. *(b)* We have, $\operatorname{cosec}\theta - \sin\theta = x^3$

$\Rightarrow$ $\frac{1}{\sin\theta} - \sin\theta = x^3$

$\Rightarrow$ $\frac{1 - \sin^2\theta}{\sin\theta} = x^3 \Rightarrow \frac{\cos^2\theta}{\sin\theta} = x^3$ ...(i)

and $\sec\theta - \cos\theta = y^3$ ...(ii)

$\Rightarrow$ $\frac{1 - \cos^2\theta}{\cos\theta} = y^3 \Rightarrow \frac{\sin^2\theta}{\cos\theta} = y^3$

From Eqs. (i) and (ii), we get

$$\frac{y^3}{x^3} = \frac{\sin^2\theta / \cos\theta}{\cos^2\theta / \sin\theta}$$

$\Rightarrow$ $\frac{y^3}{x^3} = \frac{\sin^3\theta}{\cos^3\theta}$

$\Rightarrow$ $\tan^3\theta = \frac{y^3}{x^3} \Rightarrow \tan\theta = \frac{y}{x}$

$\therefore$ $\sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$ and $\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$

$$\left[\because \sin\theta = \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}} \text{ and } \cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}}\right]$$

Now, putting the values of $\sin\theta$ and $\cos\theta$ in Eq. (i), we get

$$x^3 = \frac{x^2 / (\sqrt{x^2 + y^2})^2}{y / \sqrt{x^2 + y^2}}$$

$\Rightarrow$ $x^3 y\sqrt{x^2 + y^2} = x^2 \Rightarrow xy\sqrt{x^2 + y^2} = 1$

$\Rightarrow$ $x^2 y^2 (x^2 + y^2) = 1$ [squaring on both sides]

4. *(a)* Given, $T_n = \sin^n\theta + \cos^n\theta$. ...(i)

Now, $\frac{T_3 - T_5}{T_1} = \frac{\sin^3\theta + \cos^3\theta - \sin^5\theta - \cos^5\theta}{\sin\theta + \cos\theta}$ [from Eq. (i)]

$$= \frac{\sin^3\theta - \sin^5\theta + \cos^3\theta - \cos^5\theta}{(\sin\theta + \cos\theta)}$$

$$= \frac{\sin^3\theta(1 - \sin^2\theta) + (\cos^3\theta)(1 - \cos^2\theta)}{\sin\theta + \cos\theta}$$

$= \frac{\sin^3\theta\cos^2\theta + \cos^3\theta\sin^2\theta}{(\sin\theta + \cos\theta)}$ [$\because \sin^2 A + \cos^2 A = 1$]

$$= \frac{\sin^2\theta\cos^2\theta(\sin\theta + \cos\theta)}{\sin\theta + \cos\theta}$$

$$= \sin^2\theta\cos^2\theta$$

5. *(b)* We have, $\sin\theta + \sin^2\theta = 1$

$\Rightarrow \quad \sin\theta = 1 - \sin^2\theta$

$\Rightarrow \quad \sin\theta = \cos^2\theta \quad [\because \sin^2 A + \cos^2 A = 1]$

$\therefore \quad \cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta + 2\cos^4\theta + 2\cos^2\theta - 2$

$= (\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta) + 2(\cos^4\theta + \cos^2\theta - 1)$

$= (\cos^4\theta)^3 + 3\cos^6\theta(\cos^4\theta + \cos^2\theta) + (\cos^2\theta)^3 + 2(\cos^4\theta + \cos^2\theta - 1)$

$= (\cos^4\theta + \cos^2\theta)^3 + 2(\cos^4\theta + \cos^2\theta - 1)$

$[\because (a+b)^3 = a^3 + b^3 + 3ab(a+b)]$

$= (\sin^2\theta + \cos^2\theta)^3 + 2(\sin^2\theta + \cos^2\theta - 1)$

$[\because \cos^2\theta = \sin\theta \Rightarrow \cos^4\theta = \sin^2\theta]$

$= (1)^3 + 2(1-1) = 1 \quad [\because \sin^2 A + \cos^2 A = 1]$

6. *(d)* Given expression can be written as

$$\frac{1+\cos y - \sin^2 y}{1+\cos y} + \frac{(1-\cos^2 y) - \sin^2 y}{\sin y(1-\cos y)}$$

$$= \frac{\cos y(1+\cos y)}{1+\cos y} + \frac{1-1}{\sin y(1-\cos y)}$$

$= \cos y + 0 = \cos y$

7. *(a)* We have, $(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C)$

$= (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$

On multiplying both sides by

$(\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$, we get

$(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C)$

$\times (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$

$= (\sec A - \tan A)^2 (\sec B - \tan B)^2 (\sec C - \tan C)^2$

$\Rightarrow \quad (\sec^2 A - \tan^2 A)(\sec^2 B - \tan^2 B)(\sec^2 C - \tan^2 C)$

$= (\sec A - \tan A)^2(\sec B - \tan B)^2(\sec C - \tan C)^2$

$\Rightarrow \quad 1\times1\times1 = [(\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)]^2$

$[\because \sec^2\theta - \tan^2\theta = 1]$

$\Rightarrow \quad (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C) = \pm 1$

8. *(c)* Given, $f(x) = \cos^2 x + \sec^2 x = \cos^2 x + \sec^2 x - 2 + 2$

[adding and subtracting 2]

$= \cos^2 x + \sec^2 x - 2\cos x \cdot \sec x + 2 \quad [\because \cos x \cdot \sec x = 1]$

$= (\cos x - \sec x)^2 + 2 \quad [\because a^2 + b^2 - 2ab = (a-b)^2]$

We know that square of any equation is always greater than equal to zero.

$\therefore \quad f(x) \geq 2$

9. *(b)* Given, ABC is a right angled triangle.

So, the sum of the angles of a triangle is 180°.

$\therefore \quad A + B + C = 180° \quad$ ...(i)

Now, $\tan\left(\frac{A-B-C}{2}\right) = \tan\left[\frac{A-(180°-A)}{2}\right] \quad$ [from Eq. (i)]

$= \tan\left(\frac{2A-180°}{2}\right) = \tan(A - 90°)$

$= \tan[-(90°-A)]$

$= -\tan(90°-A) \quad [\because \tan(-\theta) = -\tan\theta]$

$= -\cot A \quad [\because \tan(90°-A) = \cot A] \quad$ ...(ii)

and $-\tan\left(\frac{A+B-C}{2}\right) = -\tan\left(\frac{(180°-C-C)}{2}\right) \quad$ [from Eq. (i)]

$= -\tan(90°-C) = \cot C \quad$ ...(iii)

From Eq. (ii) and Eq. (iii), we get

$$\tan\left(\frac{A-B-C}{2}\right) \neq -\tan\left(\frac{A+B-C}{2}\right)$$

10. *(b)* We have,

$a\sec\theta + b\tan\theta + c = 0$

and $\quad p\sec\theta + q\tan\theta + r = 0$

On solving these two equations for $\sec\theta$ and $\tan\theta$ by cross-multiplication, we get

$$\frac{\sec\theta}{br-qc} = \frac{\tan\theta}{cp-ar} = \frac{1}{aq-bp}$$

$\Rightarrow \quad \sec\theta = \frac{br-cq}{aq-bp}$ and $\tan\theta = \frac{cp-ar}{aq-bp}$

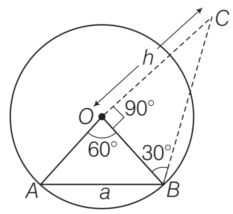
Now, $\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow \quad \left(\frac{br-cq}{aq-bp}\right)^2 - \left(\frac{cp-ar}{aq-bp}\right)^2 = 1$

$\Rightarrow \quad (br-cq)^2 - (cp-ar)^2 = (aq-bp)^2$

Chapter 19 Heights and Distances

1. *(b)* Let $OC = h$ m be the height of the tower at the centre of the circular park.



Given, $AB = a$ and $\angle AOB = 60°$.

Also, $\quad OA = OB \quad$ [same radius of a circle]

$\therefore \quad \angle OAB = \angle OBA \quad$ ...(i)

We know that, sum of all angles in a triangle is 180°.

$\therefore \quad \angle AOB + \angle OAB + \angle OBA = 180°$

$\Rightarrow \quad 60° + \angle OAB + \angle OAB = 180° \quad$ [from Eq. (i)]

$\Rightarrow \quad 60° + 2\angle OAB = 180°$

$\Rightarrow \quad 2\angle OAB = 120°$

$\Rightarrow \quad \angle OAB = 60°$

$\therefore \quad \angle OAB = \angle OBA = 60° \quad$ [from Eq.(i)]

Hence, ΔOAB is an equilateral triangle.

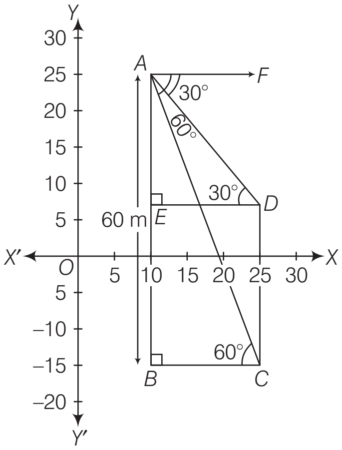
$\Rightarrow \quad OA = AB = OB = a$

In right angled ΔCOB,

$$\tan 30° = \frac{OC}{OB} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{a} \Rightarrow h = \frac{a}{\sqrt{3}}\text{ m}$$

Hence, height of the tower is $a/\sqrt{3}$ m.

Solutions (Q. Nos. 2-4) In the given figure, draw ED perpendicular to the line AB.



From the figure,

$$\angle FAD = \angle EDA = 30^\circ \quad \text{[alternate angles]}$$

and $$\angle FAC = \angle ACB = 60^\circ \quad \text{[alternate angles]}$$

2. *(c)* In right angled ΔABC, $\tan 60^\circ = \dfrac{AB}{BC}$

$$\Rightarrow \quad \sqrt{3} = \frac{60}{BC}$$

$$\Rightarrow \quad BC = \frac{60}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$\Rightarrow \quad BC = \frac{60\sqrt{3}}{3} = 20\sqrt{3} \text{ m}$$

Hence, the horizontal distance between BA and CD is $20\sqrt{3}$ m.

3. *(b)* In right angled ΔAED, $\tan 30^\circ = \dfrac{AE}{ED}$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{AE}{20\sqrt{3}} \Rightarrow AE = 20 \text{ m}$$

Now, $CD = BE = AB - AE = 60 - 20 = 40$ m

Hence, height of the lamp-post is 40 m.

4. *(d)* We know that, if two tangents of a circle are parallel, then distance between two tangents is equal to the diameter of a circle.

In the above figure, distance between Y-axis and AB is 10 m, which is equal to the diameter of a circle.

$\therefore$ Radius of required circle $= \dfrac{10}{2} = 5$ m

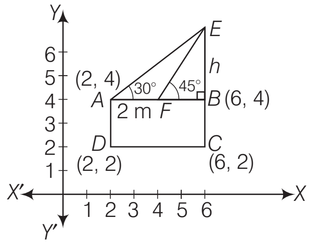
5. *(c)* When we plot all the points $A(2, 4)$, $B(6, 4)$, $C(6, 2)$ and $D(2, 2)$ on a graph paper and join them, a rectangle $ABCD$ is formed.

Let a pole of height $BE = h$ m be standing at point B.

Now, $AB = \sqrt{(6-2)^2 + (4-4)^2} = \sqrt{4^2 + 0^2} = 4$ m

$$[\because \text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}]$$

and $BC = \sqrt{(6-6)^2 + (2-4)^2} = \sqrt{(-2)^2} = \sqrt{4} = 2$ m



In right angled ΔABE, $\tan 30^\circ = \dfrac{BE}{AB} = \dfrac{h}{4}$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{4} \Rightarrow h = \frac{4}{\sqrt{3}} \text{ m}$$

Now, area of rectangle $ABCD = AB \times BC$

$$= 4 \times 2 = 8 \text{ m}^2$$

and area of $\Delta ABE = \dfrac{1}{2} \times AB \times BE$

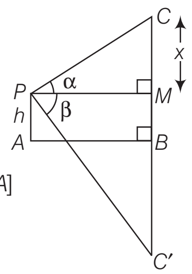
$$= \frac{1}{2} \times 4 \times \frac{4}{\sqrt{3}} = \frac{8}{\sqrt{3}} \text{ m}^2$$

$\therefore$ Total area of the figure = Area of rectangle $ABCD$ + Area of ΔABE

$$= 8 + \frac{8}{\sqrt{3}} = \frac{8(\sqrt{3}+1)}{\sqrt{3}} \text{ m}^2$$

6. *(a)* Let AB be the surface of the lake and P be a point of observation, such that $AP = h$ m.

Let C be the position of the cloud and C' be its reflection in the lake. Then, $CB = C'B$.



Let $PM \perp CB$.

Then, $\angle CPM = \alpha$ and $\angle MPC' = \beta$.

Let $CM = x$

Then, $CB = CM + MB = x + h$

$= CM + PA$ $\quad [\because MB = PA]$

In right angled ΔCMP,

$$\tan\alpha = \frac{CM}{PM}$$

$$\Rightarrow \quad \tan\alpha = \frac{x}{AB} \quad [\because PM = AB]$$

$$\Rightarrow \quad x = AB \cdot \tan\alpha$$

$$\Rightarrow \quad AB = \frac{x}{\tan\alpha}$$

$$\Rightarrow \quad AB = x\cot\alpha \quad \left[\because \tan\alpha = \frac{1}{\cot\alpha}\right] \quad \ldots\text{(i)}$$

Again, in right angled $\Delta PMC'$,

$$\tan\beta = \frac{C'M}{PM}$$

$$\Rightarrow \quad \tan\beta = \frac{x + 2h}{AB} \quad [\because C'M = C'B + BM = x + h + h = x + 2h]$$

$$\Rightarrow \quad AB = \frac{x + 2h}{\tan\beta}$$

$$\Rightarrow \quad AB = (x + 2h)\cot\beta \quad \left[\because \tan\beta = \frac{1}{\cot\beta}\right] \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$x\cot\alpha = (x + 2h)\cot\beta$$

$$\Rightarrow \quad x\cot\alpha = x\cot\beta + 2h\cot\beta$$

$$\Rightarrow \quad x(\cot\alpha - \cot\beta) = 2h\cot\beta$$

$$\Rightarrow \quad x\left(\frac{1}{\tan\alpha}-\frac{1}{\tan\beta}\right)=\frac{2h}{\tan\beta}$$

$$\Rightarrow \quad x\left(\frac{\tan\beta-\tan\alpha}{\tan\alpha\cdot\tan\beta}\right)=\frac{2h}{\tan\beta}$$

$$\therefore \quad x=\frac{2h\cdot\tan\alpha}{\tan\beta-\tan\alpha} \quad \ldots\text{(iii)}$$

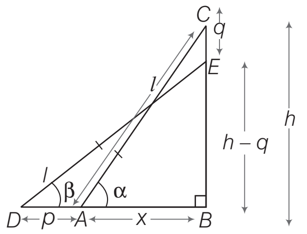
Hence, the height of the cloud, $CB = x + h$

$$=\left(\frac{2h\cdot\tan\alpha}{\tan\beta-\tan\alpha}\right)+h \quad \text{[from Eq. (iii)]}$$

$$=\left(\frac{2h\cdot\tan\alpha+h\tan\beta-h\tan\alpha}{\tan\beta-\tan\alpha}\right)$$

$$=\frac{h(\tan\alpha+\tan\beta)}{\tan\beta-\tan\alpha}\text{ m}$$

7. *(b)* Let $BC = h$ m be the height of the wall and $AC = l$ m be the length of the ladder, which rests against a vertical wall at $\angle BAC = \alpha$.



We have to prove that $\dfrac{\cos\beta-\cos\alpha}{\sin\alpha-\sin\beta}=\dfrac{p}{q}$.

When the ladder pulled away from the wall, its new position will be $DE = l$ m and $AD = p$ m.

Let $AB = x$ and $EC = q$.

In right angled ΔABC,

$$\sin\alpha=\frac{BC}{AC}=\frac{h}{l}\text{ and }\cos\alpha=\frac{AB}{AC}=\frac{x}{l}$$

In right angled ΔEBD,

$$\sin\beta=\frac{BE}{DE}=\frac{BC-EC}{DE}=\frac{h-q}{l}$$

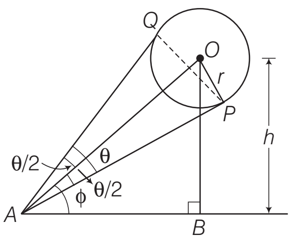
and $$\cos\beta=\frac{BD}{DE}=\frac{AB+DA}{DE}=\frac{p+x}{l}$$

Now, $$\frac{\cos\beta-\cos\alpha}{\sin\alpha-\sin\beta}=\frac{\frac{p+x}{l}-\frac{x}{l}}{\frac{h}{l}-\frac{h-q}{l}}=\frac{p+x-x}{h-(h-q)}=\frac{p}{q}$$

8. *(c)* Let A be the eye of the observer and O be the centre of the balloon.

Given, $OP = r$, $\angle PAQ = \theta$ and $\angle OAB = \phi$.

Let height of the centre of balloon be $OB = h$ m



In right angled ΔOAP, $\angle OPA = 90°$

[$\because$ radius is perpendicular to the tangent]

$$\Rightarrow \quad \sin\frac{\theta}{2}=\frac{OP}{OA}=\frac{r}{OA} \quad \ldots\text{(i)}$$

and in right angled ΔOBA,

$$\sin\phi=\frac{OB}{OA}=\frac{h}{OA} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{\sin\phi}{\sin\frac{\theta}{2}}=\frac{\frac{h}{OA}}{\frac{r}{OA}}=\frac{h}{r}\Rightarrow h=\frac{r\sin\phi}{\sin\frac{\theta}{2}}$$

$$\Rightarrow \quad h=r\sin\phi\,\text{cosec}\,\frac{\theta}{2}$$

Hence, the height of the centre of balloon is $r\sin\phi\,\text{cosec}\,\dfrac{\theta}{2}$.

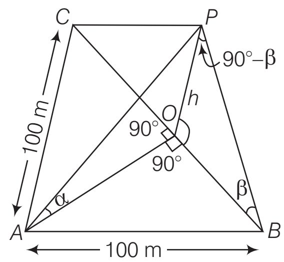
9. *(b)* Let OP be the clock tower of height h m, standing at the mid-point O of side BC of ΔABC.

Given, $\alpha = \angle PAO$ and $\beta = \angle PBO$.

$\because \quad \cot\alpha = 3.2$ and $\text{cosec}\,\beta = 2.6$

$$\therefore \quad \cot\beta=\sqrt{\text{cosec}^2\beta-1}=\sqrt{(2.6)^2-1}$$

$$=\sqrt{6.76-1}=\sqrt{5.76}=2.4$$



In right angled ΔPOA and ΔPOB, we have

$$AO = h\cot\alpha = 3.2h$$

and $$BO = h\cot\beta = 2.4h$$

In right angled ΔAOB, $AB^2 = OA^2 + OB^2$

$$\Rightarrow \quad 100^2 = (3.2h)^2 + (2.4h)^2$$

$$\Rightarrow \quad 100^2 = 16h^2$$

$$\Rightarrow \quad h^2=\frac{10000}{16}=625$$

$$\therefore \quad h = 25\text{ m}$$

[taking positive square root as side cannot be negative]

Hence, the height of the tower is 25 m.

10. *(a)* In right angled ΔAOP, we have

$$\tan\alpha=\frac{OP}{OA}=\frac{h}{OA}$$

Since, $\tan\alpha = 6$

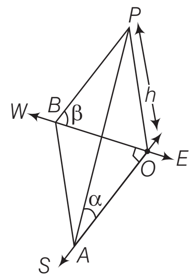
$$\therefore \quad OA=\frac{h}{6} \quad \ldots\text{(i)}$$

Also, in right angled ΔBOP, we have

$$\tan\beta=\frac{OP}{OB}=\frac{h}{OB}$$

Since, $\tan\beta = 0.75$

$$\therefore \quad OB=\frac{h}{0.75}=\frac{4h}{3} \quad \ldots\text{(ii)}$$



Now, in right angled ΔAOB, we have

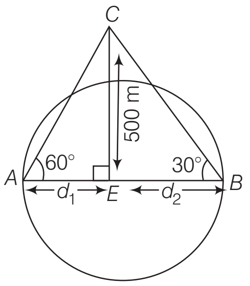
$$OA^2 + OB^2 = AB^2$$

$$\Rightarrow \quad AB^2 = \frac{h^2}{36} + \frac{16h^2}{9} \quad \text{[from Eqs. (i) and (ii)]}$$

$$\Rightarrow \quad \lambda^2 h^2 = h^2 \left(\frac{1+64}{36}\right) \quad [\because AB = \lambda h]$$

$$\Rightarrow \quad \lambda^2 = \frac{65}{36}$$

11. *(b)* According to the given information, we have the following figure



In ΔAEC, $\tan 60° = \dfrac{500}{d_1}$

$$\Rightarrow \quad d_1 = \frac{500}{\sqrt{3}} \text{ m} \quad [\because \tan 60° = \sqrt{3}] \text{ ...(i)}$$

and in ΔBEC, $\tan 30° = \dfrac{500}{d_2}$

$$\Rightarrow \quad d_2 = 500\sqrt{3} \text{ m} \quad \left[\because \tan 30° = \frac{1}{\sqrt{3}}\right] \text{...(ii)}$$

$\therefore$ Required diameter, $AB = d_1 + d_2$

$$= \frac{500}{\sqrt{3}} + 500\sqrt{3} \quad \text{[from Eqs. (i) and (ii)]}$$

$$= \frac{2000}{\sqrt{3}} \text{ m}$$

Chapter 20 Statistics

1. *(a)* Let the observations are $x_1, x_2, x_3, x_4, \ldots, x_n$. Now, if we divide each observation by x and increased each observation by y, then the observations become

$$\frac{x_1}{x} + y, \frac{x_2}{x} + y, \frac{x_3}{x} + y, \ldots, \frac{x_n}{x} + y.$$

$$\therefore \quad \text{Mean} = \frac{\left(\frac{x_1}{x} + y\right) + \left(\frac{x_2}{x} + y\right) + \ldots + \left(\frac{x_n}{x} + y\right)}{n}$$

$$= \frac{\frac{1}{x}(x_1 + x_2 + \ldots + x_n) + ny}{n}$$

$$= \frac{1}{x}\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) + y$$

$$= \frac{1}{x}m + y = \frac{m + xy}{x} \quad [\because \text{mean of } x_1, x_2, \ldots, x_n \text{ is } m]$$

2. *(d)* As x, y and $2x$ are in ascending order, therefore median is y.

$$\Rightarrow \quad y = 27$$

Also, $\quad \text{mean} = \dfrac{x + y + 2x}{3} = 33$

$$\Rightarrow \quad \frac{3x + 27}{3} = 33 \Rightarrow x = 24$$

$\therefore$ Mean of x and $y = \dfrac{x + y}{2} = \dfrac{24 + 27}{2} = \dfrac{51}{2} = 25.5$

3. *(b)* Let n_1 be the number of boys and n_2 be the number of girls.

Then, total ages of boys $= 14 \times n_1$

and total ages of girls $= 17 \times n_2$

Now, average of children $= \dfrac{14n_1 + 17n_2}{n_1 + n_2}$

$$\Rightarrow \quad 15 = \frac{14n_1 + 17n_2}{n_1 + n_2}$$

$$\Rightarrow \quad 15n_1 + 15n_2 = 14n_1 + 17n_2$$

$$\Rightarrow \quad n_1 = 2n_2 \Rightarrow \frac{n_1}{n_2} = \frac{2}{1}$$

or $\quad n_1 : n_2 = 2 : 1$

4. *(d)* Given, 10 boys brought 11 books each, therefore total number of books brought by 10 boys $= 10 \times 11 = 110$

Also, as 6 girls brought 13 books each.

$\therefore$ Total number of books brought by 6 girls $= 6 \times 13 = 78$

Clearly, the remaining number of students $= 4$

Now, let the number of books brought by 4 students be x.

Then, $x \geq 10$

[$\because$ no two students brought the same number of books]

Thus, the average number of books brought in the class

$$= \frac{x + 78 + 110}{20}$$

It is given that, average number of books brought in the class is a positive integer, therefore the minimum value of x is 12.

5. *(d)* Sum of expenditure from Monday to Wednesday

$= 250 \times 3 =$ ₹ 750

Sum of expenditure from Wednesday to Friday

$= 400 \times 3 =$ ₹ 1200

$\because$ Expenditure on Wednesday = ₹ 300

$\therefore$ Expenditure from Monday to Friday

$= 750 + 1200 - 300 = 1650$

Hence, the mean expenditure from Monday to Friday

$= \dfrac{1650}{5} =$ ₹ 330

6. *(c)* Mean of the observations of sets

$$P = \frac{3 + 5 + 9 + 12 + x + 7 + 2}{7} = \frac{38 + x}{7}$$

$$Q = \frac{8 + 2 + 1 + 5 + 7 + 9 + 3}{7} = \frac{35}{7} = 5$$

and $\quad R = \dfrac{5 + 9 + 8 + 3 + 2 + 7 + 1}{7} = \dfrac{35}{7} = 5$

$\therefore$ Ratio of means of sets P and $Q = 7 : 5$ [given]

Let P's mean $= 7y$ and Q's mean $= 5y$.

$$\Rightarrow \quad 5y = 5$$

$$\Rightarrow \quad y = 1$$

$\therefore$ Q's mean = 5

Now, P's mean = $7y$

$$\Rightarrow \frac{38+x}{7} = 7 \times 1 \Rightarrow 38 + x = 48 \Rightarrow x = 11$$

$\therefore$ Mean of P : Mean of $R = 7y : 5 = 7 \times 1 : 5 = 7 : 5$

7. *(b)* We know that,

$$\text{Mode} = 3\text{ Median} - 2\text{ Mean}$$

On dividing both sides by median, we get

$$\frac{\text{Mode}}{\text{Median}} = 3 - 2\frac{\text{Mean}}{\text{Median}}$$

$$\Rightarrow \frac{6}{5} = 3 - 2\frac{\text{Mean}}{\text{Median}} \quad \left[\because \frac{\text{mode}}{\text{median}} = \frac{6}{5}, \text{given}\right]$$

$$\Rightarrow \frac{6}{5} - 3 = -2\frac{\text{Mean}}{\text{Median}}$$

$$\Rightarrow \frac{6-15}{5} = -2\frac{\text{Mean}}{\text{Median}}$$

$$\Rightarrow \frac{-9}{5} = -2\frac{\text{Mean}}{\text{Median}}$$

$$\Rightarrow \frac{\text{Mean}}{\text{Median}} = \frac{9}{10}$$

8. *(c)* Let the least possible value of x be n. Then,

$$\text{Range} = \text{Maximum value} - \text{Minimum value}$$

$$\Rightarrow 23 = 35 - n$$

$$\Rightarrow n = 35 - 23 = 12$$

Hence, the least value of x is 12.

Chapter 21 Probability

1. *(a)* In a wall clock, the minute hand cover the 60 min in one complete round.

$\therefore$ Total number of possible outcomes = 60

The minute hand cover the time from 5 to 15 min,

Number of outcomes favourable to E

= Distance from 5 to 15 min = 10

$$\therefore \text{ Required probability} = \frac{10}{60} = \frac{1}{6}$$

Solution (Q.Nos. 2-4) Total number of circles = 34

$\therefore$ Total number of possible outcomes = 34

2. *(a)* Number of circles having diameter less than 28.

Thus, number of outcomes favourable to $E = 3 + 5 = 8$

[$\because$ number of circles having diameter 14 – 20 = 3 and 21 – 27 = 5]

$$\therefore \quad P(E) = \frac{8}{34} = \frac{4}{17}$$

3. *(b)* Number of circles having radius lying between 14 to 17.

Thus, number of outcomes favourable to $F = 8$

$$\therefore \quad P(F) = \frac{8}{34} = \frac{4}{17}$$

4. *(c)* Number of circles having diameter above 50.

Thus, number of outcomes favourable to $G = 0$

$$\therefore \quad P(G) = \frac{0}{34} = 0$$

5. *(c)* Area of the square $ABCD = (BC)^2 = 10^2 = 100\text{ cm}^2$

and area of the square $PQRS = (\text{Side})^2 = (5)^2$

[$\because$ side = 5 cm, given]

$= 25\text{cm}^2$

$\therefore$ P (that the point will be chosen from the shaded part)

$$= \frac{\text{Area of the square } PQRS}{\text{Area of the square } ABCD} = \frac{25}{100} = 0.25$$

6. *(b)* Area of rectangle = $3 \times 2 = 6\text{ m}^2$

$$E = \text{Area of circle} = \pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}\text{ m}^2$$

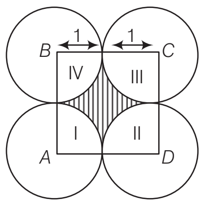
$$\left[\because \text{diameter} = 1\text{m} \Rightarrow \text{radius} = \frac{1}{2}\text{m and area of circle} = \pi r^2\right]$$

$\therefore$ Probability that the die land inside the circle

$$= \frac{\text{Area of circle}}{\text{Area of rectangle}} = \frac{\pi/4}{6} = \frac{\pi}{24}$$

7. *(b)* Since, radius of each of the given circles is 1 unit.

$\therefore$ Side of square $ABCD = 1 + 1 = 2$ units



Now, area of square = $2^2 = 4$ sq units

Clearly, area of shaded region = area of square
– area of quadrant I – area of quadrant II
– area of qudrant III – area of quadrant IV

$= 4 - 4 \times$ area of a quadrant

$$= 4 - 4 \times \frac{\pi(1)^2}{4} \quad \left[\because \text{area of a quadrant} = \frac{1}{4}\pi r^2\right]$$

$= 4 - \pi$

$\therefore$ Required probability

$$= \frac{\text{Area of shaded region}}{\text{Area of square}} = \frac{4-\pi}{4}$$

Internal Assessment of Project Work

Project 1

Topic

Comparative newspaper coverage of different items

1. Introduction
2. Importance of Newspaper
3. Different Items Covered by Newspaper
4. Comparative Study of Different Items
5. Conclusion

1. Introduction

Newspaper is defined as 'a printed paper or publication, reporting local or general news, or printed editorial comment, announcement, miscellaneous, reading matter, commercial advertising, classical advertising, legal advertising and other notices and comment on public news'.

Each newspaper tries to establish its identity and win the loyalty of its readers through a combination of words, pictures, cartoons, presentations, techniques and exclusive news stories. Newspaper also contains public grievances and reflects public opinion.

In India, newspapers have undeniably a vital role to play and an important duty to perform both as voice of people and a builder of public opinion.

It is believed that not more than 17% people in India read newspapers. Young men mostly read the sports news and film reviews, the traders and other business-minded people fix themselves to the commercial pages which provides them the latest market quotations of shares, stock, etc.

2. Importance of Newspaper

Newspaper performs its true role as a guardian of the public interest, a watch dog and a source of all kinds of information.

Newspapers give us all types of news about our country and the world. All the government policies are published by the newspapers. Newspapers not only give us the latest news but also publish advertisements for jobs and many other things.

People of all profession need newspaper. Engineers, doctors, scientists, professors and other professionals need them to know the day to day developments in their respective fields.

Newspapers give us knowledge about many things of the past and present. Without these, people cannot get knowledge and news about the country and the world. Thus, in our society, newspapers are very important.

3. Different Items Covered by Newspapers

(i) Local News
(ii) State News
(iii) National News
(iv) International News
(v) Political News
(vi) Sports News
(vii) Classified News
(viii) Business News
(ix) Educational News
(x) Entertainment News

4. Comparative Study of Different Items of a News

Sports News

(i) **Headline of Main Sports News** (Cricket News) News on the match played on 26.02.2015 between India and South Africa.

	Newspaper	Date	Headline
a	The Hindu	23.10.2015	India won the match against South Africa by 6 wickets
b	The Times of India	23.10.2015	Virat Kohli made another century
c	The Indian Express	23.10.2015	India *vs* South Africa : Virat Kohli lead Indian Cricket Team

(ii) **Presentation of News**

	Newspaper	Headline
a	The Hindu	Presents this news very exclusively and in details.
b	The Times of India	Present this news in short.
c	The Indian Express	Presents this news effectively and in a remarkable manner.

(iii) **Pictures of News**

	Newspaper	Headline
a	The Hindu	Highlights the picture of Indian batsman Kohli who celebrates his century against South Africa during the fourth ODI in Chennai in coloured mode.
b	The Times of India3	Also presents the picture of Indian bastman (Virat Kohli, Dhoni) in coloured mode.
c	The Indian Express	Presents the picture of Virat Kohli and AB de villiers in coloured mode.

(iv) **Views of Experts about the News**

	Newspaper	Headline
a	The Hindu	Kohli said he knew he would have to push himself more than usual given the conditions he was batting in.
b	The Times of India	Virat Kohli should be allowed to bat No. 3 said Sunil Gavaskar
c	The Indian Express	Aakash Chopra a sports critic says that "Duminy's absence has meant that SA forced to go with Morris and not Abbott picked a bowel for his batting huge toss to win for India"

(v) **Format of the News**

	Newspaper	Headline
a	The Hindu	Presents the news in a simple way in three or four small columns with pictures.
b	The Times of India	Presents the news in good manner with the extended views of experts in four columns.
c	The Indian Express	Presents the news at almost full sports page with some small and big pictures in three column.

5. Conclusion

So, we can say that newspaper plays very important role in the society. By comparing the news of some item in different news papers, we can judge the truthfulness and standard of that news and can choose the correct and genuine newspaper for reading.

Project 2

Topic

Survey of various types of Bank Accounts, rates of interest offered

Table for Contents

1. Types of Bank Accounts
2. Calculation of Interest
3. Prevailing Rate of Interest by State Bank of India
4. Prevailing Rate of Interest by Bank of Baroda.

1. Types of Bank Accounts

Bank is an institution, which attracts money on deposits for the purpose of lending to trade, industry, etc. Receiving deposits from the public is an important function of a commercial bank. A bank provides the facility to open different kinds of deposit accounts with various facilities to suit the needs of various customers or depositors.

There are mainly five types of accounts which a person can open with a bank

(i) **Fixed or Time Deposit Account** Fixed deposit account is one where money is deposited for a fixed period and is not supposed to be withdrawn before the expiry of the commit period. This period usually varies from 15 days to five years. The rate of interest allowed on such accounts generally increases with the period of deposit. This account is also known as "Term Deposit Account".

This is the most suitable form of accepting deposits for a commercial bank.

Banks allow a higher rate of interest to attract such deposits. Customers deposit their money in such account with a view to earn interest as well as to withdraw the same on the expiry of the period of the deposit.

(ii) **Savings Bank Deposit Account** This is an account into which small savings are deposited into bank by the customers. This account is meant for the benefit of middle class and low-income group people. Any person with a minimum specific deposit can open a Saving Bank Deposit Account.

The special feature of this account is deposits can be made for any number of times in a week but withdrawals can be made only once or twice a week. Banks, to discourage the habit of frequent withdrawals, impose restrictions on withdrawals.

(iii) **Current Deposit Account or Current Account** It is one in which money can be deposited and withdrawn at any time during the working hours without giving any notice to the bank. This account may be defined as running account between a banker and a customer. Current accounts suit the requirements of businessmen, companies, corporations, institutions, firms, etc.

It is also known as "Open Account".

Since customers can deposit money into or withdraw money from their current accounts whenever they like.

(iv) **Recurring Deposit Account** This is a special type of bank accounts, in which a depositor deposits fixed amount every month for a fixed period of time. At the end of the period depositor gets all the principal sum, alongwith the interest earned during that period.

(v) **Flexible Accounts** These accounts combine the characteristics and benefits of savings account and fixed deposit accounts. In such accounts a fixed deposit account is opened for the maturity periods available and a savings account is also opened. The amount in fixed deposit account is treated as units of ₹1000 each. In case of customer needs money in excess of balance in the savings account, the requisite amount will be transferred out of fixed deposit account in units of ₹1000 each along with interest payable on the amount.

2. Calculation of Interest

Calculating Interest on a Savings Bank Account

To calculate the interest on a savings account proceed as under

(i) Interest for the month is calculated on the minimum balance between the 10th day and the last day of the month.

(ii) Convert the minimum balance of each month as a multiple of ₹10. No interest is paid up to ₹ 5.

(iii) Add all these balances. However, if the same balance continues for n months then multiply this balance by n, rather than writing it n times and then adding.

(iv) Find the simple interest on this sum for one month.

(v) If the interest is less than ₹ 1, neglect it.

(vi) No interest is paid for the month in which the account is closed. e.g. A page from savings bank passbook of Mr. Brar is given below

Date, Year 2006	Particulars	Withdrawals (in ₹)	Deposits (in ₹)	Balance (in ₹)
Jan 1	B/F			3142.35
Jan 1	By Salary	...	2718.50	5860.85
Jan 6	To self	2500.00	...	3360.85
Jan 13	To cheque	255.00	...	3105.85
Feb 3	By salary	...	2805.40	5911.25
March 2	By salary	...	2805.40	8716.65
March 10	To self	3000.00	...	5716.65
April 4	By salary	...	2805.40	8522.05
April 23	To cheque	2625.00	...	5897.05
May 5	By salary	...	3060.70	8957.75
May 8	To self	5000.00	...	3957.75
May 13	By cash	...	650.00	4607.75

Mr. Brar closes his account in the period ending May, 2006. Calculate the interest earned for this period if the rate of interest is 5%.

Sol.

Month	Minimum balance between 10th and last day	Qualifying amount for interest
Jan	3105.85	3110
Feb	5911.25	5910
March	5716.65	5720
April	5897.05	5900
May	3957.75	3960
Total		**₹ 24600**

Principal for one month = ₹ 24600,

Time = $\frac{1}{12}$ yr, Rate = 5% per annum

$$\therefore \text{ Interest} = \frac{P \times R \times T}{100}$$

$$= \frac{24600 \times 5 \times 1}{100 \times 12} = ₹102.50$$

(i) **Calculating Interest on a Recurring Deposit Account**

e.g. Aman has a recurring deposit account of ₹2500 per month for a period of 6 yr. What interest will he get at the time of maturity if the rate of interest is 11.5% per annum?

Sol. Equivalent principal for one month = ₹ 2500

Time = 6 yr (72 months), Rate of interest = 11.5% per annum

$$\text{Interest} = \frac{xI \times (x+1)}{2} \times \frac{r}{100} \times \frac{1}{12}$$

[where, I = monthly instalment, x = number of months, r = rate of interest]

$$\therefore \text{ Interest} = \frac{2500 \times 72 \times (72+1)}{2} \times \frac{11.5}{100} \times \frac{1}{12} = ₹62962.50$$

3. Prevailing Rates of Interest by State Bank of India : Fixed Deposit

Duration (in years)	Rate of interest
6 Months	5%
1-2 yr	5.5%
3-5 yr	5.75%
Above 5 yr	6.25%

Recurring Deposit = 5.5 %

Savings Account = 3.5 % per annum

4. Prevailing Rates of Interest by Bank of Baroda : Fixed Deposit

Duration	Rate of interest
7-14 days	3.75%
15-19 days	4.25%
30-45 days	4.50%
76-89 days	4.70%
90-179 days	5.00%
180 days-9 months	5.25%
9 months-1 yr	5.50%
1-2 yr	5.75%
2-5 yr	6.00%
5 yr and above	6.25%
Senior Citizens Extra	0.50%

Recurring Deposit = 5.75%

Savings Account = 3.5%

Project 3

Topic

Planning a Home Budget

1. Introduction
2. Monthly income of family
3. Monthly expenditure of family
4. Family size
5. Bar graph
6. Pie chart
7. Savings
8. Pie chart
9. Conclusion

1. **Introduction** A home budget is a snapshot of income and expenses, usually expressed over a period of one month. After adding up all of the sources of income, you can calculate total amount available to spend.

2. **Monthly Income of Family**

I. Father → (Mr. Amit Chhabra)

Income from salary	₹ 65000
Income from other sources	₹ 10000
Deductions (GPF, LIC, etc.)	₹ 10000
Taxes	₹ 15000
Total income	₹ 50000

II. Mother → (M/s Sunita Chhabra)

Income from salary	₹ 20000
Deductions (GPF, LIC etc.)	₹ 3000
Taxes	₹ 4000
Total income	₹ 13000

Grand total income of family

= ₹ 50000 + ₹ 13000 = 63000

3. **Monthly Expenditure of Family**

Category	**Expenditure (in ₹)**
House Maintainance	4000
Grocery + fruits + vegetables	12000
Milk	5000
Electricity + water	2500
Conveyance	1000
Clothing	2500
Education	5000
Entertainment	2000
Automobiles	2000
Telephone + Mobile	2000
Home helpers	1500
Repairs	500
Cable + Internet connection	500
Travelling	2000
Laundary	500
Shopping	2500
Stationery	500
Medical	3000
Grand total	₹ 53000

∴ Monthly expenditure = ₹ 53000

4. **Family Size** Our income and expenditure are totally depend on the size of family. Family size varies from family-to-family. The general family size is as follows:

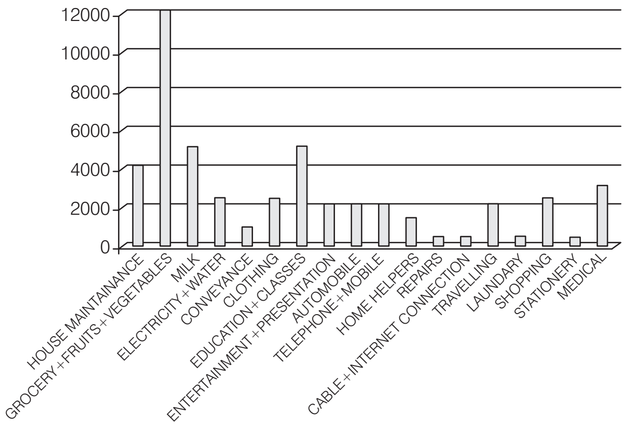
Father → One (Mr. Amit Chhabra)

Mother → One (M/s Sunita Chhabra)

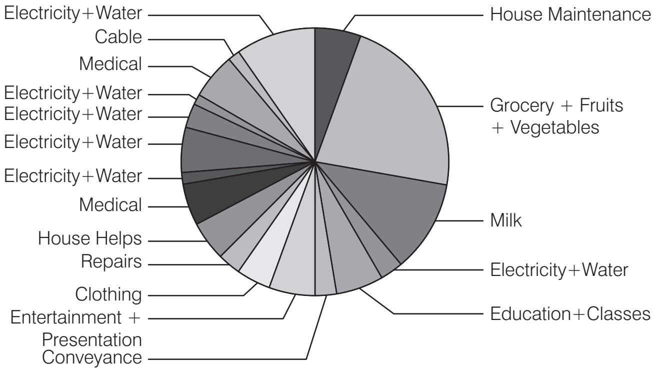
Children → Two (Pawan and Sheena)

Total members of family → Four

5. **Bar Graph Comparing all the Expenses**



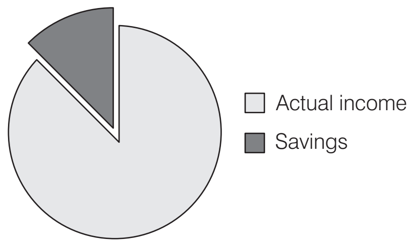
6. **Pie Chart Comparing all the Expenses**



7. **Savings** Some saving is important for the better future of the family. This saving can be helpful at the time of requirement or need. So, every family make saving according to their need and reach.

Total savings = Total income − Total expenditure

= ₹ 63000 − ₹ 51000 = ₹ 12000



Conclusion

From the above we conclude that planning a Home-Budget is an essential for a family with the help of this planning, we can actuals estimate that how much we should spend on different elements.

Project 4

Topic

Conduct a survey in your locality to study the mode of conveyance/prices of various essential commodities favourite sports. Represent the data, using a bar graph/ histogram and estimate the mode.

1. Mode of conveyance
2. Prices of various essential commodities
3. Favourite sports

1. Mode of Conveyance

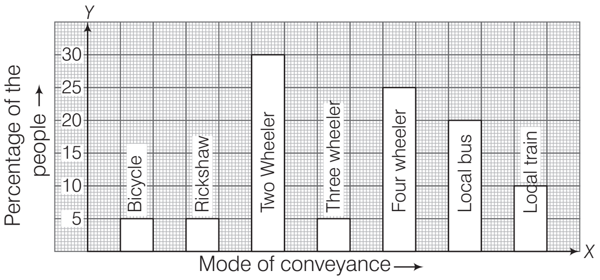
(i) **Introduction** In our locality people use various type of mode of conveyance such as bicycle rickshaw, two wheeler, three wheeler, four-wheeler, local bus, local train and etc for doing their day-to-day work.

On the basis of their work age and financial conditions they use various types of modes of conveyance.

(ii) **Description** The percentage (%) of various types of modes of conveyance used by the people of our locality are given as

Mode of conveyance	Percentage of people
Bicycle	5%
Rickshaw	5%
Two wheeler	30%
Three wheeler	5%
Four Wheeler	25%
Local bus	20%
Local train	10%

Representation of the above data using bar graph is given as

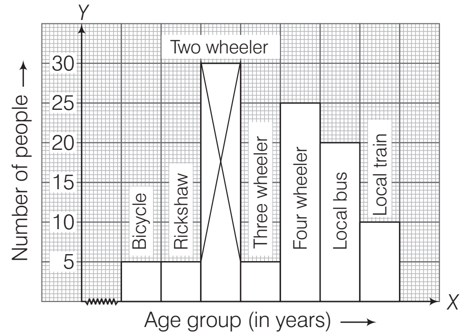


The number of people used various types of modes of conveyance according to their age are given as below

Let take sample of 100 people.

Age group (in years)	Mode of conveyance	Number of people
15-20	Bicycle	5
20-25	Rickshaw	5
25-30	Two wheeler	30
30-35	Three wheeler	5
35-40	Four wheeler	25
40-45	Local bus	20
45-50	Local train	10

Representation of the above data using histogram is given as



(iii) **Conclusion** Mode = 27 yr

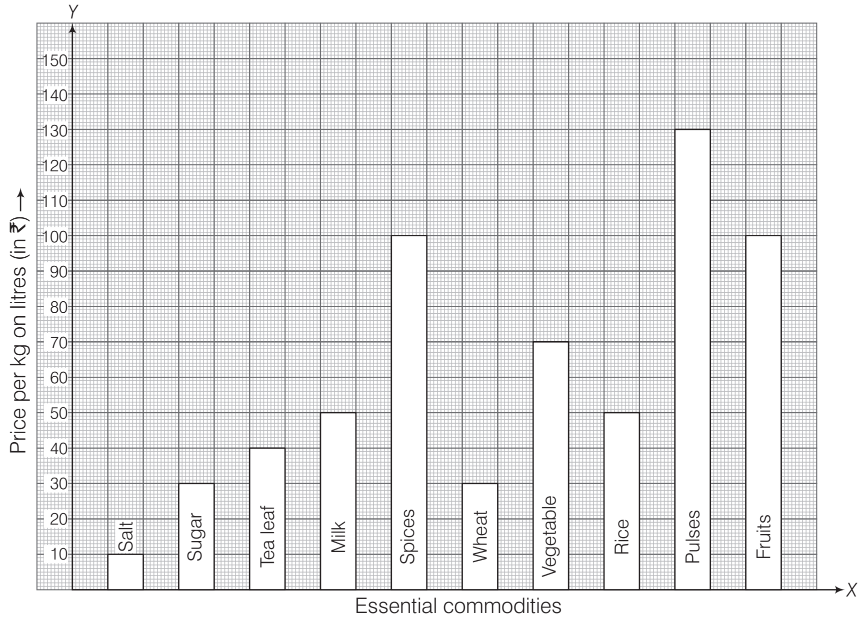
2. Prices of Various Essential Commodities

(i) **Introduction** In our locality people use various essential commodities in their daily life to fulfil their daily needs. On the basis of their essential commodities.

(ii) **Description** The price of various essential commodities used by people of our locality are given as

Essential Commodities	Price per kg /litre (in ₹)
Salt	10
Sugar	30
Tea leaf	40
Milk	50
Spices	100
Wheat	30
Vegetable	70
Rice	50
Pluses	130
Fruits	100

Representation of above data using bar graph is given as



(iii) **Conclusion** The price of various essential commodities indicate the rate of inflation in our country.

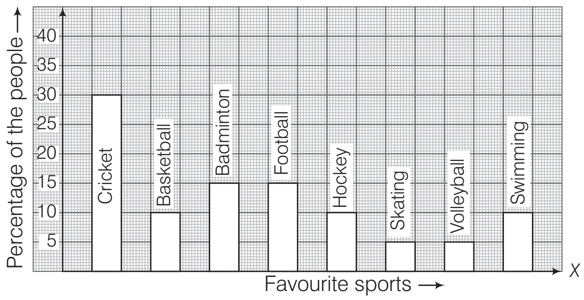
3. Favourite Sports

(i) **Introduction** In our locality people plays different sports in their day-to-day life to maintain their good health as per their age and ability.

(ii) **Description** The percentage (%) of the people playing different sports in their day-to-day life are given as

Favourite sports	Percentage (%) of people
Cricket	30%
Basketball	10%
Badminton	15%
Football	15%
Hockey	10%
Skating	5%
Volleyball	5%
Swimming	10%

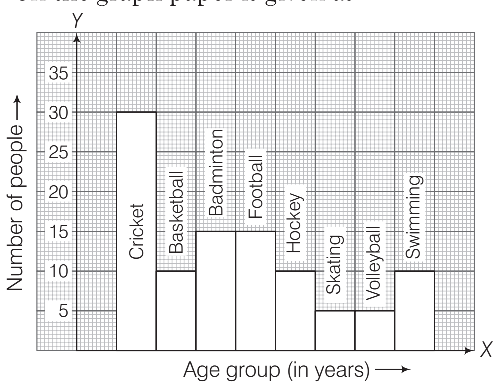
Representation of above data using bar graph is given as



The number of people, play various sports as per their age are given below. Let us take as sample of 100 people

Age group (in years)	Favourite sports	Number of people
5-10	Cricket	30
10-15	Basketball	10
15-20	Badminton	15
20-25	Football	15
25-30	Hockey	10
30-35	Skating	5
35-40	Volleyball	5
40-45	Swimming	10

Representation of above data using histogram on the graph paper is given as



(iii) **Conclusion** Mode = 7 yr

Project 5

Topic

To use a Newspaper to Study and Report on Shares and Dividends

Important Points

1. Introduction
2. Business news
3. Meaning of shares and dividends
4. Discussion of various shares and dividends of different companies.
5. Comparative study of shares of different companies.
6. Comparative study of profit on sale/purchase of shares of different companies.
7. Conclusion

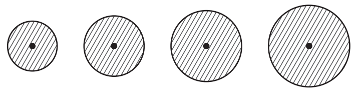
Now, do same as Project 1.

Project 6

Topic

Set up a dropper with ink in it vertical at a height say 20 cm above a horizontally placed sheet of plain paper. Release one ink drop; observe the pattern, if any, on the paper. Vary the vertical distance and repeat. Discover any pattern of relationship between the vertical height and the ink drop observed.

1. **Material used** Ink, dropper, plain paper sheet, scale.
2. **Procedure** Place a plain paper sheet horizontally and fill the dropper with ink, now release one ink drop from dropper on the sheet from different heights at different places.
3. **Observation** We see circular shapes of released drop of ink on the paper sheet and observe that "As the height of dropper increases then the size of circular shapes also increases," as shown in figure.



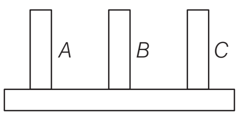
Radius	1 cm	2 cm	3 cm	4 cm
Height of dropper	10 cm	20 cm	30 cm	40 cm

4. **Conclusion** Radius of circular shapes (R) is directly proportional to the height of dropper (H) above the paper sheet.

Project 7

Topic

You are provided (or you construct a model as shown) three vertical sticks (size of a pencil) stuck to a horizontal board. You should also have discs of varying sizes with holes (like a doughnut). Start with one disc; place it on (in) stick A. Transfer it to another stick (B or C); this is one move (m). Now try with two discs placed in A such that the large disc is below and the smaller disc is above (number of discs $n = 2$). Now transfer them one at a time in B or C to obtain similar situation (larger disc below). How many moves? Try with more discs ($n = 1, 2, 3$, etc.) and generalise.

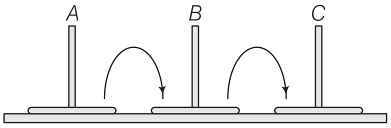


1. **Material used** Three vertical sticks, horizontal board, discs of varying size having holes.
2. **Procedure**

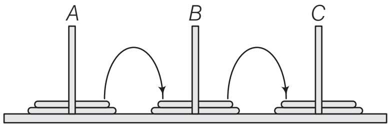
   ***Case* I** Stuck three vertical sticks on a horizontal board 30 cm apart. Now start with one disc, place it on (in) stick A.

Projects

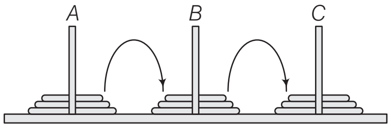
Transfer it to another stick B of C.



Case II Now try with two discs placed in A such that the larger disc is below and the smaller disc is above. Now, transfer them one at a time in B or C to obtain similar situation.



Case III Similarly, try this experiment with three, four, etc, discs to obtain similar situation.



3. **Observation**

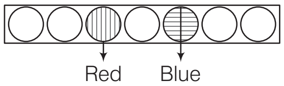
Number of discs	1	2	3	4
Number of moves	2	4	6	8

4. **Conclusion** Number of moves of disc $= 2 \times$ Number of disc used.

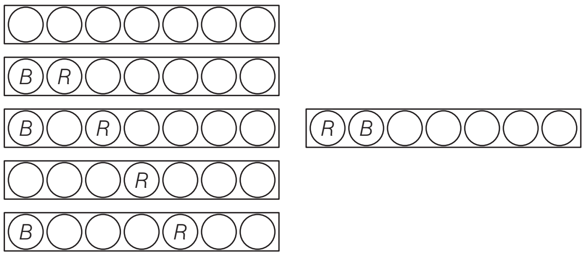
Project 8

Topic

The board has some holes to hold marbles, red on one side and blue on other. Start with one pair. Interchange the positions by making one move at a time. A marble can jump, over another to fill the hole behind. The move (m) equal 3. Try with $2\,(n = 2)$ and more. Find relationship between n and m.



1. **Material used** Card board having holes, red and blue marbles.
2. **Procedure** Set a card board having seven holes horizontally and then fill the holes by using red and blue marbles, as red one side and the blue on the other side. Now interchange the positions by making one move, at a time in any manner.



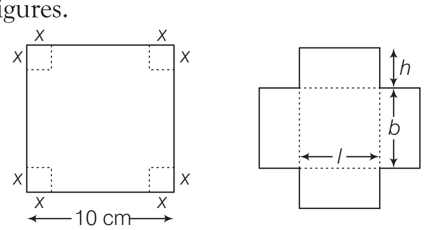
3. **Calculations** Now count the number of moves (m). So, $m = {}^7P_2 = 7 \times 6 = 42$. When $n = 2$ $n \to$ (number of marbles)
4. **Conclusion** $m = {}^qP_n$ where $m =$ Number of moves, $q =$ Number of holes, $n =$ Number of marbles.

Project 9

Topic

Take a square sheet of paper of side 10 cm four small squares are to be cut from the corners of the square sheet and then paper folded at the cuts to form an open box. What should be the size of the squares cut so that the volume of the open box is maximum?

1. **Material used** Pencil, square sheet of paper of side 20 cm, a pair of scissors, scale.
2. **Procedure**
   (i) Take a square sheet of paper of side 20 cm.
   (ii) Cut small squares from the corners of the squares sheet say
   'x' cm and then the paper folder at the cuts to form an open box as shown in the following figures.



Projects

3. **Observation** In this way, we find different open box of different volume by taking the different values of x

Calculation $V = l \times b \times h$

where, $V =$ volume of the square box,
$l =$ length of the square box,
$b =$ breadth of the square box,
and $h =$ height of the square box.

Let $h = x$ cm and each side of square $= 20$ cm.

For different cases, we have different volume which are given as

S.No.	$h = x$ cm	l (in cm)	b (in cm)	V (in cm^3) $[V = l \times b \times h]$
1.	1	18	18	324
2.	$\frac{5}{3}$	$\frac{50}{3}$	$\frac{50}{3}$	$\frac{50}{3} \times \frac{50}{3} \times \frac{50}{3} = 462.96$
3.	2	16	16	512
4.	3	11	11	363

For $x = 2$ cm the volume of the open box is maximum.

4. **Conclusion** From the above we conclude that the volume of the open box is totally depend on the value of x.

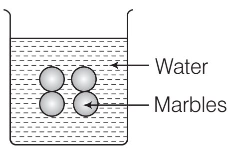
Project 10

Topic

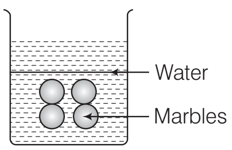
Take an open box, four sets of marbles (ensuring that marbles in each set are of same size) and some water. By placing the marbles and water in the box, attempt to answer the question do larger marbles or smaller marbles occupy more volume in a given space?

1. **Material used** An open box, four sets of marbles, some water.
2. **Procedure** Take an open box and put the set of marbles and some water into the box. (Set of marbles has 4 marbles of diameter 2 cm each). Repeat the experiment with other set of marbles of different size.
3. **Observation**

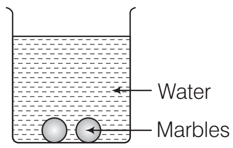
(i) Larger marbles occupy more volumes in a given space.

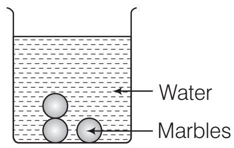


(ii) Smaller marbles occupy less volumes in a given space.

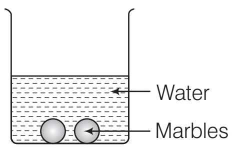


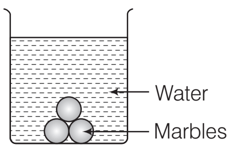
(iii) If the number of marbles, and size of marbles are different, then volumes occupied may by same.





(iv) If the number of marbles is different, then volume occupied by marbles is also different.





4. **Conclusion**

(i) If the number of marbles is same, then volume occupied by marbles is directly proportional to the size of marble.

(ii) If the size of marbles is same, then volume occupied by marbles is directly proportional to the number of marbles.

Project 11

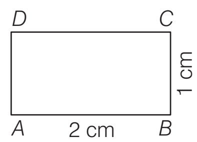
Topic

An eccentric artist says that the best paintings have the same area as their perimeter (numerically). Let us not argue whether such sizes increase the viewers appreciation, but only try and find what sides (in integers only) a rectangle must have if its area and perimeter are to be equal (note : there are only two rectangles).

1. **Introduction** An eccentric artist says that the best paintings have the same area as their perimeter in case of rectangles.
2. **Procedure** Draw different rectangles of different dimensions as

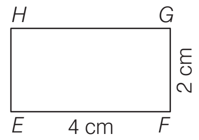
   Case **I** In rectangle *ABCD*,

   $$A = l \times b, P = 2(l + b)$$



$$\Rightarrow \quad A = 1 \times 2 = 2\,\text{cm}^2 \text{ and } P = 2(1+2)6\,\text{cm}$$

Case **II** In rectangle *EFGH*,

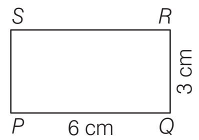


$$A = 2 \times 4 = 8\,\text{cm}^2$$

$$P = 2(2+4) = 12\,\text{cm}$$

Case **III** In rectangle *PQRS*,

$$A = 3 \times 6 = 18\,\text{cm}^2$$



$$P = 2(3+6) = 18\,\text{cm}$$

Here, we observe that in the third case area and perimeter is same.

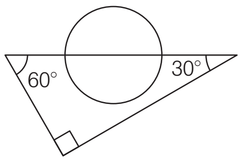
3. **Conclusion** Hence, if length of the rectangle is 6 cm and breadth of rectangle is 3 cm, then area and perimeter will be same numerically.

Project 12

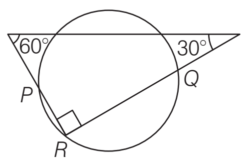
Topic

Find by construction the centre of a circle, using only a 60-30 set square and a pencil.

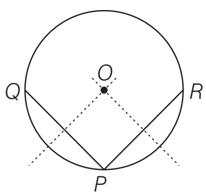
1. **Material used** Plain paper, 60°-30° set square, pencil etc.
2. **Procedure**
   (i) Draw a circle of any radius.
   (ii) Now, put 60°-30° set square on the circle as



   (iii) Now, move the set square upward so as to bring one corner of set square at the circumference of the cirlce as



   (iv) Draw the line *PQ* and *PR* along the sides of set square as chords of circle now remove the set square.
   (v) Draw perpendicular bisector of *PQ* and *PR* which meet at a point O, which is centre of the circle



3. **Conclusion** O is the required centre of the given circle.

Project 13

Topic

Various types of 'Cryptarithm'.

A cryptarithmetic is a type of mathematical puzzle in which the digits are replaced by letters of the alphabet or other symbols. To find the solution you have to find the original digits. Making and solving such puzzles are called **cryptarithmetic**.

1. **Let us understand the rules to solve such problems**

Each letter or symbol represents only one digit throughout the problem.

When letters are replaced by their digits, the resultant arithmetical operation must be correct.

Numbers must not begin with a zero.

There must be only one solution to the problem.

2. **We follow the following steps to solve such problems**

***Step* I** Write the problems in columns, leaving gaps in between letters for writing trial numbers.

***Step* II** A good hint to find zero or 9 is to look for columns containing two or three identical letters.

Look at these additions

$$\begin{array}{r} *\,*\,*A \\ +\,*\,*\,*A \\ \hline *\,*\,*A \end{array} \qquad \begin{array}{r} *\,*\,*B \\ +\,*\,*\,*A \\ \hline *\,*\,*B \end{array}$$

The columns $A+A=A$ and $B+A=B$ indicate that A = zero. In math this is called the "additive identity property of zero" ; it says that you add "0" to anything and it does not change, therefore it stays the same. Now, look at those same additions in the body of the cryptarithm.

$$\begin{array}{r} *\,A\,*\,* \\ +\,*\,A\,*\,* \\ \hline *\,A\,*\,* \end{array} \qquad \begin{array}{r} *\,B\,*\,* \\ +\,*\,A\,*\,* \\ \hline *\,B\,*\,* \end{array}$$

In these cases, we may have A = zero or $A=9$. It depends whether or not "carry 1" is received from the previous column. In other words, the "9" mimics zero every time it gets a carry-over of "1".

***Step* III** Look for 1 in addition or subtraction, observe left hand digits. If single, they are probably 1.

e.g. (i) $$\begin{array}{r} B\,*\,*\,* \\ +\,A\,*\,*\,* \\ \hline A\,*\,*\,*\,* \end{array}$$

In this example A can only be 1 as it has been carried over from the previous column. This means that whenever an addition of p digits gives a total of, $p+1$ digits the left hand digit of the total must be 1.

e.g. (ii) $$\begin{array}{r} B\,*\,*\,* \\ -\,B\,*\,*\,* \\ \hline *\,*\,*\,*\,* \end{array}$$

In this example, $B=1$ e.g. (iii) $$\begin{array}{r} 1\,B\,A \\ +\,A\,A\,2 \\ \hline B\,7\,5 \end{array}$$

Here, $A+2=5 \Rightarrow A=5-2=3$

$B+A=7 \Rightarrow B+3=7$

$\Rightarrow B=7-3=4 \Rightarrow B=4$ and $1+A=B$

$\Rightarrow 1+A=4 \Rightarrow A=4-1=3$

Thus, $\begin{array}{r} 1\,B\,A \\ +\,A\,A\,2 \\ \hline B\,7\,5 \end{array}$ is $\begin{array}{r} 143 \\ +\,332 \\ \hline 475 \end{array}$

Projects

SAMPLE QUESTION PAPER 1

A HIGHLY SIMULATED SAMPLE QUESTION PAPER FOR ICSE CLASS X

MATHEMATICS

UNSOLVED

GENERAL INSTRUCTIONS

1. You will not be allowed to write during the first 15 minutes. This time is to be spent in reading the questions paper.
2. The time given at the head of this paper is the time allowed for writing the answers.
3. Attempt **all questions** from **Section A** and **any 4 questions** from **Section B**.
4. All working, including rough work, must be clearly shown and must be done on the same sheet as the rest of the answer.
5. Omission of essential working will result in loss of marks.
6. The intended marks for questions or parts of questions are given in brackets [ ].
7. Mathematical tables are provided.

Time : 2.5 Hrs **Max. Marks : 80**

Section A [40 *Marks*]

1. (a) If both $(x-2)$ and $\left(x-\frac{1}{2}\right)$ are factors of px^2+5x+r, then show that $p=r$. [3]

(b) Ajay opened a cumulative time deposit account with OBC Bank for $\frac{3}{2}$ yr. If the rate of interest is 10% per annum and the bank pays ₹4662 on maturity, then find how much did Ajay deposit each month? [3]

(c) Manufacturer *A* sells a LED to a dealer B for ₹ 12500. The dealer B sells it to a consumer at a profit of ₹ 1500. If the sales are intrastate and the rate of GST is 12%, find

(i) the amount of tax (under GST) paid by the dealer B to the Central Government.

(ii) the amount that the consumer pays for the machine. [4]

2. (a) Solve the following inequation and represent the solution set on the number line.

$$4x-19<\frac{3x}{5}-2\le\frac{-2}{5}+x, \forall x\in R$$ [3]

(b) The sum of the 5th and 7th terms of an AP is 52 and its 10th term is 46. Find the AP. [3]

(c) Prove that the angle between two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line segments joining the points of contact at the centre. [4]

3. (a) Solve the following equation for matrix X.

$$2X+4\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}=3\begin{bmatrix}4 & -2\\ 0 & -6\end{bmatrix}$$ [3]

(b) Find the value of k for which the given equation has equal roots.

$$(k-12)x^2+2(k-12)x+2=0$$ [3]

(c) If $\operatorname{cosec}\theta-\sin\theta=l$ and $\sec\theta-\cos\theta=m$, then prove that $l^2m^2(l^2+m^2+3)=1$. [4]

4. (a) Two unbiased coins are tossed simultaneously. Find the probability of getting

(i) two heads.

(ii) one head.

(iii) one tail.

(iv) atleast one head.

(v) atmost one head.

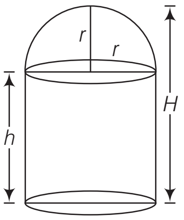
(vi) no head. [3]

(b) Compute all quartiles from the following data.

Weekly income (in ₹)	Number of workers
58	2
59	3
60	6
61	15
62	10
63	5
64	4
65	3
66	1

[3]

(c) A building is in the form of a cylinder surmounted by a hemispherical dome (see the figure). The base diameter of the dome is equal to $\frac{2}{3}$ of the total height of the building. Find the height of the building, if it contains $67\frac{1}{21}$ m^3 of air. [4]



Section B [40 *Marks*]

5. (a) If the first term of a GP is 5 and the sum of first three terms is $\frac{31}{5}$, find the common ratio. Hence find the GP. [4]

(b) Use graph paper for this question. A (0, 3), B(3, – 2) and O (0, 0) are the vertices of ΔABO.

(i) Plot the triangle on a graph sheet taking 0.5 cm = 1 unit on both the axes.

(ii) Plot D, the reflection of B in the Y-axis and write its coordinates.

(iii) Give the geometrical name of the figure ABOD. [6]

6. (a) A spherical glass vessel has a cylindrical neck 8 cm long, 2 cm in diameter and diameter of the spherical part is 8.5 cm. By measuring the amount of water it holds, Richa find its volume to be 345 cm^2. Check whether she is correct or not, taking the above as the inside measurements and $\pi = 3.14$. [3]

(b) Find the coordinates of the points which divide the line segment joining A (–2, 2) and B (2, 8) into four equal parts. [3]

(c) Two isosceles triangles have equal vertical angles and their areas are in the ratio 36 : 49.

Find the ratio of their corresponding heights. [4]

7. (a) The following table gives weekly wages in rupees of workers in a certain commercial organisation. The frequency of class 49-52 is missing. It is known that the mean of the frequency distribution is 47.2. Find the missing frequency.

Weekly wages (in ₹)	40-43	43-46	46-49	49-52	52-55
Number of workers	31	58	60	?	27

[3]

(b) A straight line makes on the coordinates axes positive intercepts whose sum is 5. If the line passes through the point A (–3, 4), then find its equation. [4]

(c) A box contains 90 discs, which are numbered from 1 to 90. If one disc is drawn at random from the box, find the probability that it bears

(i) a two-digit number.

(ii) a perfect square number.

(iii) a number divisible by 5. [3]

8. (a) If y is the mean proportional between x and z, then prove that xy + yz is the mean proportional between $x^2 + y^2$ and $y^2 + z^2$. [3]

(b) Metallic spheres of radii 6 cm, 8 cm and 10 cm respectively, are melted to form a single solid sphere. Find the radius of the resulting sphere. [3]

(c) Prove that

$$\frac{\operatorname{cosec}\theta + \cot\theta}{\operatorname{cosec}\theta - \cot\theta} = 1 + 2\cot^2\theta + 2\operatorname{cosec}^2\theta\cos\theta.$$

[4]

9. (a) Sum of the areas of two squares is 468 m^2. If the difference of their perimeters is 24m, then find the sides of the two squares. [3]

(b) The frequency distribution of scores obtained by 230 candidates in a medical entrance test is a follows

Scores	Number of candidates
400-450	20
450-500	35
500-550	40
550-600	32
600-650	24
650-700	27
700-750	18
750-800	24

Draw a cumulative frequency curve to represent the above data. [4]

(c) Using ruler and compass only. Construct a $\triangle ABC$ in which $BC = 7$ cm, $\angle A = 60°$ and the altitude through A is 2.6 cm. Draw the incircle of $\triangle ABC$. [3]

10. (a) Ashok sold a certain number of ₹20 shares paying 8% dividend at ₹18 and invested the proceeds in ₹10 shares, paying 12% dividend at 50% premium. If the change in his annual income is ₹120, then find the number of shares sold by him. [3]

(b) Use graph paper for this question.

Take 1 cm = 1 unit on the both axes.

(i) Plot the points A (1, 1), B (5, 3) and C (2, 7).

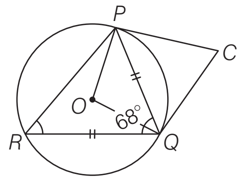
(ii) Construct the locus of points equidistant from A and B.

(iii) Construct the locus of points equidistant from AB and AC.

(iv) Locate the point P such that AP = PB and A is equidistant from AB and AC.

(v) Measure the length of PA in cm and record it. [4]

(c) In the given figure, PQ = QR, $\angle RQP = 68°$, PC and QC are tangents to the circle with centre O. Calculate the values of



(i) $\angle QOP$. (ii) $\angle QCP$. [3]

11. (a) The equation of a line is $2x - 2\sqrt{3y} + \sqrt{3} = 0$. Find its

(i) gradient or slope.

(ii) inclination.

(iii) y-intercept. [3]

(b) At the foot of a mountain the angle of elevation of its summit is 45° after ascending 1000 m towards the mountain up a slope of 30° inclination, the angle of elevation is found to be 60°. Find the height of the mountain. [4]

(c) Miss Radhika goes to a mall to purchase a saree whose cost is ₹ 1770 (list price). She tells the shopkeeper to reduce the price in such an extent that she has to pay ₹ 1770, inclusive GST which is at the rate of 18%, find the reduction of price needed in the saree. [3]

UNSOLVED

SAMPLE QUESTION PAPER 2

A HIGHLY SIMULATED SAMPLE QUESTION PAPER FOR ICSE CLASS X

MATHEMATICS

GENERAL INSTRUCTIONS See the Sample Question Paper 1.

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Section A [40 *Marks*]

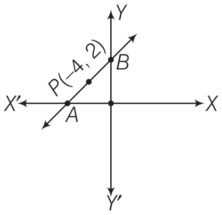
1. (a) Find the number of terms in the AP

$$18, 15\frac{1}{2}, 13, \ldots, -47.$$ [3]

(b) If A is the solution set of $5x - 4 \geq 6$, $x \in R$ and B is the solution set of $5 - x > 1$, $x \in R$, then find $A \cap B$ and $A' \cap B$. [3]

(c) Prove that $\dfrac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A} = \sin^2 A \cos^2 A$. [4]

2. (a) In the adjoining figure, line APB meets the X-axis at A and Y-axis at B. P is the point $(-4, 2)$ and $AP : PB = 1 : 2$. Write down the coordinates of A and B. [3]



(b) If

$$A = \begin{bmatrix} \sec 60^\circ & \cos 90^\circ \\ -3\tan 45^\circ & \sin 90^\circ \end{bmatrix}$$ and

$$B = \begin{bmatrix} 0 & \cot 45^\circ \\ -4\sin 30^\circ & 3\sin 90^\circ \end{bmatrix},$$ then find

(i) A^2 (ii) BA. [3]

(c) A shopkeeper in Rajasthan buys an article at the printed price of ₹ 50000 from a wholesaler in Delhi. The shopkeeper sells the article to a consumer in Rajasthan at a profit of 25% on the basic cost price. If the rate of GST is 18%, find

(i) the price of the article inclusive of tax (under GST) at which the shopkeeper bought it.

(ii) the amount of tax (under GST) paid by the shopkeeper to Government. [4]

3. (a) In a right angled ΔABC, right angled at B, the perpendicular BD on the hypotenuse AC is drawn. Prove that $AC \times CD = (BC)^2$. [3]

(b) Let A(6, 4) and B(2, 12) be two given points. Find the slope of a line

(i) parallel to AB.

(ii) perpendicular to AB. [3]

(c) Apply step-deviation method to find the AM of the following frequency distribution.

Variate (x)	5	10	15	20	25	30	35	40	45	50
Frequency (f)	20	43	75	67	72	45	39	9	8	6

[4]

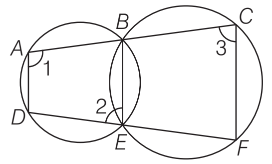
4. (a) Given $\angle BAC$, a line intersects the arms of $\angle BAC$ in P and Q. How will you locate a point on line segment PQ, which is equidistant from AB and AC? Does such point always exist? [3]

(b) Veena has a cumulative deposit account of ₹ 400 per month at 10% per annum simple interest. If she gets ₹ 30100 at the time of maturity, then find the total time for which the account was held. [3]

(c) A piece of cloth costs ₹ 200. If the piece was 5 m longer and each metre of cloth cost ₹ 2 less, then the cost of the piece would have remain unchanged. How long is the piece and what is the original rate per metre? [4]

Section B [40 *Marks*]

5. (a) In the given figure, A, B, C and D, E, F are two sets of collinear points. Prove that AD || CF.



[3]

(b) The ratio of sum of first three terms of the GP to the sum of first six terms is 125 : 152. Find the common ratio of GP. [3]

(c) Use graph paper to answer the following questions.

(i) Plot the points $P(4, 6)$ and $Q(1, 2)$.

(ii) P' is the image of P when reflected in X-axis.

(iii) Q' is the image of Q when reflected in the line PP'.

(iv) Give the geometrical name for the figure $PQP'Q'$. [4]

6. (a) Draw the cumulative frequency curve for the following frequency distribution by less than method.

Marks	Number of students
0-10	7
10-20	10
20-30	23
30-40	51
40-50	6
50-60	3

[5]

(b) Two points P and Q have the coordinates (3, 2) and (7, 6), respectively. Find the

(i) gradient of PQ.

(ii) equation of the perpendicular bisector of PQ.

(iii) value of k for which $(-2, k)$ lies on PQ. [5]

7. (a) Internal and external diameters of a hollow hemispherical vessel are 24 cm and 25 cm, respectively. The cost to paint 1 cm^2 the surface is ₹ 0.05. Find the total cost to paint the vessel all over. [use $\pi = 22/7$] [3]

(b) Find α and β, if $x+1$ and $x+2$ are factors of $x^3 + 3x^2 - 2\alpha x + \beta$. [3]

(c) Prove that

$$2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0.$$ [4]

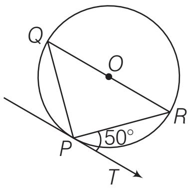
8. (a) A bag contains 5 white and 7 red balls. One ball is drawn at random. What is the probability that ball drawn is white? [3]

(b) Find the annual income derived from

(i) 260, ₹ 100 shares paying 15 % dividend.

(ii) 750, ₹ 20 shares paying 25 % dividend. [3]

(c) In the given figure, PT is tangent to the circle at P and QR is a diameter of the circle. If $\angle RPT = 50°$, find $\angle QRP$. [4]



9. (a) A tower subtends an angle α at a point A in the plane of its base and the angle of depression of the foot of the tower at a point b m just above A is β. Prove that the height of the tower is $b\tan\alpha \cot\beta$. [3]

(b) If AB and CD are two chords which, when produced meet at P and if AP = CP, show that AB = CD. [3]

(c) The monthly profits in rupees of 100 shops are distributed as follows

Profit (in ₹)	Number of shops
0-100	13
100-200	18
200-300	27
300-400	20
400-500	16
500-600	6

Find the modal value graphically. [4]

10. (a) A shopkeeper buys a washing machine at a discount of 20% from a wholesaler, the printed price of the camera being ₹ 1600. The shopkeeper sells it to a consumer at the printed price. If the sales are intrastate and rate of GST is 12%, find

(i) GST paid by the shopkeeper to the Central Government.

(ii) GST received by the Central Government.

(iii) GST received by the State Government. [3]

(b) A matrix x has $(a+b)$ rows and $(a+2)$ columns while the matrix y has $(b+1)$ rows and $(a+3)$ columns. Both matrices xy and yx exist. Find a and b. Can you say xy and yx are of the same type? Are they equal? [3]

(c) The difference between outside and inside surface areas of cylindrical metallic pipe 14 cm long is 88 cm^2. If the pipe is made of 198 cm^2 metal, then find the inner and outer radii of the pipe. [4]

11. (a) If b is the mean proportional between a and c, then prove that

$$abc(a+b+c)^3 = (ab+bc+ca)^3.$$ [3]

(b) The sum of two numbers is 15. If the sum of their reciprocals is $\frac{3}{10}$, then find the numbers. [3]

(c) Draw a regular hexagon of side 4 cm. Draw the circle about the hexagon. [4]

SAMPLE QUESTION PAPER 3

A HIGHLY SIMULATED SAMPLE QUESTION PAPER FOR ICSE CLASS X

MATHEMATICS

UNSOLVED

GENERAL INSTRUCTIONS See the Sample Question Paper 1.

Section A [40 *Marks*]

1. (a) For what values of a is $2x^3 + ax^2 + 11x + a + 3$ exactly divisible by $(2x - 1)$? [3]

(b) Rashmi deposits ₹ 240 per month at 10 % per annum in a bank and the bank pays her ₹ 4662 on maturity, then find the total time for which the account is held. [3]

(c) A shopkeeper buys a dinning table whose list price is ₹ 40000 at some rate of discount from a wholesaler. He sells the article to a consumer at the list price and charges GST at the rate of 12%. If the sales are intrastate and the shopkeeper has to pay tax (under GST) of ₹ 48 to the State Government, find the rate of discount at which he bought the article from the wholesaler.

2. (a) If x is a negative integer, then find the solution set of $\frac{2}{3} + \frac{1}{3}(x + 1) > 0$. [3]

(b) The third term of a GP is 4. Find the product of first five terms. [3]

(c) If all the side of a parallelogram touch a circle, show that the parallelogram is a rhombus. [4]

3. (a) If $B = \begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix}$, then find a matrix *A* satisfying $B^2 = B + 2A$. [3]

(b) Without using trigonometric tables, evaluate

$$\frac{\sin^2 29^\circ + \sin^2 61^\circ}{\cos^2 29^\circ + \cos^2 61^\circ} + \frac{\sin(90^\circ - \theta)\sin\theta}{\tan\theta} + \frac{\cos(90^\circ - \theta)\cos\theta}{\cot\theta}.$$ [4]

(c) Find the value of k for which the roots are real and equal in the following equation.

$$(k + 1)x^2 - 2(3k + 1)x + 8k + 1 = 0$$ [3]

4. (a) 17 cards numbered 1,2,3,...,17 are put in a box and mixed throughly. One person draws a card from the box. Find the probability that the number on the card is

(i) odd (ii) a prime

(iii) divisible by 3.

(iv) divisible by 3 and 2 both. [3]

(b) The mean of the following frequency table 50. But the frequencies f_1 and f_2 in class 20-40 and 60-80 are missing. Find the missing frequencies.

Class	Frequency
0-20	17
20-40	f_1
40-60	32
60-80	f_2
80-100	19
Total	120

[3]

(c) A hollow metal sphere of internal and external radii 2 cm and 4 cm, respectively is melted into a solid cone of base radius 4 cm. Find the height and slant height of the cone. [4]

Section B [40 *Marks*]

5. (a) The eigth term of an AP is half its second term and the eleventh term exceeds one-third of its fourth term by 1. Find the 15th term. [4]

(b) If $A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$, then show that $A^2 - 2A + 3I_2 = 0$. [3]

(c) Using graph paper and taking 1 cm = 1 unit along both the axes.

(i) Plot the points $A(-4, 4)$ and $B(2, 2)$.

(ii) Reflect *A* and *B* in the origin to get the image A' and B' respectively.

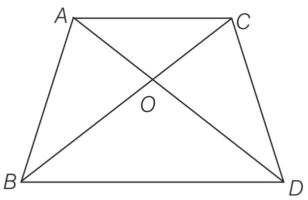
(iii) Write down the coordinates of A' and B'. [3]

6. (a) Find the ratio in which the line $3x + y - 9 = 0$ divides the segments joining the points (1, 3) and (2, 7). [4]

(b) A solid wooden toy is in the shape of a right circular cone mounted on a hemisphere. If the radius of the hemisphere is 4.2 cm and the total height of the toy is 10.2 cm, then find the volume of the wooden toy. [3]

(c) In the given figure, ABC and DBC are two triangles on the same base BC. If AD intersects BC at O, then show that

$$\frac{\text{ar}(\Delta ABC)}{\text{ar}(\Delta DBC)} = \frac{AO}{DO}.$$ [3]



7. (a) For the following distribution, draw a histogram.

Weight (in kg)	Number of shops
44-47	20
48-51	28
52-55	36
56-59	16
60-63	8
64-67	4

From the histogram estimate the mode. [4]

(b) (i) A lot of 20 bulbs contain 4 defective ones. One bulb is drawn at random from the lot. What is the probability that this bulb is defective?

(ii) Suppose the bulb drawn in (i) is not defective and is not replaced. Now, one bulb is drawn at random from the rest. What is the probability that this bulb is not defective?

(iii) Suppose the bulb drawn in (i) is defective and it is not replaced. Now, one bulb is drawn at random from the rest. What is the probability that this bulb is defective? [6]

8. (a) Prove that $\dfrac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = \dfrac{\cos^2\theta}{(1+\sin\theta)^2}$

$= 1 - 2\sec\theta\tan\theta + 2\tan^2\theta.$ [4]

(b) If $a = \dfrac{b+c}{2}$, $c = \dfrac{a+b}{2}$ and b is the mean proportional between a and c, then prove that $\dfrac{1}{a} + \dfrac{1}{c} = \dfrac{2}{b}$. [3]

(c) Find the equations of the altitudes of the triangle whose vertices are A (7, – 1), B (– 2, 8) and C (1, 2). [3]

9. (a) Draw a ΔABC in which BC = 6 cm, $\angle B = 45°$ and AB = 7 cm. Draw the circumcircle of ΔABC. [3]

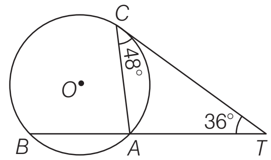
(b) The sum of a number and its positive square root is $\dfrac{6}{25}$. Find the number. [3]

(c) Find the mode and median of the following frequency distribution.

x	10	11	12	13	14	15
f	1	4	7	5	9	3

[4]

10. (a) A, B and C are three points on a circle. The tangent at C meets BA produced at T. Given that $\angle ATC = 36°$ and that $\angle ACT = 48°$, then calculate the angle subtended by AB at the centre of the circle.



[3]

(b) The line passing through (0, 3) and (–4, – 2) is parallel to line passing through (– 2, 6) and (6, a). Find a. [3]

(c) The price of a spider toy is ₹ 3136 inclusive tax (under GST) at the rate of 12% on its listed price. A buyer asks for a discount on the listed price, so that after charging GST, the selling price becomes equal to the listed price. Find the amount of discount which the seller has to allow for the deal. [4]

11. (a) A man sold four hundred, ₹ 20 shares paying 5% at ₹ 18 and invested the proceeds in ₹ 10 shares paying 7% at ₹ 12. How many ₹ 10 shares did he buy and what was the change in income? [3]

(b) The angle of elevation of a cliff from a fixed point is θ. After going up a distance of k m towards the top of the cliff at an angle of ϕ, it is found that the angle of elevation is α. Show that the height of the cliff is

$$\frac{k(\cos\phi - \sin\phi\cos\alpha)}{\cot\theta - \cot\alpha}\text{ m.}$$ [4]

(c) Construct a ΔABC having given AB = 4.8 cm, AC = 4 cm and $\angle A = 75°$. Locate the point P

(i) inside the ΔABC.

(ii) outside the ΔABC.

Equidistant from B and C and at a distance of 1.2 cm from BC. [3]

SAMPLE QUESTION PAPER 4

A HIGHLY SIMULATED SAMPLE QUESTION PAPER FOR ICSE CLASS X

MATHEMATICS

GENERAL INSTRUCTIONS See the Sample Question Paper 1.

UNSOLVED

Section A [40 *Marks*]

1. (a) Factorise $9z^3 - 27z^2 - 100z + 300$, if it is given that $(3z + 10)$ is a factor of it. [3]

(b) Mr. Kumar has recurring deposite account for 2 yr at 10% per annum. If he receives ₹ 1900 as interest, then find the value of monthly instalment paid by him. [3]

(c) A retailer buys an bicycle at a discount of 15% on the printed price from a wholesaler. He marks up the price by 10% on the printed price but due to competition in the market, he allows a discount of 5% on the market price to a buyer. If the rate of GST is 12% and the buyer pays ₹ 468.16 for the article inclusive of tax (under GST), find

(i) the printed price of the article.

(ii) the profit percentage of the retailer. [4]

2. (a) Find three consecutive largest positive integers, such that the sum of one-third of first, one-fourth of second and one-fifth of third is atmost 20. [3]

(b) Find the common difference of an AP whose first term is $\frac{1}{2}$ and 8th term is $\frac{17}{6}$. Also, write its 4th term. [3]

(c) Two circles intersect at P and Q. *A* straight line through P meets the circles in A and B. Prove that QA = QB. [4]

3. (a) Show that $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \tan\theta + \cot\theta$

$= 1 + \sec\theta \operatorname{cosec}\theta$. [4]

(b) Let a, b, c and d be non-zero real numbers. Find the values of x and y in the following.

$$\begin{bmatrix} ax + by & 0 \\ 0 & cx + dy \end{bmatrix} = \begin{bmatrix} a - b & 0 \\ 0 & c - d \end{bmatrix}$$ [3]

(c) Find the values of *k* for which the equation $x^2 + 5kx + 16 = 0$ has no real roots. [3]

4. (a) A family has 3 children. Find the probability that the family has

(i) atleast one girl child.

(ii) atmost one girl child.

(iii) exactly one girl child.

(iv) no girl child. [3]

(b) Following is the distribution of IQ of 100 students. Find the median IQ.

IQ	Number of students
55-64	1
65-74	2
75-84	9
85-94	22
95-104	33
105-114	22
115-124	8
125-134	2
135-144	1

[3]

(c) Metal spheres, each of radius 2 cm, are packed into a rectangular box of internal dimension 16 cm × 8 cm × 8 cm. When 16 spheres are packed, the box is filled with preservative liquid. Find the volume of this liquid. $\left[\text{Take, } \pi = \frac{669}{213}\right]$ [4]

Section B [40 *Marks*]

5. (a) Find the three numbers in GP, whose product is 8 and sum is $\frac{26}{3}$. [4]

(b) If $\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} y & 11 \\ 4 & z \end{bmatrix}$, then find the values of x, y and z. [3]

(c) P' and Q' be the images of $P(-3, 5)$ and $Q(-5, 3)$ respectively on reflection in Y-axis.

(i) Find the coordinates of P' and Q'.

(ii) Assign special name of quadrilateral $PP'Q'Q$.

(iii) Are PQ' and QP' equal in length? [3]

6. (a) In parallelogram ABCD, E is a point on AB, CE intersects the diagonal BD at O and EF || BC. If AE : EB = 2 : 3, then find

(i) EF : AD.

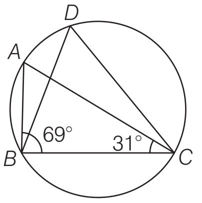
(ii) ar (ΔBEF) : ar (ΔABD).

(iii) ar (ΔABD) : ar (Trapezium AEFD).

(iv) ar (ΔFEO) : ar (ΔOBC). [5]

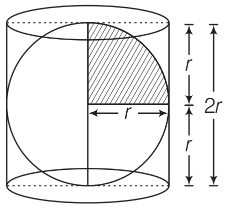
(b) If the coordinates of the mid-points of the sides of a triangle are (1, 2), (0, – 1) and (2, – 1), then find the coordinates of its vertices. [5]

7. (a) In the given figure, $\angle ABC = 69°$ and $\angle ACB = 31°$. Find $\angle$ BDC.



[3]

(b) A right circular cylinder just encloses a sphere of radius r (see the figure). Find the



(i) surface area of the sphere.

(ii) curved surface area of the cylinder.

(iii) ratio of the areas obtained in (i) and (ii). [3]

(c) If the mean of the following distribution is 7.5, find the missing frequency 'f'.

Variable	5	6	7	8	9	10	11	12
Frequency	20	17	*f*	10	8	6	7	6

[4]

8. (a) If $\dfrac{\cos\alpha}{\cos\beta} = m$ and $\dfrac{\cos\alpha}{\sin\beta} = n$, then show that

$$(m^2 + n^2)\cos^2\beta = n^2.$$ [4]

(b) Find the equation of the diagonals of a rectangle, whose sides are $x = -1$, $x = 4$, $y = -1$ and $y = 2$. [3]

(c) If a : b :: c : d, then show that $(4a + 5b) : (4a - 5b) :: (4c + 5d) : (4c - 5d)$. [3]

9. (a) If $x = \dfrac{\sqrt{a^2 + b^2} + \sqrt{a^2 - b^2}}{\sqrt{a^2 + b^2} - \sqrt{a^2 - b^2}}$, then show that $b^2x^2 - 2a^2x + b^2 = 0$. [3]

(b) Draw a circle of diameter 9 cm. Mark a point at a distance of 7.5 cm from the centre of the circle. Draw tangents to the given circle from this exterior point. Measure the length of each tangent. [4]

(c) Find the mode of the following frequency distribution by drawing a histogram.

Mid values	12	18	24	30	36	42	48
Frequency	20	12	8	24	16	8	12

[3]

10. (a) Using ruler and compass only.

(i) Construct a Δ *ABC*, having BC = 7 cm, AB – AC = 1 cm and $\angle ABC = 45°$.

(ii) Inscribe a circle in Δ*ABC* constructed in (i). [4]

(b) (i) Plot the points P(1, 0), Q(4, 0) and S(1, 3). Find the coordinates of the point R such that PQRS is a square.

(ii) Determine the length of line segment SR in figure PQRS.

(iii) Find the equation of the diagonal PR of square PQRS. [3]

(c) Ramesh purchased a machine for ₹ 31860, which includes 10% rebate on the list price and 18% tax (under GST) on the remaining price. Find the marked price of the digital camera. [3]

11. (a) A company gives *x*% dividends on its ₹ 60 shares, whereas the return on the investment in these shares is $(x + 3)$%. If the market value of each share is ₹ 50, then find the value of x. [3]

(b) The angle of elevation of a jet plane from point A on the ground is 60°. After a flight of 15 s, the angle of elevation changes to 30°. If the jet plane is flying at a constant height of $1500\sqrt{3}$ m, then find the speed of the jet plane. [4]

(c) Draw the locus of a point in the circle, so that it is equidistant from A and B, where AB is a chord of a circle. [3]

SAMPLE QUESTION PAPER 5

A HIGHLY SIMULATED SAMPLE QUESTION PAPER FOR ICSE CLASS X

MATHEMATICS

GENERAL INSTRUCTIONS See the Sample Question Paper 1.

Section A [40 *Marks*]

1. (a) Using factor theorem, factorise the polynomial $x^3 - 3x^2 - 9x - 5$. [3]

(b) Ashu deposits a certain sum of money every month in a recurring deposit account for a period of 12 months. If the bank pays interest at the rate of 11% per annum and Ashu gets ₹ 12715 as the maturity value of this account, then what sum of money he pay every month? [3]

(c) The printed price of an article is ₹ 2500. A wholesaler in Kerala buys the article from a manufacturer in Rajasthan at a discount of 12% on the printed price. The wholesaler sells the article to a retailer in Odisha at 32% above the marked price. If the rate of GST on the article is 5%, find

(i) the price inclusive of tax (under GST) at which the wholesaler bought the article.

(ii) the price inclusive of tax (under GST) at which the retailer bought the article.

(iii) the tax (under GST) paid by wholesaler to the Government. [4]

2. (a) Solve the following inequation and graph the solution set on the number line.

$$2x - 3 < x + 2 \le 3x + 5;\ x \in R.$$ [3]

(b) Find the sum of the following AP.

$$\frac{1}{15}, \frac{1}{12}, \frac{1}{10}, \ldots \text{ to 11 terms}$$ [3]

(c) PQR is a triangle and $G(4, 3)$ is the centroid of the triangle. If $P \equiv (1, b)$, $Q \equiv (a, 5)$ and $R \equiv (7, 1)$, find 'a' and 'b'. Find the length of side PQ. [4]

3. (a) Solve the following equation.

$$\frac{2x+3}{x+3} = \frac{x+4}{x+2}$$ [3]

(b) Find the matrix A of order 2×2, such that $A\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = 6I_2$. [3]

(c) Prove that $(\sin^8 \theta - \cos^8 \theta)$

$$= (\sin^2 \theta - \cos^2 \theta)(1 - 2\sin^2 \theta \cos^2 \theta)$$ [4]

4. (a) Vinod and Ravi are friends. What is the probability that both will have

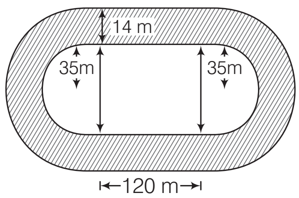
(i) the same birthday?

(ii) different birthdays?

(Ignoring a leap year) [3]

(b) The median of the observations 11, 12, 14, 18, $x + 2$, $x + 4$, 30, 32, 35 and 41 arranged in ascending order, is 24. Find the value of x. [3]

(c) An athletic track 14 m wide consists of two straight sections 120 m long joining semi-circle ends, whose inner radius is 35 m. Calculate the area of the shaded region.



[4]

Section B [40 *Marks*]

5. (a) The ΔABC, when $A(1, 2)$, $B(4, 8)$ and $C(6, 8)$, is reflected in the X-axis to $\Delta A'B'C'$. The $\Delta A'B'C'$ is then reflected in the origin to $\Delta A''B''C''$.

(i) Write the coordinates of A′, B′ and C′.

(ii) Write the coordinates of A″B″ and C″.

(iii) Write down the single transformation, that maps ABC into A″B″C″. [3]

(b) Let $A = \begin{bmatrix} \cot 45^\circ & 0 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & \sec 60^\circ \\ 4 & 5 \end{bmatrix}$.

Find

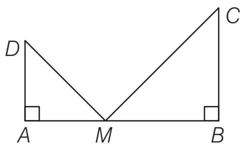
(i) AB (ii) $3A - 2B$ (iii) $B^2 - A^2$. [3]

(c) If the 4th and 8th terms of a GP are 24 and 384 respectively, find the first term and common ratio. [4]

6. (a) The surface areas of a sphere and a cube are equal. Prove that their volumes are in the ratio $1 : \sqrt{\pi/6}$. [3]

(b) In what ratio, is the line joining the points (2, 3) and (4, 1) divides the line segment joining the points (1,2) and (4,3)? [3]

(c) In the given figure, M is the mid-point of AB and $\angle A = \angle B = 90^\circ = \angle CMD$.



Prove that

(i) $\Delta DAM \sim \Delta MBC$ (ii) $\frac{\text{ar}(\Delta DAM)}{\text{ar}(\Delta MBC)} = \frac{AD}{BC}$

(iii) $\frac{AD}{BC} = \frac{MD^2}{MC^2}$. [4]

7. (a) Estimate the median for the given data by drawing ogive. [3]

Class	0-10	10-20	20-30	30-40	40-50
Frequency	4	9	15	14	8

(b) In right angled ΔABC, right angled at B, a circle with side AB as diameter is drawn to intersect the hypotenuse AC at P. Prove that tangent to the circle at P bisects the side BC. [3]

(c) Rajeev had a *RD* Account in State Bank of India and deposited ₹ 600 per month. If the maturity value of this account was ₹ 24930 and the rate of interest was 10% per annum, then find the time (in years) for which the account was held. [4]

8. (a) Find the number which bears the same ratio of $\frac{7}{33}$, that $\frac{8}{21}$ does to $\frac{4}{9}$. [3]

(b) A joker's cap is in the form of a right circular cone of base radius 7 cm and height 24 cm. Find the area of sheet required to make 10 such caps. [3]

(c) If A, B and C are the interior angles of a ΔABC, then show that

$$\cos^2 \frac{A}{2} + \cos^2 \left(\frac{B+C}{2}\right) = 1.$$ [4]

9. (a) If $x = 2/3$ is a solution of the quadratic equation $7A^2 + mx - 3 = 0$, then find the value of m. [3]

(b) Calculate the mean, median and mode for the following data.

23, 25, 28, 25, 16, 23, 17, 22, 25, 25 [3]

(c) Draw a line segment AB of length 8 cm. Taking A as centre, draw a circle of radius 4 cm and taking B as centre, draw another circle of radius 3 cm. Construct tangents to each circle from the centre of the other circle. [4]

10. (a) Prove that the circle drawn with any side of a rhombus as a diameter, passes through the point of intersection of its diagonals. [4]

(b) Use a ruler and a compass for this question.

(i) Construct a ΔABC in which BC = 6.5 cm, $\angle ABC = 60^\circ$, AB = 5 cm.

(ii) Construct the locus of points at a distance of 3.5 cm from A.

(iii) Construct the locus of points equidistant from AC and BC. [6]

11. (a) A shopkeeper in Rajasthan buys scooty at the printed price of ₹30000 from a wholesaler in Kerala. The shopkeeper sells scooty to a consumer in Rajasthan at a profit of 15% on the basic cost price. If the rate of GST is 12%, find

(i) the price inclusive of tax (under GST) at which the wholesaler bought the scooty.

(ii) the amount which the consumer pays for the scooty. [3]

(b) On plotting the points O (0, 0), A (3, 0), B (3, 4), C (0, 4) and joining OA, AB, BC and CO, which shape is obtained? Also, find the equations of OB and AC? [3]

(c) Determine the height of a mountain, if the elevation of its top at an unknown distance from the base is 30° and at a distance 10 km further off from the mountain, along the same line, the angle of elevation is 15°.

[Take, $\tan 15^\circ = 0.27$] [4]

UNSOLVED

ANSWERS

Sample Paper 1

1. (b) ₹ 240 per month (c) (i) ₹ 90 (ii) ₹ 15680

2. (a) $\{x : -4 \le x < 5,\ x \in R\}$

Representation of solution set in the real number line

(b) 1, 6, 11, 16, ...111

3. (a) $X = \begin{bmatrix} 4 & -3 \\ 0 & -7 \end{bmatrix}$ (b) $k = 14$

4. (a) (i) $\frac{1}{4}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{2}$ (iv) $\frac{3}{4}$ (v) $\frac{3}{4}$ (vi) $\frac{1}{4}$

(b) $Q_1 =$ ₹ 61, $Q_2 =$ ₹ 61 and $Q_3 =$ ₹ 63

(c) 6 m

5. (a) Common ratio $= -\frac{6}{5}$ and GP is $\frac{1}{5}$; 5, 1, $\frac{1}{5}$, ... or 5, –6, $\frac{36}{5}$

(b) (ii) $D\,(-3, -2)$

(iii) Quadrilateral $ABOD$ is a kite or arrow head.

6. (a) 346.5 cm^3; No (b) $\left(-1, \frac{7}{2}\right)$, (0, 5), $\left(1, \frac{13}{2}\right)$ (c) $\frac{6}{7}$

7. (a) 44 (b) $2x + 3y = 6$

(c) (i) $\frac{9}{10}$ (ii) $\frac{1}{10}$ (iii) $\frac{1}{5}$

8. (b) 12 cm

9. (a) 18 m, 12 cm

10. (a) 750 (b) (v) $PA = 2.5$ cm

(c) (i) 112° (ii) 68°

11. (a) (i) $\frac{1}{\sqrt{3}}$ (ii) 30° (iii) $\frac{1}{2}$

(b) 1.366 km (c) ₹ 270

Sample Paper 2

1. (a) 27

(b) $A \cap B = [2, 4)$ and $A' \cap B = (-\infty, 2)$

2. (a) $A(-6, 0)$ and $B(0, 6)$.

(b) $A^2 = \begin{bmatrix} 4 & 0 \\ -9 & 1 \end{bmatrix}$, $BA = \begin{bmatrix} -3 & 1 \\ -13 & 3 \end{bmatrix}$

(c) (i) ₹ 59000 (ii) ₹ 2250

3. (b) (i) –2 (ii) $\frac{1}{2}$ (c) 22.2

4. (a) Required point is the point of intersection of the angle bisector and the line PQ, Yes

(b) 5 yr (c) 20 m and ₹ 10 per m

5. (b) $\frac{3}{5}$

(c) (ii) $P'\,(4, -6)$; (iii) $Q'\,(7, 2)$ (iv) $PQP'Q'$ is a kite.

6. (b) (i) 1 (ii) $x + y = 9$ (iii) – 3

7. (a) ₹ 96.28 (b) $\alpha = -1, \beta = 0$

8. (a) $\frac{5}{12}$ (b) (i) ₹ 3900 (ii) ₹ 3750

(c) 40°

9. (c) 256 (approx.)

10. (a) (i) ₹ 19.20 (ii) ₹ 96 (iii) ₹ 96

(b) (i) $a = 2$, $b = 3$ (ii) No (iii) No

(c) Outer radius $= \frac{11}{4}$ and inner radius $= \frac{7}{4}$

11. (b) 5 and 10

Sample Paper 3

1. (a) $a = -7$ (b) 18 months (c) 2%

2. (a) {–1, – 2} (b) 1024

3. (a) $\begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix}$ (b) 2

(c) $k = 0$ or 3

4. (a) (i) $\frac{9}{17}$ (ii) $\frac{7}{17}$ (iii) $\frac{5}{17}$ (iv) $\frac{2}{17}$

(b) $f_1 = 28$, $f_2 = 24$

(c) $h = 14$ cm, $l = 14.56$ cm

5. (a) $a_{15} = 3$ (c) (iii) $A'\,(4, -4)$ and $B'\,(-2, -2)$

6. (a) 3 : 4 (b) 265.870 cm^3

7. (a) 52.7 kg (c) (i) $\frac{1}{5}$ (ii) $\frac{15}{19}$ (iii) $\frac{3}{19}$

8. (c) $2y - x + 9 = 0$, $y - 2x - 12 = 0$, $y - x - 1 = 0$

9. (b) $\frac{1}{25}$ (c) Median = 13 and mode = 14

10. (a) 96° (b) $a = 16$ (c) ₹ 300

11. (a) 600, ₹ 20

Sample Paper 4

1. (a) $(3z + 10)(3z - 10)(z - 3)$

(b) ₹ 760 (c) (i) ₹ 400 (ii) ₹ $22\frac{16}{17}$%

2. (a) 24, 25, 26 (b) $d = \frac{1}{3}$, $a_4 = \frac{3}{2}$

3. (b) $x = 1$, $y = -1$ (c) $-\frac{8}{5} < k < \frac{8}{5}$

4. (a) (i) $\frac{7}{8}$ (ii) $\frac{1}{2}$ (iii) $\frac{3}{8}$ (iv) $\frac{1}{8}$

(b) 99.35

(c) 487.96 cm^3 (approx.)

5. (a) $6, 2, \frac{2}{3}$ or $\frac{2}{3}, 2, 6$ (b) $x = 3, y = 9, z = 3$

(c) (i) $P' = (3, 5), Q' = (5, 3)$

(ii) Isosceles trapezium (iii) Yes

6. (a) (i) 3 : 5 (ii) 9 : 25 (iii) 25 : 16 (iv) 9 : 25

(b) $(1, -4), (3, 2), (-1, 2)$

7. (a) 80°

(b) (i) $4\pi r^2$ (ii) $4\pi r^2$ (iii) 1 : 1

(c) $f = 16$

8. (b) $3x + 5y - 7 = 0,\ 5y - 3x + 2 = 0$

9. (b) 6 cm (c) 30.9

10. (b) (i) $R(4, 3)$ (ii) $SR = 3$ units

(iii) $y - x + 1 = 0$

(c) ₹ 30000

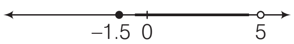
11. (a) 15 (b) 720 km/h

Sample Paper 5

1. (a) $(x+1)\ (x+1)\ (x-5)$ (b) ₹ 1000

(c) (i) ₹ 110 (ii) ₹ 3465 (iii) ₹ 55

2. (a) $-1.5 \le x < 5$



(b) $\frac{33}{20}$ (c) $a = 4,\ b = 3$ and $PQ = \sqrt{13}$ units

3. (a) $\pm\sqrt{6}$ (b) $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

4. (a) (i) $\frac{1}{365}$ (ii) $\frac{364}{365}$ (b) 21 (c) 7056 m^2

5. (a) (i) $(1, -2), (4, -8), (6, -8)$

(ii) $(-1, 2), (-4, 8), (-6, 8)$

(iii) Reflection in Y-axis

(b) (i) $\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -4 \\ -11 & -7 \end{bmatrix}$ (iii) $\begin{bmatrix} 8 & 12 \\ 26 & 32 \end{bmatrix}$

(c) $a = 3,\ r = 2$

6. (b) 1 : 1

7. (a) 28 (c) 3 yr

8. (a) $\frac{2}{11}$ (b) 5500 cm^2

9. (a) $\frac{-1}{6}$ (b) Mean = 22.9, median = 24, mode = 25

11. (a) (i) ₹ 36000 (ii) ₹ 38640

(b) Rectangle; $3y = 4x$; $4x + 3y - 12 = 0$

(c) 5 km

ICSE EXAMINATION PAPER 2019

MATHEMATICS (FULLY SOLVED)

GENERAL INSTRUCTIONS

1. You will not be allowed to write during the first 15 minutes. This time is to be spent in reading the questions paper.
2. The time given at the head of this paper is the time allowed for writing the answers.
3. Attempt **all questions** from **Section A** and **any 4 questions** from **Section B**.
4. All working, including rough work, must be clearly shown and must be done on the same sheet as the rest of the answer.
5. Omission of essential working will result in loss of marks.
6. The intended marks for questions or parts of questions are given in brackets [ ].
7. Mathematical tables are provided.

Time : 2.5 Hrs **Max. Marks : 80**

FULLY SOLVED

Section A [40 *Marks*]

1. (a) Solve the following inequation and write down the soluton set.

$$11x - 4 < 15x + 4 \leq 13x + 14, x \in W$$

Represent the solution on a real number line. [3]

(b) A man invests ₹ 4500 in shares of a company which is paying 7.5% dividend.

If ₹ 100 shares are available at a discount of 10%.

Find

(i) Number of shares he purchases.

(ii) His annual income. [3]

(c) In a class of 40 students, marks obtained by the students in a class test (out of 10) are given below

Marks	1	2	3	4	5	6	7	8	9	10
Number of students	1	2	3	3	6	10	5	4	3	3

Calculate the following for the given distribution.

(i) Median

(ii) Mode [4]

2. (a) Using the factor theorem, show that $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.

Hence factorise the polynomial completely. [3]

(b) Prove that

$(\text{cosec}\,\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1.$ [3]

(c) In an Arithmetic Progression (AP) the fourth and sixth terms are 8 and 14 respectively. Find the

(i) first term.

(ii) common difference.

(iii) sum of the first 20 terms. [4]

3. (a) Simplify

$$\sin A \begin{bmatrix} \sin A & -\cos A \\ \cos A & \sin A \end{bmatrix} + \cos A \begin{bmatrix} \cos A & \sin A \\ -\sin A & \cos A \end{bmatrix}$$

. [3]

(b) M and N are two points on the X-axis and Y-axis, respectively. $P(3, 2)$ divides the line segment MN in the ratio 2 : 3.

Find

(i) the coordinates of M and N.

(ii) slope of the line MN. [3]

(c) A solid metallic sphere of radius 6 cm is melted and made into a solid cylinder of height 32 cm. Find the

(i) radius of the cylinder.

(ii) curved surface area of the cylinder.

[Take $\pi = 3.1$] [4]

4. (a) The numbers $K+3$, $K+2$, $3K-7$ and $2K-3$ are in proportion. Find K. [3]

(b) Solve for x the quadratic equation $x^2 - 4x - 8 = 0$.

Give your answer correct to three significant figures. [3]

(c) Use ruler and compass only for answering this question.

Draw a circle of radius 4 cm. Mark the centre as O. Mark a point P outside the circle at a distance of 7 cm from the centre. Construct two tangents to the circle from the external point P. Measure and write down the length of any one tangent. [4]

Section B [40 *Marks*]

5. (a) There are 25 discs numbered 1 to 25. They are put in a closed box and shaken thourughly. A disc is drawn at random from the box. Find the probability that the number on the disc is

(i) an odd number.

(ii) divisible by 2 and 3 both.

(iii) a number less than 16. [3]

(b) Rekha opened a recurring deposit account for 20 months. The rate of interest is 9% per annum and Rekha receives ₹ 441 as interest at the time of maturity. Find the amount Rekha deposited each month. [3]

(c) Use a graph sheet for this question.

Take 1 cm = 1 unit along both X and Y-axis.

(i) Plot the following points.

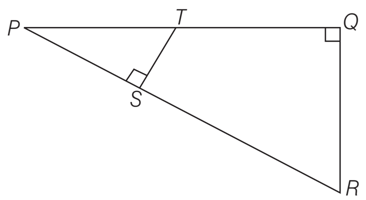
$A(0, 5)$, $B(3, 0)$, $C(1, 0)$ and $D(1, -5)$

(ii) Reflect the points B, C and D on the Y-axis and name them as B', C' and D' respectively.

(iii) Write down the coordinates of B', C' and D'.

(iv) Join the points A, B, C, D, D', C', B', A' in order and give a name to the closed figure $ABCDD'C'B'$. [4]

6. (a) In the given figure, $\angle PQR = \angle PST = 90°$, $PQ = 5$ cm and $PS = 2$ cm. [3]



(i) Prove that $\Delta PQR \sim \Delta PST$.

(ii) Find Area of ΔPQR : Area of quadrilateral $SRQT$.

(b) The first and last term of a Geometric Progression (GP) are 3 and 96 respectively. If the common ratio is 2, then find

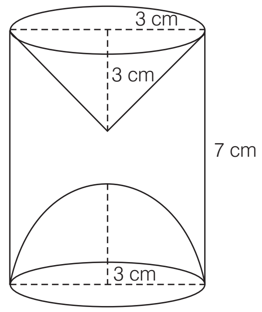
(i) n in the number of terms of the GP.

(ii) Sum of the n terms. [3]

(c) A hemispherical and a conical hole is scooped out of a solid wooden cylinder. Find the volume of the remaining solid where the measurements are as follows

The height of the solid cylinder is 7 cm, radius of each of hemisphere, cone and cylinder is 3 cm. Height of cone is 3 cm. Give your answer correct to the nearest whole number.

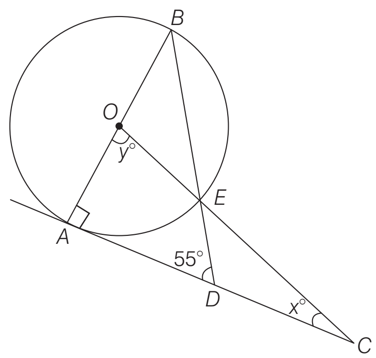
$\left[\text{Take } \pi = \frac{22}{7}\right]$.



[4]

7. (a) In the given figure AC is a tangent to the circle with centre O.

If $\angle ADB = 55°$, find x and y. Give reasons for your answers.



[3]

(b) The model of a building is constructed with the scale factor 1 : 30. [3]

(i) If the height of the model is 80 cm, then find the actual height of the building in metres.

(ii) If the actual volume of a tank at the top of the building is 27 m³, then find the volume of the tank on the top of the model.

(c) Given $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} M = 6I$, where M is a matrix and I is unit matrix of order 2×2.

(i) State the order of matrix M.

(ii) Find the matrix M. [3]

8. (a) The sum of the first three terms of an Arithmetic Progression (AP) is 42 and the product of the first and third term is 52. Find the first term and the common difference. [3]

(b) The vertices of a ΔABC are $A(3, 8)$, $B(-1, 2)$ and $C(6, -6)$. Find the

(i) slope of BC.

(ii) equation of a line perpendicular to BC and passing through A. [3]

(c) Using ruler and a compass only construct a semi-circle with diameter $BC = 7$ cm. Locate a point A on the circumference of the semi-circle such that A is equidistant from B and C. Complete the cyclic quadrilateral $ABCD$, such that D is equidistant from AB and BC. Measure $\angle ADC$ and write it down. [4]

9. (a) The data on the number of patients attending a hospital in a month are given below. Find the average (mean) number of patients attending the hospital in a month by using the shortcut method.

Take the assumed mean as 45. Give your answer correct to 2 decimal places.

Number of patients	10-20	20-30	30-40	40-50	50-60	60-70
Number of days	5	2	7	9	2	5

[3]

(b) Using properties of proportion solve for x, given $\dfrac{\sqrt{5x} + \sqrt{2x-6}}{\sqrt{5x} - \sqrt{2x-6}} = 4$. [3]

(c) Sachin invests ₹ 8500 in 10%, ₹ 100 shares at ₹ 170. He sells the shares when the price of each share rises by ₹ 30. He invests the proceeds in 12% ₹ 100 shares at ₹ 125.

Find

(i) the sale proceeds.

(ii) the number of ₹ 125 shares he buys.

(iii) the change in his annual income. [4]

10. (a) Use graph paper for this question.

The marks obtained by 120 students in an English test are given below

Marks	Number of students
0-10	5
10-20	9
20-30	16
30-40	22
40-50	26
50-60	18
60-70	11
70-80	6
80-90	4
90-100	3

Draw the ogive and hence, estimate

(i) the median marks.

(ii) the number of students who did not pass the test if the pass percentage was 50.

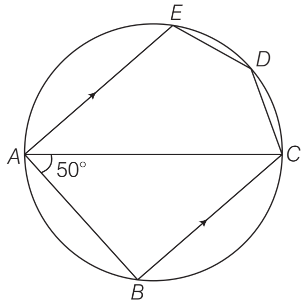
(iii) the upper quartile marks. [6]

(b) A man observes the angle of elevation of the top of the tower to be 45°. He walks towards it in a horizontal line through its base. On covering 20 m the angle of elevation changes to 60°. Find the height of the tower correct to 2 significant figures. [4]

11. (a) Using the remainder theorem find the remainders obtained when $x^3 + (kx + 8)x + k$ is divided by $x + 1$ and $x - 2$. Hence find k, if the sum of the two remainders is 1. [3]

(b) The product of two consecutive natural numbers which are multiples of 3 is equal to 810. Find the two numbers. [3]

(c) In the given figure, $ABCDE$ is a pentagon inscribed in a circle such that AC is a diameter and side $BC \parallel AE$. If $\angle BAC = 50°$, then find giving reasons



(i) $\angle ACB$ (ii) $\angle EDC$

(iii) $\angle BEC$

Hence prove that BE is also a diameter. [4]

SOLUTIONS

1. (a) We have, $11x - 4 < 15x + 4 \le 13x + 14,\ x \in W$

Consider, $11x - 4 < 15x + 4$

$\Rightarrow \quad 11x - 4 - 4 < 15x + 4 - 4$ [subtract 4 from both sides]

$\Rightarrow \quad 11x - 8 < 15x$

$\Rightarrow \quad 11x - 8 - 11x < 15x - 11x$ [subtract $11x$ from both sides]

$\Rightarrow \quad -8 < 4x$

$\Rightarrow \quad \dfrac{-8}{4} < \dfrac{4x}{4}$

[divide both sides by 4]

$-2 < x$

or $\quad x > -2 \quad$ ...(i)

and $\quad 15x + 4 \le 13x + 14$

$\Rightarrow \quad 15x + 4 - 13x \le 13x + 14 - 13x$ [subtract $13x$ from both sides]

$\Rightarrow \quad 2x + 4 \le 14$

$\Rightarrow \quad 2x + 4 - 4 \le 14 - 4$ [subtract 4 from both sides]

$\Rightarrow \quad 2x \le 10$

$\Rightarrow \quad \dfrac{2x}{2} \le \dfrac{10}{2}$ [divide both sides by 5]

$\Rightarrow \quad x \le 5 \quad$ ...(ii)

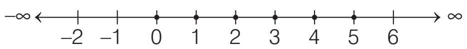
$\therefore$ From Eqs. (i) and (ii), we get

$-2 < x \le 5$

Since, $x \in W$

$\therefore$ Solution set = {0, 1, 2, 3, 4, 5}

On the number line it can be represented as



(b) Given, total investment = ₹ 4500

and rate of dividend = 7.5%

(i) Nominal value of one share = ₹ 100 − 10% of 100

$= 100 - 10 = ₹\ 90$

$\therefore$ Number of shares, he purchase $= \dfrac{4500}{90} = 50$

(ii) Annual income

= Number of shares × Nominal value of share × Rate of dividend

$= 50 \times 90 \times 7.5 = ₹\ 337.5$

(c)

Marks (x)	Number of students (f)	cf
1	1	1
2	2	3
3	3	6
4	3	9
5	6	15
6	10	25
7	5	30
8	4	34
9	3	37
10	3	40
Total	$\Sigma f_i = 40$	

(i) Here, $N = 40$, which is even.

$$\therefore \text{Median} = \frac{\left(\frac{40}{2}\right)\text{th} + \left(\frac{40}{2} + 1\right)\text{th}}{2} = \frac{20\text{th} + 21\text{ th}}{2} = \frac{6 + 6}{2} = 6$$

(ii) It is clear from the table that highest number of students is 10, i.e. it has highest frequency 10. Therefore corresponding value of 10 is 6.

Hence, the mode is 6.

2. (a) Let $f(x) = x^3 + x^2 - 4x - 4 \quad$ ...(i)

Using factor theorem, if $(x - 2)$ is a factor of $f(x)$, then $f(2) = 0$

Now, put $x = 2$ in Eq. (i), we get

$f(2) = (2)^3 + (2)^2 - 4 \times 2 - 4$

$= 8 + 4 - 8 - 4 = 0$

Hence, factor theorem is satisfied.

Now, using long division divide $f(x)$ by $(x - 2)$, we get

$x - 2\ )\ x^3 + x^2 - 4x - 4\ (\ x^2 + 3x + 2$

$x^3 - 2x^2$

$-\quad +$

$3x^2 - 4x$

$3x^2 - 6x$

$-\quad +$

$2x - 4$

$2x - 4$

$-\quad +$

0

$\therefore f(x) = (x - 2)(x^2 + 3x + 2)$

$= (x - 2)[x^2 + 2x + x + 2]$

$= (x - 2)[x(x + 2) + 1(x + 2)]$

$= (x - 2)[(x + 2)(x + 1)]$

(b) To prove,

$(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$

$$\text{LHS} = \left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= \left(\frac{1 - \sin^2\theta}{\sin\theta}\right)\left(\frac{1 - \cos^2\theta}{\cos\theta}\right)\left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)$$

$$= \frac{\cos^2\theta}{\sin\theta} \times \frac{\sin^2\theta}{\cos\theta}\left(\frac{1}{\sin\theta\cos\theta}\right) \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$= 1 = \text{RHS}$

Hence proved.

(c) Let a and d be the first term and common difference of an AP.

Then, $T_4 = 8$ and $T_6 = 14$

$\Rightarrow \quad a + (4 - 1)d = 8$

and $\quad a + (6 - 1)d = 14 \quad [\because T_n = a + (n - 1)d]$

$\Rightarrow \quad a + 3d = 8 \quad$ ...(i)

and $\quad a + 5d = 14 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$a = -1$ and $d = 3$

(i) First term $(a) = -1$

(ii) Common difference $(d) = 3$

(iii) Sum of first 20 terms $= \frac{20}{2}[2 \times (-1) + (20-1)3]$

$$= 10[-2 + 57]$$
$$= 10 \times 55 = 550$$

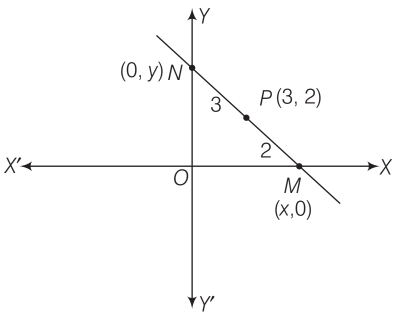
3. (a) $\sin A \begin{bmatrix} \sin A & -\cos A \\ \cos A & \sin A \end{bmatrix} + \cos A \begin{bmatrix} \cos A & \sin A \\ -\sin A & \cos A \end{bmatrix}$

$$= \begin{bmatrix} \sin^2 A & -\sin A \cos A \\ \sin A \cos A & \sin^2 A \end{bmatrix} + \begin{bmatrix} \cos^2 A & \cos A \sin A \\ -\sin A \cos A & \cos^2 A \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2 A + \cos^2 A & -\sin A \cos A + \cos A \sin A \\ \sin A \cos A - \sin A \cos A & \sin^2 A + \cos^2 A \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad [\because \cos^2\theta + \sin^2\theta = 1]$$

(b) Let the coordinates of M and N be $(x, 0)$ and $(0, y)$.



(i) Using internal section formula,

$$P(3, 2) = \left(\frac{3x + 0}{2 + 3}, \frac{0 + 2y}{2 + 3}\right)$$

$\Rightarrow \quad (3, 2) = \left(\frac{3x}{5}, \frac{2y}{5}\right)$

$\therefore \quad \frac{3x}{5} = 3$ and $\frac{2y}{5} = 2$

$\Rightarrow \quad x = 5$ and $y = 5$

Hence, the coordinates of M and N are $(5, 0)$ and $(0, 5)$, respectively.

(ii) Slope of line $MN = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 0}{0 - 5} = \frac{5}{-5} = -1$

(c) Given, radius of sphere $(r) = 6$ cm

and height of cylinder $(h) = 32$ cm

According to the given condition,

(i) Volume of solid sphere = Volume of cylinder

$\Rightarrow \quad \frac{4}{3}\pi r^3 = \pi R^2 h \Rightarrow \frac{4}{3}(6)^3 = R^2 \times (32)$

$\Rightarrow \quad R^2 = \frac{4}{3} \times \frac{6 \times 6 \times 6}{32} = \frac{2 \times 6 \times 6}{8} = 3^2$

$\Rightarrow \quad R = 3$ cm

Hence, the radius of cylinder is 3 cm.

(ii) Curved surface area of the cylinder $= 2\pi Rh$

$$= 2 \times 3.1 \times 3 \times 32$$
$$= 595.2 \text{ cm}^2$$

4. (a) Given numbers $K + 3$, $K + 2$, $3K - 7$ and $2K - 3$ are in proprotion.

$\therefore \quad (K + 3) : (K + 2) :: (3K - 7) : (2K - 3)$

$\Rightarrow \quad \frac{K + 3}{K + 2} = \frac{3K - 7}{2K - 3}$

$\Rightarrow \quad (K + 3)(2K - 3) = (K + 2)(3K - 7)$

$\Rightarrow \quad 2K^2 + 3K - 9 = 3K^2 - K - 14$

$\Rightarrow \quad K^2 - 4K - 5 = 0$

$\Rightarrow \quad K^2 - 5K + K - 5 = 0$ [by splitting middle term]

$\Rightarrow \quad K(K - 5) + 1(K - 5) = 0$

$\Rightarrow \quad (K - 5)(K + 1) = 0$

$\Rightarrow \quad K = -1, 5$

(b) Given, $x^2 - 4x - 8 = 0$

which is a quadratic equation of the form

$$ax^2 + bx + c = 0.$$

Here, $a = 1$, $b = -4$ and $c = -8$

By using quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{+4 \pm \sqrt{(-4)^2 - 4 \times 1 \times (-8)}}{2 \times 1}$$

$$= \frac{4 \pm \sqrt{16 + 32}}{2} = \frac{4 \pm \sqrt{48}}{2}$$

$$= \frac{4 \pm 4\sqrt{3}}{2} = \frac{2(2 \pm 2\sqrt{3})}{2}$$

$$= 2 \pm 2\sqrt{3} = 2 + 2\sqrt{3},\ 2 - 2\sqrt{3}$$

$$= 2 + 2(1.732),\ 2 - 2(1.732)$$

$$= 2 + 3.464,\ 2 - 3.464$$

$$= 5.464,\ -1.464$$

$$= 5.46,\ -1.46$$

[correct to two significant figures]

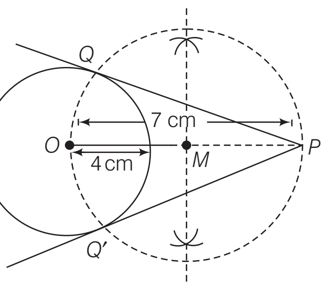
(c) Given, a circle of radius 4 cm and distance between the point and centre of a circle is 7 cm.

Steps of construction

(i) Draw a circle with centre O and radius 4 cm and take a point P outside the circle at a distance of 7 cm from its centre.

(ii) Join OP.

(iii) Draw a perpendicular bisector of OP, which intersects OP at (the mid-point) M.



(iv) Taking M as centre and $OM = MP$ as radius, draw a dotted circle, which intersects the given circle at Q and Q'.

(v) Join PQ and PQ', which are the required tangents.

On measuring the length of tangent with the help of ruler, we get $PQ = PQ' = 5.7$ cm.

5. (a) (i) Total number of discs in a box, $n(S) = 25$

Odd number of discs,

$E = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25\}$

$\therefore \quad n(E) = 13$

$\therefore$ Probability of getting odd number $= \dfrac{n(E)}{n(S)} = \dfrac{13}{25}$

(ii) Number divisible by 2 and 3 i.e. 6,

$E_1 = \{6, 12, 18, 24\}$

$\therefore \quad n(E_1) = 4$

$\therefore$ Probability of getting the number divisible by 2 and 3 $= \dfrac{n(E_1)}{n(S)} = \dfrac{4}{25}$

(iii) Number less than 16, $E_2 = \{1, 2, \ldots, 15\}$

$\therefore \quad n(E_2) = 15$

$\therefore$ Probability of getting a number less than 16

$= \dfrac{n(E_2)}{n(S)} = \dfrac{15}{25} = \dfrac{3}{5}$

(b) Let money deposit per month be $P = ₹x$.

Number of months $(n) = 20$

and rate of interest $(r) = 9\%$ per annum

$\because$ Maturity amount $= P \times n\left[1 + \dfrac{r(n+1)}{2400}\right]$

$\Rightarrow \quad 441 = x \times 20\left[1 + \dfrac{9(20+1)}{2400}\right]$

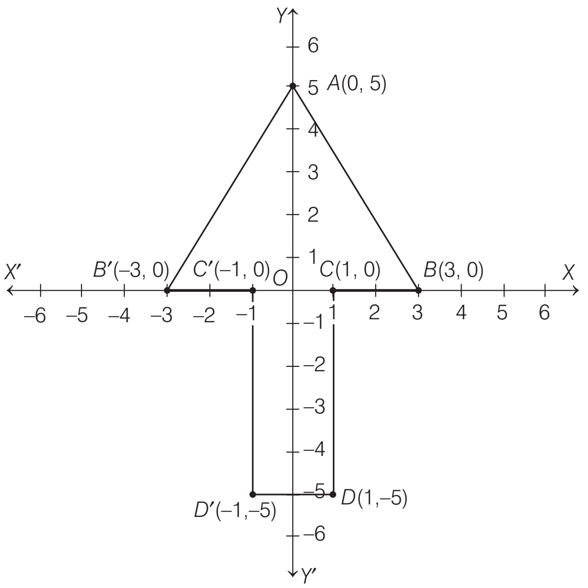
$\Rightarrow \quad 441 = 20x\left[1 + \dfrac{9 \times 21}{2400}\right] \Rightarrow 441 = 20x\left[1 + \dfrac{63}{800}\right]$

$\Rightarrow \quad 441 = 20x\left[\dfrac{863}{800}\right] \Rightarrow x = \dfrac{441 \times 800}{20 \times 863} = ₹\ 20.44$

Hence, Rekha deposit each month of ₹ 20.44.

(c) (i) We can plot the points $A(0, 5)$, $B(3, 0)$, $C(1, 0)$ and $D\ (1, -5)$

(ii) The reflection of the points B, C and D on the Y-axis are B', C' and D' which are shown below



(iii) We know that the image of points (x, y) on Y-axis has the coordinates $(-x, y)$.

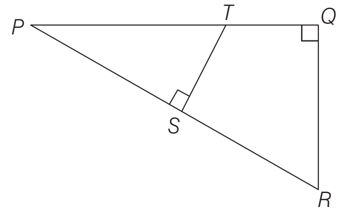
$\therefore$ Image of point $B(3, 0)$ on Y-axis has the coordinate $B'(-3, 0)$.

Image of point $C(1, 0)$ on Y-axis has the coordinate $C'(-1, 0)$.

Image of point $D(1, -5)$ on Y-axis has the coordinate $D'(-1, -5)$.

(iv) When we join the points A, B, C, D, D', C', B', A in order we get the figure of an arrow.

6. (a) Given, $\angle PQR = \angle PST = 90°$, $PQ = 5$ cm and $PS = 2$ cm



(i) To prove, $\Delta PQR \sim \Delta PST$

In ΔPQR and ΔPST,

$\angle PQR = \angle PST = 90°$ [given]

$\angle QPR = \angle SPT$ [common angle]

$\therefore \quad \Delta PQR \sim \Delta PST$ [by AA similarity criterian]

(ii) Since, $\Delta PQR \sim \Delta PST$

$\therefore \quad \dfrac{PQ}{PS} = \dfrac{QR}{TS} = \dfrac{PR}{PT}$

$\Rightarrow \quad \dfrac{5}{2} = \dfrac{QR}{TS} = \dfrac{PR}{PT}$

$\Rightarrow TS = 2k$ and $QR = 5k$, where k is any non-zero constant.

Area of $\Delta PST = \dfrac{1}{2} \times PS \times TS = \dfrac{1}{2} \times 2 \times 2k = 2k$

Area of $\Delta PQR = \dfrac{1}{2} PQ \times QR$

$= \dfrac{1}{2} \times 5 \times 5k = \dfrac{25k}{2}$

Now, area of quadrilateral $SRQT$

$=$ Area of $\Delta PQR -$ Area of ΔPST

$= \dfrac{25}{2}k - 2k = \dfrac{21k}{2}$

$\therefore$ Area of ΔPQR : Area of quadrilateral $SRQT$

$= \dfrac{\frac{25}{2}k}{\frac{21}{2}k} = \dfrac{25}{21}$

(b) Given, $a = 3$, $l = a_n = 96$ and $r = 2$

(i) nth term of a GP is $a_n = ar^{n-1}$

$\therefore \quad 96 = 3(2)^{n-1}$

$\Rightarrow \quad 32 = 2^{n-1}$

$\Rightarrow \quad 2^5 = 2^{n-1}$

On equating the power 2, we get

$5 = n - 1$

$\Rightarrow \quad n = 6$

Hence, the number of terms in a GP is 6.

(ii) Now, sum of n terms of a GP is

$$S_n = \frac{a(r^n - 1)}{(r-1)}, r > 1 = \frac{3(2^6 - 1)}{2-1}$$

$$= \frac{3 \times (64-1)}{1} = 3 \times 63 = 189$$

(c) Given, for cylinder, $h_1 = 7$ cm and $r_1 = 3$ cm

For hemisphere, $r_2 = 3$ cm

For cone,

$$r_3 = 3 \text{ cm and } h_2 = 3 \text{ cm}$$

Now, volume of cone,

$$V_1 = \frac{1}{3}\pi r_3^2 h_2 = \frac{1}{3} \times \frac{22}{7} \times (3)^2 \times 3$$

$$= \frac{22 \times 9}{7} \qquad = \frac{198}{7} \text{cm}^3$$

Volume of hemisphere, $V_2 = \frac{2}{3}\pi r_2^3$

$$= \frac{2}{3} \times \frac{22}{7} \times (3)^3$$

$$= \frac{44 \times 9}{7} = \frac{396}{7} \text{cm}^3$$

and volume of cylinder, $V_3 = \pi r_1^2 h_1$

$$= \frac{22}{7} \times (3)^2 \times 7 = \frac{1386}{7} \text{ cm}^3$$

$\therefore$ Volume of remaining solid figure

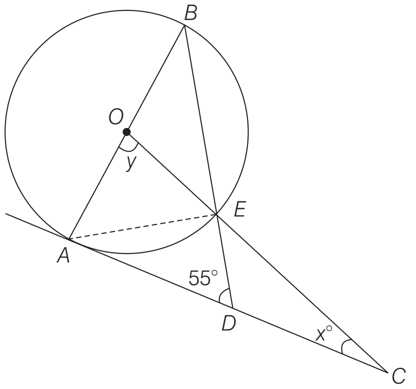
= Volume of cylinder – Volume of hemisphere – Volume of cone

$$= V_3 - V_2 - V_1$$

$$= \frac{1386}{7} - \frac{396}{7} - \frac{198}{7}$$

$$= \frac{792}{7} = 113 \text{ cm}^3$$

7. (a) Given, $\angle ADB = 55°$



Join AE.

Here, AC is a tangent.

$\therefore \quad \angle OAC = 90°$

[$\because$ tangent at any point of a circle and radius through the point of contact are perpendicular to each other]

In ΔABD,

$$\angle ABD + \angle BAD + \angle ADB = 180°$$

[by angle sum property of a triangle]

$\Rightarrow \quad \angle ABD + 90° + 55° = 180°$

$\Rightarrow \quad \angle ABD = 180° - (90° + 55°) = 35°$

We know that angle made by a chord at centre is double the angle made by it any remaining part

$\therefore \quad y = 2\angle ABD = 2 \times 35° = 70°.$

In ΔOAC,

$$y + \angle OAC + \angle ACO = 180°$$

$\Rightarrow \quad 70° + 90° + x° = 180°$

$\Rightarrow \quad x° = 180° - (70° + 90°) = 20°$

(b) Given scale factor is 1 : 30 i.e. $k = \frac{1}{30}$

(i) $\because$ Height of building on the model

= k (Height of actual model)

$\Rightarrow \quad 80 = \frac{1}{30}$ (height of actual model)

$\therefore$ Height of actual model $= 80 \times 30$

$= 2400$ cm

(ii) Volume of model tank of the building

= k^3 (Volume of actual tank of the building)

$$= \left(\frac{1}{30}\right)^3 (27) = \frac{27}{27000} \text{ m}^3$$

$$= \frac{1}{1000} \text{ m}^3$$

(c) We have, $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} M = 6I$ ...(A)

(i) Since, $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ is a order of 2×2 and right side of the equation I is also a 2×2 order. So, the matrix M should be the order of 2×2.

(ii) Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

From Eq. (A), we get

$$\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4a + 2c & 4b + 2d \\ -a + c & -b + d \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

On equating both sides, we get

$4a + 2c = 6$ ...(i)

$4b + 2d = 0$ ...(ii)

$-a + c = 0$ ...(iii)

and $-b + d = 6$ ...(iv)

On solving Eqs. (i) and (iii), we get

$$a = 1, c = 1$$

Now, on solving Eqs. (ii) and (iii), we get

$$b = -2, d = 4$$

$\therefore \quad$ Matrix $M = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

8. (a) Let a and d be the first term and common difference of an AP and first three terms of an AP are a, $a+d$ and $a+2d$.

According to the question,

$$a+(a+d)+(a+2d)=42$$
$$\Rightarrow \quad 3a+3d=42$$
$$\Rightarrow \quad a+d=14$$

Also, $\quad a\times(a+2d)=52$

$$\Rightarrow \quad a\times[a+2(14-a)]=52$$
$$\Rightarrow \quad a\times(28-a)=52$$
$$\Rightarrow \quad 28a-a^2=52$$
$$\Rightarrow \quad a^2-28a+52=0$$
$$\Rightarrow \quad a^2-26a-2a+52=0$$
$$\Rightarrow \quad a(a-26)-2(a-26)=0$$
$$\Rightarrow \quad (a-2)(a-26)=0$$
$$\Rightarrow \quad a=2,\ a=26$$

When $a=2$, then $d=14-a$
$$=14-2=12$$

When $a=26$, then $d=14-26=-12$

(b) Given vertices of a ΔABC are
$$A(3,8),\ B(-1,2) \text{ and } C(6,-6).$$

(i) Now, slope of BC, $m=\dfrac{y_2-y_1}{x_2-x_1}$
$$=\frac{-6-2}{6-(-1)}=\frac{-8}{7}$$

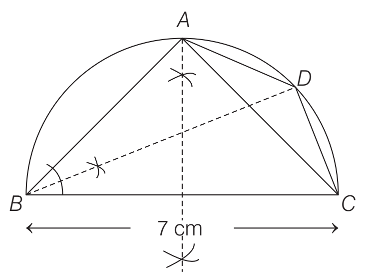
(ii) Now, equation of line passing through $A(3,8)$ and perpendicular to BC is
$$y-8=-\frac{1}{m}(x-3)$$
$$\Rightarrow \quad y-8=+\frac{7}{8}(x-3)$$
$$\Rightarrow \quad 8y-64=7x-21$$
$$\Rightarrow \quad 7x-8y+43=0$$

(c) Given, diameter of a semi-circle $BC=7$ cm

Steps of construction

(i) First, draw a line segment $BC=7$ cm.

(ii) Draw a perpendicular bisector on BC which meets the circle at A.



(iii) Since, D is equidistant from A and C, therefore D lies on the bisector of $\angle ABC$.

As point A is on semicircle, therefore
$$\angle BAC=90^\circ$$

Also A is equidistant from points B and C

$$\therefore \quad \angle ABC=\angle ACB$$

In ΔABC,
$$\angle BAC+\angle ABC+\angle ACB=180^\circ$$
$$\Rightarrow \quad 90^\circ+2\angle ABC=180^\circ$$
$$\Rightarrow \quad 2\angle ABC=90^\circ \Rightarrow \angle ABC=45^\circ$$

Since, $ABCD$ is a cyclic quadrilateral.
$$\therefore \quad \angle ABC+\angle ADC=180^\circ$$
[sum of opposite angles of a cyclic quadrilateral is 180°]
$$\Rightarrow \quad \angle ADC=180^\circ-45^\circ=135^\circ$$

9. (a) Given, assumed mean, $A=45$

and class width, $h=20-10=10$

The table for the given data is

Class	x_i	f_i	$u_i=\dfrac{x_i-A}{h}$	$f_i u_i$
10-20	$\frac{10+20}{2}=15$	5	$\frac{15-45}{10}=-3$	−15
20-30	$\frac{20+30}{2}=25$	2	$\frac{25-45}{10}=-2$	−4
30-40	$\frac{30+40}{2}=35$	7	$\frac{35-45}{10}=-1$	−7
40-50	$\frac{40+50}{2}=45$	9	$\frac{45-45}{10}=0$	0
50-60	$\frac{50+60}{2}=55$	2	$\frac{55-45}{10}=1$	2
60-70	$\frac{60+70}{2}=65$	5	$\frac{65-45}{10}=2$	10
Total		$\Sigma f_i=30$		$\Sigma f_i u_i=-14$

Here, $\Sigma f_i u_i=-14$, $\Sigma f_i=30$, $A=45$ and $h=10$

$$\therefore \quad \text{Mean } (\bar{x})=A+\frac{\Sigma f_i u_i}{\Sigma f_i}\times h$$
$$=45+\frac{-14}{30}\times(10)$$
$$=45-\frac{14}{3}=\frac{135-14}{3}=\frac{121}{3}=40.33$$

(b) We have, $\dfrac{\sqrt{5x}+\sqrt{2x-6}}{\sqrt{5x}-\sqrt{2x-6}}=\dfrac{4}{1}$

Applying componendo and dividendo rule,
$$\frac{(\sqrt{5x}+\sqrt{2x-6})+(\sqrt{5x}-\sqrt{2x-6})}{(\sqrt{5x}+\sqrt{2x-6})-(\sqrt{5x}-\sqrt{2x-6})}=\frac{4+1}{4-1}$$
$$\Rightarrow \quad \frac{2\sqrt{5x}}{2\sqrt{2x-6}}=\frac{5}{3}$$
$$\Rightarrow \quad 3\sqrt{5x}=5\sqrt{2x-6}$$

On squaring both sides, we get
$$9(5x)=25(2x-6)$$
$$\Rightarrow \quad 45x=50x-150$$
$$\Rightarrow \quad 5x=150$$
$$\Rightarrow \quad x=30$$

(c) Given, money invested by Sachin = ₹ 8500

Nominal value of one share = ₹ 100

Market value of one share = ₹ 170

∴ Number of shares he bought

$$= \frac{\text{Money invested}}{\text{Market value of one share}}$$

$$= ₹ \frac{8500}{170} = 50$$

Total nominal value = Number of shares × Nominal value of one share

$= 50 \times 100 = ₹ 5000$

∴ Dividend = 10% of total nominal value

$= \frac{10}{100} \times 5000 = ₹ 500$

(i) Sale proceed = Amount received on selling 50 shares = Number of shares × Market value

$= 50 \times 200 = ₹ 10000$

(ii) If he bought a share at ₹ 125, then the number of shares he bought $= \frac{10000}{125} = 80$

(iii) Annual income of these shares

$= \frac{12}{100} \times 100 \times 80$

$= ₹ 960$

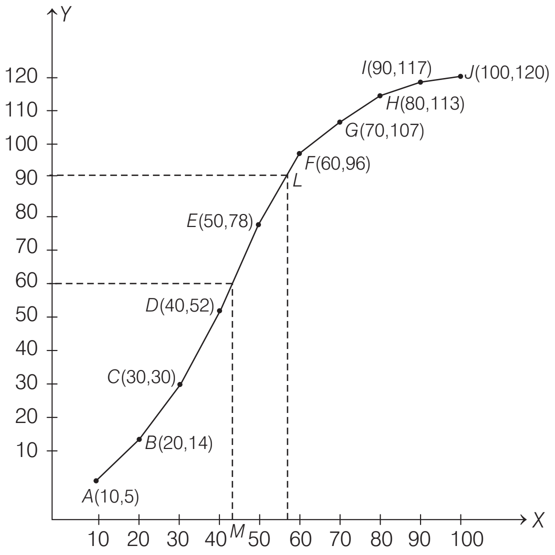
∴ Change in his annual income

$= 960 - 500 = ₹ 460$

10. (a) The cumulative frequency table for the given continuous distribution is given below

Marks	Number	Cumulative frequency (cf)
0-10	5	5
10-20	9	14
20-30	16	30
30-40	22	52
40-50	26	78
50-60	18	96
60-70	11	107
70-80	6	113
80-90	4	117
90-100	3	120
Total	120	

On the graph paper, we plot the following points $A(10, 5)$, $B(20, 14)$, $C(30, 30)$, $D(40, 52)$, $E(50, 78)$, $F(60, 96)$, $G(70, 107)$, $H(80, 113)$, $I(90, 117)$, $J(100, 120)$. Join all these points by a free hand drawing. The required ogive is shown on the graph paper, which is given below



(i) Here $n = 120$

Now, $\frac{n}{2} = \frac{120}{2} = 60$

Taking point 60 on Y-axis, draw a horizontal line parallel to Y-axis, which intersect the curve at point P. Now, draw a line parallel to Y-axis from point P, which meets the X-axis at M.

∴ The distance of point M from O is 43, which is the required median marks.`

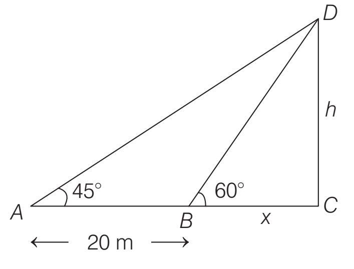
(ii) The number of students, who did not pass the test is 78.

(iii) Here, $n = 120$

For upper quartile, $\frac{3n}{4} = \frac{3 \times 120}{4} = 90$

Taking point 90 on Y-axis, draw a horizontal line parallel to X-axis, which meets the curve at point L. Further draw a line at point L parallel to Y-axis, which meets the X-axis at point M'. Hence, the distance from origin to the point M is 57, which is the required upper quartile marks.

(b) Let height of tower be $CD = h$ m and distance $BC = x$ m.



In right angled ΔBCD,

$$\tan 60° = \frac{CD}{BC} \Rightarrow \sqrt{3} = \frac{h}{x}$$

$$\Rightarrow \quad x = \frac{h}{\sqrt{3}} \quad \text{...(i)}$$

and in right angled ΔACD,

$$\tan 45^\circ = \frac{h}{20 + x} \Rightarrow 1 = \frac{h}{20+x}$$

$$\Rightarrow \quad 20 + x = h \Rightarrow 20 + \frac{h}{\sqrt{3}} = h \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 20 = h\left(1 - \frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \quad 20 = h\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)$$

$$\Rightarrow \quad h = \frac{20\sqrt{3}}{(\sqrt{3}-1)} \times \frac{(\sqrt{3}+1)}{(\sqrt{3}+1)}$$

$$= \frac{20\sqrt{3}(\sqrt{3}+1)}{(\sqrt{3})^2 - (1)^2} = \frac{20\sqrt{3}(\sqrt{3}+1)}{3-1}$$

$$= \frac{20\sqrt{3}(\sqrt{3}+1)}{2} = 10(3+\sqrt{3})$$

$$= 10(3 + 1.732) = 10 \times 4.732$$

$$= 47.32 \text{ m} = 47 \text{ m}$$

[correct to two significant figure]

11. (a) Let $f(x) = x^3 + (kx + 8)x + k$

Then, the remainder, when $f(x)$ is divided by $(x + 1)$ and $(x - 2)$ respectively is given by

$$f(-1) = (-1)^3 + [k(-1) + 8](-1) + k$$
$$= -1 + (-k + 8)(-1) + k$$
$$= -1 + k - 8 + k = 2k - 9$$

and $f(+2) = (2)^3 + (k \times 2 + 8)2 + k$

$$= 8 + (4k + 16) + k = 5k + 24$$

According to the given condition,

$$f(-1) + f(2) = 1$$
$$\Rightarrow \quad 2k - 9 + 5k + 24 = 1$$
$$\Rightarrow \quad 7k + 15 = 1$$
$$\Rightarrow \quad 7k = -14$$
$$\Rightarrow \quad k = -2$$

(b) Let two consecutive natural numbers having multiples of 3 are $3x$ and $3x + 3$.

According to the given condition,

$$3x(3x+3) = 810$$
$$\Rightarrow \quad 9(x^2 + x) = 810$$
$$\Rightarrow \quad x^2 + x = 90$$
$$\Rightarrow \quad x^2 + x - 90 = 0$$
$$\Rightarrow \quad x^2 + 10x - 9x - 90 = 0$$
$$\Rightarrow \quad x(x+10) - 9(x+10) = 0$$
$$\Rightarrow \quad (x-9)(x+10) = 0$$
$$\Rightarrow \quad x = 9, x = -10$$

Since, numbers are natural, so we discard negative value.

$\therefore$ Required numbers are $3x = 3 \times 9 = 27$ and $3x + 3 = 27 + 3 = 30$.

(c) (i) Given, $\angle BAC = 50^\circ$ and AC is a diameter of a circle.

$\therefore$ ΔABC is a right angled triangle at B,

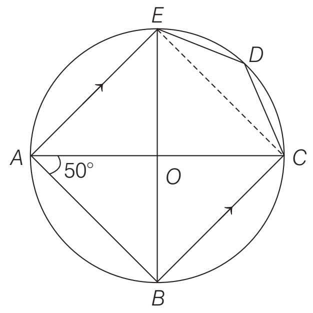
i.e. $\angle ABC = 90^\circ$.

We know that the sum of all angles of a triangle is 180°.

$$\therefore \quad \angle CAB + \angle ABC + \angle BCA = 180^\circ$$
$$\Rightarrow \quad 50^\circ + 90^\circ + \angle BCA = 180^\circ$$
$$\Rightarrow \quad \angle BCA = 180^\circ - (90^\circ + 50^\circ)$$
$$= 180^\circ - 140^\circ = 40^\circ$$

(ii) Also, $BC \parallel AE$

$$\therefore \quad \angle EAB + \angle ABC = 180^\circ$$
$$\Rightarrow \quad \angle EAO + 50^\circ + 90^\circ = 180^\circ$$
$$\Rightarrow \quad \angle EAO = 180^\circ - 140^\circ = 40^\circ$$



Here, $AEDC$ is a cyclic quadrilateral.

$$\therefore \quad \angle EAO + \angle EDC = 180^\circ$$
$$\Rightarrow \quad 40^\circ + \angle EDC = 180^\circ$$
$$\Rightarrow \quad \angle EDC = 140^\circ$$

(iii) Since, AC is a diameter of a circle.

Therefore, $\angle AEC = 90^\circ$

$$\Rightarrow \quad \angle ECB = 90^\circ$$

[$\because$ sum of all angle of a quadrilateral is 360°]

It is clear that $ABCE$ is a square.

So, EB is the angle bisector of $\angle AEC$.

$$\therefore \quad \angle BEC = \frac{90^\circ}{2} = 45^\circ$$

Since, $AE \parallel BC$ and AB is a transversal.

$$\therefore \quad \angle EAB + \angle ABC = 180^\circ$$
$$\Rightarrow \quad \angle EAB + 90^\circ = 180^\circ$$
$$\Rightarrow \quad \angle EAB = 90^\circ$$

It implies that line joining BE is the diameter of the circle.

FULLY SOLVED

LATEST ICSE SPECIMEN PAPER

A SAMPLE QUESTION PAPER FOR ICSE CLASS X ISSUED BY COUNCIL OF INDIAN SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS (FULLY SOLVED)

FULLY SOLVED

GENERAL INSTRUCTIONS

1. You will not be allowed to write during the first 15 minutes. This time is to be spent in reading the questions paper.
2. The time given at the head of this paper is the time allowed for writing the answers.
3. Attempt **all questions** from **Section A** and **any 4 questions** from **Section B**.
4. All working, including rough work, must be clearly shown and must be done on the same sheet as the rest of the answer.
5. Omission of essential working will result in loss of marks.
6. The intended marks for questions or parts of questions are given in brackets [ ].
7. Mathematical tables are provided.

Time : 2.5 Hrs **Max. Marks : 80**

Section A [40 *Marks*]

1. (a) Find the value of 'k', if $4x^3 - 2x^2 + kx + 5$ leaves remainder -10 when divided by $2x + 1$. [3]

(b) Amit deposits ₹ 1600 per month in a bank for 18 months in a recurring deposit account. If he gets ₹ 31080 at the time of maturity, what is the rate of interest per annum? [3]

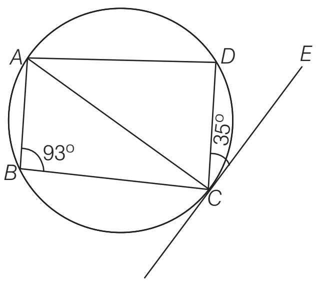
*(c) The price of an article is ₹ 9350 which includes VAT at 10%. Find how much less a customer pays for the article, if the VAT on the article decreases by 3%. [4]

2. (a) Solve the following inequation and represent your solution on the real number line.

$$-5\frac{1}{2} - x \le \frac{1}{2} - 3x \le 3\frac{1}{2} - x,\ x \in R$$ [3]

(b) Find the 16th term of the AP 7, 11, 15, 19,.... Find the sum of the first 6 terms. [3]

(c) In the given figure, CE is a tangent to the circle at point C. ABCD is a cyclic quadrilateral. If $\angle ABC = 93°$ and $\angle DCE = 35°$.



Find

(i) $\angle ADC$ (ii) $\angle CAD$

(iii) $\angle ACD$ [4]

3. (a) Prove the following identity.

$$\frac{\sec A}{\sec A - 1} + \frac{\sec A}{\sec A + 1} = 2\ \text{cosec}^2 A$$ [3]

(b) Find x and y, if

$$3\begin{bmatrix} 5 & -6 \\ 4 & x \end{bmatrix} - \begin{bmatrix} 6 & y \\ 0 & 6 \end{bmatrix} = 3\begin{bmatrix} 3 & -2 \\ 4 & 0 \end{bmatrix}.$$ [3]

(c) For what value of 'k' will the quadratic equation $(k+1)x^2 - 4kx + 9 = 0$ have real and equal roots? Solve the equations. [4]

* These questions are related to the chapter 'Value Added Tax (VAT)', which has removed in the latest syllabus, so we have not given solutions to these questions.

4. (a) A box consists of 4 red, 5 black and 6 white balls. One ball is drawn out at random. Find the probability that the ball drawn is

(i) black. (ii) red or white [3]

(b) Calculate the median and mode for the following distribution.

Weight (in kg)	35	47	52	56	60
Number of students	4	3	5	3	2

[3]

(c) A solid cylinder of radius 7 cm and height 14 cm is melted and recast into solid spheres each of radius 3.5 cm. Find the number of spheres formed. [4]

Section B [40 *Marks*]

5. (a) The 2nd and 45th term of an arithmetic progression are 10 and 96 respectively. Find the first term and the common difference and hence find the sum of the first 15 terms. [3]

(b) If $A = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$, find matrix B such that $A^2 - 2B = 3A - 5I$, where I is 2×2 identity matrix. [3]

(c) With the help of a graph paper, taking 1 cm = 1 unit along both X and Y-axes.

(i) Plot points A (0, 3), B (2, 3), C (3, 0), D (2, –3), E (0, –3)

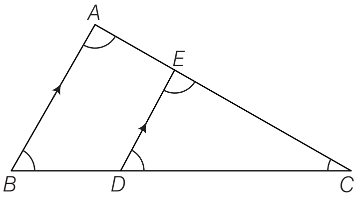
(ii) Reflect points B, C and D on the Y-axis and name them as B′, C′ and D′ respectively.

(iii) Write the coordinates of B′, C′ and D′.

(iv) Write the equation of line BD′.

(v) Name the figure BCDD′ C′ B′ B. [4]

6. (a) In ΔABC and ΔEDC, AB is parallel to ED. BD = $\frac{1}{3}$BC and AB = 12.3 cm.



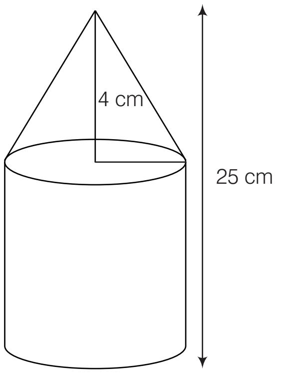
(i) Prove that $\Delta ABC \sim \Delta EDC$.

(ii) Find DE.

(iii) Find $\frac{\text{area of } \Delta EDC}{\text{area of } \Delta ABC}$. [3]

(b) Find the ratio in which the line joining (– 2, 5) and (– 5, – 6) is divided by the line $y = -3$. Hence find the point of intersection. [3]

(c) The given solid figure is a cylinder surmounted by a cone. The diameter of the base of the cylinder is 6 cm. The height of the cone is 4 cm and the total height of the solid is 25 cm. $\left[\text{Take } \pi = \frac{22}{7}\right]$



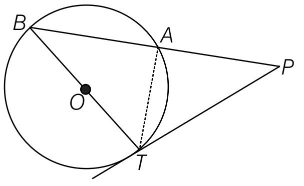
Find the

(i) volume of the solid.

(ii) curved surface area of the solid.

Give your answers correct to the nearest whole number. [4]

7. (a) In the given figure, PAB is a secant and PT a tangent to the circle with centre O. If $\angle ATP = 40°$, PA = 9 cm and AB = 7 cm



Find

(i) $\angle APT$ (ii) length of PT [3]

(b) The 1st and the 8th term of a GP are 4 and 512 respectively.

Find

(i) the common ratio.

(ii) the sum of its first 5 terms. [3]

(c) The mean of the following distribution is 49. Find the missing frequency 'a'.

Class	0-20	20-40	40-60	60-80	80-100
Frequency	15	20	30	a	10

[4]

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8. (a) Prove the following identity.

$$(\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 = 5 + \sec^2 A \cdot \operatorname{cosec}^2 A$$ [3]

(b) Find the equation of the perpendicular bisector of line segment joining $A(4, 2)$ and $B(-3, -5)$. [3]

(c) Using properties of proportion, find $x : y$, if

$$\frac{x^3 + 12x}{6x^2 + 8} = \frac{y^3 + 27y}{9y^2 + 27}.$$ [4]

9. (a) The difference of the squares of two natural numbers is 84. The square of the larger number is 25 times the smaller number. Find the numbers. [4]

(b) The following table shows the distribution of marks in Mathematics

Marks (Less than)	Number of students
10	7
20	28
30	54
40	71
50	84
60	105
70	147
80	180

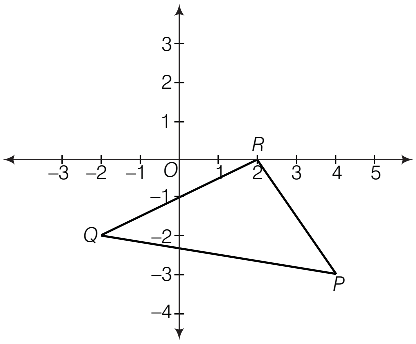
With the help of a graph paper, taking 2 cm = 10 units along one axis and 2 cm = 20 units along the other axis, plot an ogive for the above distribution and use it to find the

(i) median.

(ii) number of students who scored distinction marks (75% and above).

(iii) number of students, who passed the examination if pass marks is 35%. [6]

10. (a) Prove that two tangents drawn from an external point to a circle are of equal length. [3]



(b) From the above figure, find the

(i) coordinates of points P, Q, R.

(ii) equation of the line through *P* and parallel to QR. [3]

*(c) A manufacturer sells an article to a wholesaler with marked price ₹ 2000 at a discount of 20% on the marked price. The wholesaler sells it to a retailer at a discount of 12% on the marked price. The retailer sells the article at the marked price. If the VAT paid by the wholesaler is ₹ 11.20, find the

(i) rate of VAT.

(ii) VAT paid by the retailer. [4]

11. (a) Mr. Sharma receives an annual income of ₹ 900 in buying ₹ 50 shares selling at ₹ 80. If the dividend declared is 20%, find the

(i) amount invested by Mr. Sharma.

(ii) percentage return on his investment. [3]

(b) Two poles AB and PQ are standing opposite each other on either side of a road 200 m wide. From a point R between them on the road, the angles of elevation of the top of the poles AB and PQ are 45° and 40° respectively. If height of AB = 80 m, find the height of PQ correct to the nearest metre. [3]

(c) Construct a triangle PQR, given RQ = 10 cm, $\angle PRQ = 75°$ and base RP = 8 cm.

Find by construction

(i) the locus of points which are equidistant from QR and QP.

(ii) the locus of points which are equidistant from P and Q.

(iii) mark the point O which satisfies conditions (i) and (ii). [4]

SOLUTIONS

1. (a) Let $p(x) = 4x^3 - 2x^2 + kx + 5$ and $q(x) = 2x + 1$.

Put $q(x) = 0$

$\Rightarrow \quad 2x + 1 = 0$

$\Rightarrow \quad x = -\frac{1}{2}$

According to the question, when p(x) is divided by q(x), then remainder is – 10.

$\therefore \quad p\left(-\frac{1}{2}\right) = -10$ [by remainder theorem]

$\Rightarrow \quad 4\left(-\frac{1}{2}\right)^3 - 2\left(-\frac{1}{2}\right)^2 + k\left(-\frac{1}{2}\right) + 5 = -10$

$\Rightarrow \quad -\frac{4}{8} - \frac{2}{4} - \frac{k}{2} + 5 = -10$

$\Rightarrow \quad -1 - \frac{k}{2} + 5 = -10$

$\Rightarrow \quad \frac{k}{2} = 14$

$\Rightarrow \quad k = 28$

(b) We have,

Amount deposited per month, P = ₹ 1600

Total months, n = 18 months

and maturity amount = ₹ 31080

Let rate of interest per annum = r%

Now, we know that

$$\text{Maturity value} = P \times n + \left(P \times \frac{n(n+1)}{2 \times 12} \times \frac{r}{100}\right)$$

$\Rightarrow \quad 31080 = 1600 \times 18 + \frac{1600 \times 18\,(18+1)}{2 \times 12} \times \frac{r}{100}$

$\Rightarrow \quad 31080 - 28800 = \frac{1600 \times 18 \times 19}{2 \times 12} \times \frac{r}{100}$

$\Rightarrow \quad 2280 = \frac{1600 \times 18 \times 19}{2 \times 12} \times \frac{r}{100}$

$\Rightarrow \quad r = \frac{2280 \times 2 \times 12 \times 100}{1600 \times 18 \times 19} = 10\%$

Hence, the rate of interest per annum is 10%.

2. (a) We have,

$$-5\frac{1}{2} - x \le \frac{1}{2} - 3x \le 3\frac{1}{2} - x, \quad x \in R$$

$\Rightarrow \quad -5\frac{1}{2} - x \le \frac{1}{2} - 3x$

and $\quad \frac{1}{2} - 3x \le 3\frac{1}{2} - x$

$\Rightarrow \quad 3x - x \le \frac{1}{2} + 5\frac{1}{2}$

and $\quad -3x + x \le 3\frac{1}{2} - \frac{1}{2}$

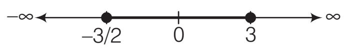
$\Rightarrow \quad 2x \le \frac{1}{2} + \frac{11}{2}$

and $\quad -2x \le \frac{7}{2} - \frac{1}{2}$

$\Rightarrow \quad 2x \le 6$ and $-2x \le 3$

$\Rightarrow \quad x \le 3$ and $x \ge -\frac{3}{2}$

$\Rightarrow \quad x \in \left[-\frac{3}{2}, 3\right]$



(b) Given AP is 7, 11, 15, 19, ...

$\therefore$ First term, a = 7 and common difference, d = 11 – 7 = 4

$\therefore \quad a_{16} = a + (16 - 1)d \quad [\because a_n = a + (n-1)d]$

$= 7 + 15 \times 4$

$= 7 + 60 = 67$

Again, $\quad S_6 = \frac{6}{2}[2 \times 7 + (6-1)4]$

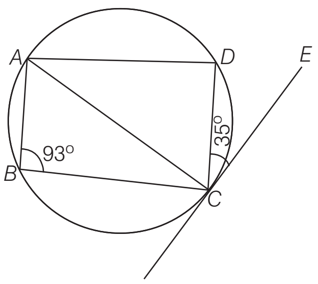
$\left[\because S_n = \frac{n}{2}(2a + (n-1)d\right]$

$= 3(2 \times 7 + 5 \times 4)$

$= 3(14 + 20)$

$= 3 \times 34 = 102$

(c) According to the question,



(i) As, ABCD is a cyclic quadrilateral.

$\therefore \quad \angle ABC + \angle ADC = 180°$

[$\therefore$ sum of opposite angles of a cyclic quadrilateral is 180°]

$\Rightarrow \quad 93° + \angle ADC = 180° \quad [\because \angle ABC = 93°]$

$\Rightarrow \quad \angle ADC = 180° - 93° = 87°$

(ii) $\angle CAD = \angle DCE$ [by alternate segment theorem]

$\Rightarrow \quad \angle CAD = 35° \quad [\because \angle DCE = 35°]$

(iii) In Δ ACD

$\angle ACD + \angle ADC + \angle CAD = 180°$

[angle sum property of triangle]

$\Rightarrow \quad \angle ACD + 87° + 35° = 180°$

$\Rightarrow \quad \angle ACD = 180° - (87° + 35°)$

$= 180° - 122° = 58°$

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3. (a) LHS $= \dfrac{\sec A}{\sec A - 1} + \dfrac{\sec A}{\sec A + 1}$

$$= \frac{\sec A(\sec A + 1) + \sec A(\sec A - 1)}{(\sec A - 1)(\sec A + 1)}$$

$$= \frac{\sec A(\sec A + 1 + \sec A - 1)}{\sec^2 A - 1}$$

$[\because (a-b)(a+b) = a^2 - b^2]$

$$= \frac{2\sec^2 A}{\tan^2 A}$$

$[\because 1 + \tan^2\theta = \sec^2\theta]$

$$= \frac{2 \times \left(\dfrac{1}{\cos^2 A}\right)}{\left(\dfrac{\sin^2 A}{\cos^2 A}\right)}$$

$\left[\because \tan A = \dfrac{\sin A}{\cos A}, \sec A = \dfrac{1}{\cos A}\right]$

$$= \frac{2}{\sin^2 A}$$

$= 2\ \text{cosec}^2 A$ $\left[\because \text{cosec}\ A = \dfrac{1}{\sin A}\right]$

(b) We have,

$$3\begin{bmatrix} 5 & -6 \\ 4 & x \end{bmatrix} - \begin{bmatrix} 6 & y \\ 0 & 6 \end{bmatrix} = 3\begin{bmatrix} 3 & -2 \\ 4 & 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 15 & -18 \\ 12 & 3x \end{bmatrix} + \begin{bmatrix} -6 & -y \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ 12 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 15-6 & -18-y \\ 12+0 & 3x-6 \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ 12 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 9 & -18-y \\ 12 & 3x-6 \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ 12 & 0 \end{bmatrix}$

$\Rightarrow$ $-18 - y = -6$ and $3x - 6 = 0$ [by equality of matrix]

$\Rightarrow$ $y = -12$ and $x = 2$

(c) We have,

$$(k+1)x^2 - 4kx + 9 = 0$$

On comparing with $ax^2 + bx + c = 0$, we get

$$a = k+1,\ b = 4k, \text{ and } c = 9$$

Since, the above equation have real and equal roots.

$\therefore$ Discriminant, $D = 0$

$\Rightarrow (-4k)^2 - 4(k+1)(9) = 0$ $[\because D = b^2 - 4ac]$

$\Rightarrow 16k^2 - 36(k+1) = 0$

$\Rightarrow 4k^2 - 9(k+1) = 0$

$\Rightarrow 4k^2 - 9k - 9 = 0$

$\Rightarrow 4k^2 - 12k + 3k - 9 = 0$

$\Rightarrow 4k(k-3) + 3(k-3) = 0$

$\Rightarrow (k-3)(4k+3) = 0$

$\Rightarrow k = 3, \dfrac{-3}{4}$

Now, if $k = 3$, then equation becomes

$$4x^2 - 12x + 9 = 0$$

$\Rightarrow 4x^2 - 6x - 6x + 9 = 0$

$\Rightarrow 2x(2x-3) - 3(2x-3) = 0$

$\Rightarrow (2x-3)(2x-3) = 0$

$\Rightarrow x = \dfrac{3}{2}, \dfrac{3}{2}$

Again, if $k = \dfrac{-3}{4}$, then equation becomes

$$\frac{1}{4}x^2 + 3x + 9 = 0$$

$\Rightarrow x^2 + 12x + 36 = 0$

$\Rightarrow (x+6)^2 = 0$

$\Rightarrow x = -6, -6$

4. (a) We have, 4 Red, 5 Black and 6 White balls.

$\therefore$ Total number of balls $= 4 + 5 + 6 = 15$.

(i) P (Black) $= \dfrac{\text{Number of black balls}}{\text{Total number of balls}}$

$$= \frac{5}{15} = \frac{1}{3}$$

(ii) P (Red or White)

= P (Red) + P (White)

$$= \frac{\text{Number of red balls}}{\text{Total number of balls}} + \frac{\text{Number of white balls}}{\text{Total number of balls}}$$

$$= \frac{4}{15} + \frac{6}{15} = \frac{10}{15} = \frac{2}{3}$$

(b) **Median** Cumulative frequency table for given distribution is given below

x_i	f_i	cf
35	4	4
47	3	7
52	5	12
56	3	15
60	2	17

Here, $N = 17$

$\therefore$ Median $= \left[\dfrac{1}{2}(N+1)\right]$th value [$\because$ N is odd]

$$= \left[\frac{1}{2}(17+1)\right] \text{th value}$$

= 9th value

$\therefore$ Median = 52

Mode Since, mode is the data having heighest frequency. So, 52 is mode as it has heighest frequency i.e. 5 among all the data.

(c) For cylinder,

Radius $(R_1) = 7$ cm, Height $(H_1) = 14$ cm

$\therefore$ Volume of cylinder $= \pi R_1^2 H_1 = \pi (7)^2 \times 14$

For sphere,

Radius $(R_2) = 3.5$ cm

$\therefore$ Volume of sphere $= \dfrac{4}{3}\pi R_2^3 = \dfrac{4}{3}\pi(3.5)^3$

Let n sphere are formed.

Then, according to the question

$n \times$ Volume of sphere = Volume of cylinder

$$\Rightarrow \quad n \times \frac{4}{3}\pi(3.5)^3 = \pi(7)^2 \times 14$$

$$\Rightarrow \quad n = \frac{7 \times 7 \times 14 \times 3}{4 \times 3.5 \times 3.5 \times 3.5} = 12$$

Hence, the number of spheres so formed is 12.

5. (a) Let the first term be a and common difference be d of the AP.

Now,

$$a_2 = 10 \Rightarrow a + d = 10 \quad \text{...(i)}$$

and $a_{45} = 96$

$\Rightarrow \quad a + 44d = 96$

...(ii)

$[\because a_n = a + (n-1)d]$

From Eq. (i), put the value of $a = 10 - d$ in Eq.^{n} (ii).

$$\therefore \quad (10 - d) + 44d = 96$$

$$\Rightarrow \quad 43d = 86$$

$$\Rightarrow \quad d = 2$$

$$\therefore \quad a = 10 - d = 10 - 2 = 8.$$

Hence, $a = 8$ and $d = 2$.

Again, we know that,

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\therefore \quad S_{15} = \frac{15}{2}[2 \times 8 + (15-1) \times 2]$$

$$= \frac{15}{2}[16 + 28] = \frac{15 \times 44}{2}$$

$$= 15 \times 22 = 330$$

Hence, the sum of first 15 terms is 330.

(b) We have,

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}, \; I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^2 = A \cdot A$$

$$= \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times 3 + (-1) \times 0 & 3 \times (-1) + (-1) \times 2 \\ 0 \times 3 + 2 \times 0 & 0 \times (-1) + 2 \times 2 \end{bmatrix}$$

$$\therefore \quad A^2 = \begin{bmatrix} 9 & -5 \\ 0 & 4 \end{bmatrix}$$

Now, we have

$$A^2 - 2B = 3A - 5I$$

$$\Rightarrow \quad 2B = A^2 - 3A + 5I$$

$$\Rightarrow \quad 2B = \begin{bmatrix} 9 & -5 \\ 0 & 4 \end{bmatrix} - 3\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

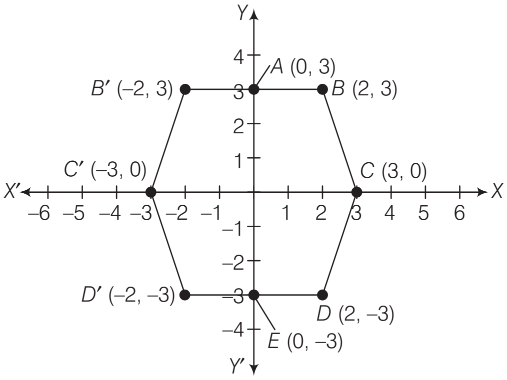
$$\Rightarrow \quad 2B = \begin{bmatrix} 9 & -5 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} -9 & 3 \\ 0 & -6 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\Rightarrow \quad 2B = \begin{bmatrix} 9 - 9 + 5 & -5 + 3 + 0 \\ 0 + 0 + 0 & 4 - 6 + 5 \end{bmatrix}$$

$$\Rightarrow \quad 2B = \begin{bmatrix} 5 & -2 \\ 0 & 3 \end{bmatrix}$$

$$\therefore \quad B = \frac{1}{2}\begin{bmatrix} 5 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5/2 & -1 \\ 0 & 3/2 \end{bmatrix}$$

(c)



(i) See the graph.

(ii) See the graph.

(iii) $B' = (-2, 3)$, $C' = (-3, 0)$ and $D' = (-2, -3)$

(iv) Equation of line passing through $B'(-2, 3)$ and $D'(-2, -3)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

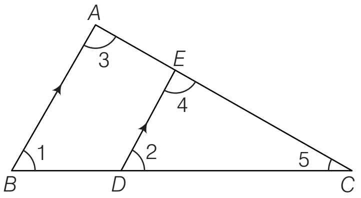
$$\Rightarrow \quad y - 3 = \frac{-3 - 3}{-2 - (-2)}(x - (-2))$$

$$\Rightarrow \quad y - 3 = \frac{-6}{0}(x + 2)$$

$$\Rightarrow \quad x + 2 = 0$$

(v) From the graph it is clear that $BCDD'C'B'B$ is hexagon.

6. (a) According to the question,



(i) Since, $AB \parallel DE$

$\therefore \angle 1 = \angle 2$ and $\angle 3 = \angle 4$ [$\because$ corresponding angles]

Now, in ΔABC and ΔEDC

$\angle 1 = \angle 2$

$\angle 3 = \angle 4$ [proved above]

$\angle 5 = \angle 5$ [common]

$\therefore$ By AAA criteria, $\Delta ABC \sim \Delta EDC$

(ii) Since, $\Delta ABC \sim \Delta EDC$

$$\therefore \quad \frac{AB}{ED} = \frac{BC}{DC}$$

$$\Rightarrow \quad \frac{AB}{ED} = \frac{BC}{BC - DB}$$

$$\Rightarrow \quad \frac{AB}{ED} = \frac{BC}{BC - \frac{1}{3}BC} \quad \left[\because BD = \frac{1}{3}BC\right]$$

$$\Rightarrow \quad \frac{AB}{ED} = \frac{BC}{\frac{2}{3}BC}$$

$$\Rightarrow \quad \frac{AB}{ED} = \frac{3}{2}$$

$$\Rightarrow \quad ED = \frac{2}{3}AB = \frac{2}{3} \times 12.3$$

$$= 8.2\text{ cm} \qquad [\because AB = 12.3\text{ cm}]$$

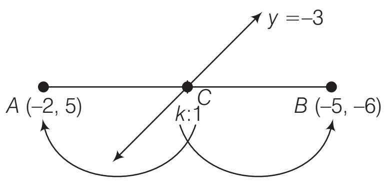
(iii) We know that, if two triangles are similar then the ratio of their areas is equal to the square of the ratio of their corresponding sides.

$$\therefore \quad \frac{\text{area of } \Delta\, EDC}{\text{area of } \Delta\, ABC} = \left(\frac{ED}{AB}\right)^2$$

$$= \left(\frac{8.2}{12.3}\right)^2$$

$$= \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

(b) Let the line $y = -3$ divides the line segment joining $A(-2, 5)$ and $B(-5, -6)$ in the ratio $k : 1$ at the point C.



Then, the coordinates of C are

$$\left(\frac{-5k-2}{k+1}, \frac{-6k+5}{k+1}\right)$$

But C lies on $y = -3$, therefore

$$\frac{-6k+5}{k+1} = -3$$

$$\Rightarrow \quad -6k + 5 = -3k - 3$$

$$\Rightarrow \quad -6k + 3k = -3 - 5$$

$$\Rightarrow \quad -3k = -8$$

$$\Rightarrow \quad k = \frac{8}{3}$$

Hence, the required ratio is $8 : 3$ internally.

On putting $k = \frac{8}{3}$ in the coordinates of C, we get

$$C = \left(\frac{-5 \times \frac{8}{3} - 2}{\frac{8}{3} + 1}, \frac{-6 \times \frac{8}{3} + 5}{\frac{8}{3} + 1}\right)$$

$$= \left(\frac{-40-6}{8+3}, \frac{-48+15}{8+3}\right)$$

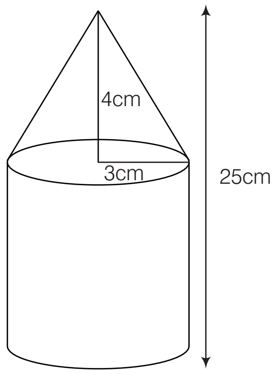
$$= \left(-\frac{46}{11}, \frac{-33}{11}\right)$$

$$= \left(\frac{-46}{11}, -3\right)$$

(c) We have,

for cylinder, Radius $(R_1) = 3$ cm, Height (H_1) $= 25 - 4 = 21$ cm

for cone, Radius $(R_2) = 3$ cm, Height $(H_2) = 4$ cm



(i) Volume of the solid = Volume of cylinder + Volume of cone

$$= \pi R_1^2 H_1 + \frac{1}{3}\pi R_2^2 H_2$$

$$= \pi (3)^2 \times 21 + \frac{1}{3}\pi(3)^2 \times 4$$

$$= 189\pi + 12\pi = 201\pi$$

$$= 201 \times \frac{22}{7}$$

$$= 631.714 \approx 632\text{ cm}^3$$

(ii) Let l be the slant height of the cone.

Then, $\quad l = \sqrt{R_2^2 + H_2^2}$

$$= \sqrt{3^2 + 4^2} = \sqrt{9 + 16}$$

$$= \sqrt{25} = 5\text{ cm}$$

So, curved surface area of the solid = curved surface area of cylinder + curved surface area of cone

$$= 2\pi R_1 H_1 + \pi R_2 l$$

$$= 2\pi \times 3 \times 21 + \pi \times 3 \times 5$$

$$= 126\pi + 15\pi$$

$$= 141\pi = 141 \times \frac{22}{7}$$

$$= 443.142 \approx 443\text{ cm}^2$$

7. (a) (i) Given, $\angle ATP = 40°$

Now, we know that angle in semi-circle is 90°.

$$\therefore \quad \angle TAB = 90°$$

Again, $\angle TAB + \angle TAP = 180°$ [linear pair]

$$\Rightarrow \quad 90° + \angle TAP = 180°$$

$$\Rightarrow \quad \angle TAP = 180° - 90° = 90°$$

Now, in Δ APT,

$$\angle ATP + \angle TAP + \angle APT = 180°$$

[$\because$ sum of all the angles of a triangle is 180°]

$$\Rightarrow \quad 40° + 90° + \angle APT = 180°$$

$$\Rightarrow \quad \angle APT = 180° - 130° = 50°$$

FULLY SOLVED

(ii) We know that,

$$PT^2 = PA \times PB$$

$$\Rightarrow \quad PT^2 = 9 \times 16$$

[$\because$ PA = 9 cm, PB = PA + AB = 9 + 7 = 16 cm]

$$\Rightarrow \quad PT^2 = 144$$

$$\Rightarrow \quad PT = 12 \text{ cm}$$

(b) Let a be the first term and r be the common ratio of the given GP, then $a_n = ar^{n-1}$.

(i) Now, according to the question

$$a_1 = 4 \Rightarrow a = 4$$

and $a_8 = 512$

$$\Rightarrow \quad ar^7 = 512$$

$$\Rightarrow \quad 4r^7 = 512$$

[$\because a = 4$]

$$\Rightarrow \quad r^7 = 128$$

$$\Rightarrow \quad r^7 = 2^7$$

$$\Rightarrow \quad r = 2$$

$\therefore$ Hence, the common ratio is 2.

(ii) We know that,

$$S_n = a\left[\frac{r^n - 1}{r - 1}\right] \quad [\because r > 1]$$

$$\Rightarrow \quad S_5 = 4\left(\frac{2^5 - 1}{2 - 1}\right)$$

$$= 4(32 - 1)$$

$$= 4 \times 31 = 124$$

(c)

Class	x_i	Frequency f_i	$f_i x_i$
0-20	10	15	150
20-40	30	20	600
40-60	50	30	1500
60-80	70	a	$70a$
80-100	90	10	900
Total		$\Sigma f_i = 75 + a$	$\Sigma f_i x_i = 3150 + 70a$

$$\therefore \quad \text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i}$$

$$\therefore \quad 49 = \frac{3150 + 70a}{75 + a}$$

$$\Rightarrow \quad 49 \times 75 + 49a = 3150 + 70a$$

$$\Rightarrow \quad 3675 + 49a = 3150 + 70a$$

$$\Rightarrow \quad 70a - 49a = 3675 - 3150$$

$$\Rightarrow \quad 21a = 525$$

$$\Rightarrow \quad a = 25$$

8. (a) $\text{LHS} = (\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2$

$$= \sin^2 A + \operatorname{cosec}^2 A + 2\sin A \operatorname{cosec} A$$

$$+ \cos^2 A + \sec^2 A + 2\cos A \sec A$$

$$[\because (a + b)^2 = a^2 + b^2 + 2ab]$$

$$= (\sin^2 A + \cos^2 A) + \left(2\sin A \times \frac{1}{\sin A}\right)$$

$$+ \left(2\cos A \times \frac{1}{\cos A}\right) + \sec^2 A + \operatorname{cosec}^2 A$$

$$\left[\because \sec A = \frac{1}{\cos A}, \operatorname{cosec} A = \frac{1}{\sin A}\right]$$

$$= 1 + 2 + 2 + \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A}$$

$$= 5 + \frac{\sin^2 A + \cos^2 A}{\cos^2 A \sin^2 A}$$

$[\because \sin^2 A + \cos^2 A = 1]$

$$= 5 + \frac{1}{\cos^2 A \sin^2 A}$$

$$= 5 + \sec^2 A \operatorname{cosec}^2 A$$

$$= \text{RHS}$$

(b) We have, $A(4, 2)$ and $B(-3, -5)$.

Mid-point of the line segment joining the points A and B $= \left(\frac{4-3}{2}, \frac{2-5}{2}\right) = \left(\frac{1}{2}, \frac{-3}{2}\right)$

$$\left[\because \text{mid-point} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\right]$$

Slope of the line joining points $A(4, 2)$ and $B(-3, -5)$

$$= \frac{-5-2}{-3-4} = \frac{-7}{-7} = 1 \quad \left[\because \text{slope} = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

Since, the required line is perpendicular to the AB, slope of the required line is $\frac{1}{-1} = -1$. We have the equation of a line passing through (x_0, y_0) and slope m is $y - y_0 = m(x - x_0)$.

Since, the required line passes through point $\left(\frac{1}{2}, \frac{-3}{2}\right)$ and having slope -1.

$\therefore$ Equation of the required line is

$$y + \frac{3}{2} = -1\left(x - \frac{1}{2}\right)$$

$$\Rightarrow \quad 2y + 3 = -2x + 1$$

$$\Rightarrow \quad 2x + 2y + 2 = 0$$

$$\Rightarrow \quad x + y + 1 = 0$$

(c) We have,

$$\frac{x^3 + 12x}{6x^2 + 8} = \frac{y^3 + 27y}{9y^2 + 27}$$

Using componendo and dividendo, we get

$$\frac{x^3 + 12x + 6x^2 + 8}{x^3 + 12x - 6x^2 - 8} = \frac{y^3 + 27y + 9y^2 + 27}{y^3 + 27y - 9y^2 - 27}$$

$$\Rightarrow \quad \frac{(x+2)^3}{(x-2)^3} = \frac{(y+3)^3}{(y-3)^3} \Rightarrow \frac{x+2}{x-2} = \frac{y+3}{y-3}$$

Using componendo and dividendo again, we get

$$\frac{(x+2)+(x-2)}{(x+2)-(x-2)} = \frac{(y+3)+(y-3)}{(y+3)-(y-3)}$$

$$\Rightarrow \quad \frac{2x}{4} = \frac{2y}{6}$$

$$\Rightarrow \quad \frac{x}{2} = \frac{y}{3} \Rightarrow \frac{x}{y} = \frac{2}{3}$$

$$\therefore \quad x : y = 2 : 3$$

9. (a) Let the two numbers be x and y, $(x > y)$.

Now, according to the question.

$$x^2 - y^2 = 84 \quad \text{...(i)}$$

and $$x^2 = 25y \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$25y - y^2 = 84$$

$$\Rightarrow \quad y^2 - 25y + 84 = 0$$

$$\Rightarrow \quad y^2 - 4y - 21y + 84 = 0$$

$$\Rightarrow \quad y(y-4) - 21(y-4) = 0$$

$$\Rightarrow \quad (y-21)(y-4) = 0$$

$$\Rightarrow \quad y = 21, 4$$

When $y = 21$, then

$$x^2 = 25 \times 21 \Rightarrow x = \pm 5\sqrt{21}$$

which is not possible as x is natural number.

When $y = 4$, then

$$x^2 = 25 \times 4$$

$$\Rightarrow \quad x^2 = 100$$

$$\Rightarrow \quad x = \pm 10$$

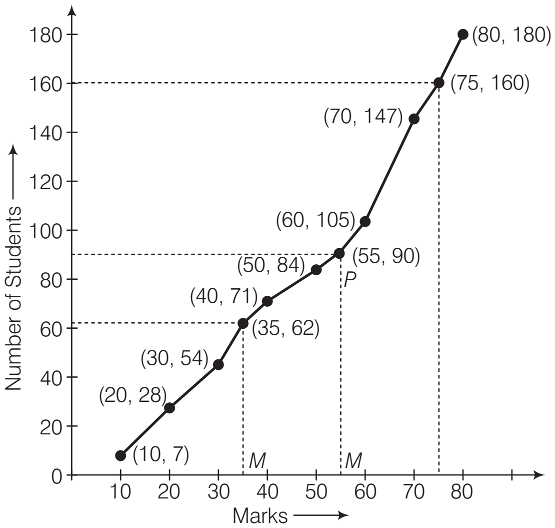
$$\Rightarrow \quad x = 10 \quad [\because x \text{ is a natural number}]$$

Hence, the required numbers are 10 and 4.

(b) **Scale**

On X-Axis; 2 cm = 10 units.

On Y-Axis; 2 cm = 20 units.



(i) Here, $N = 180$

$$\therefore \quad \frac{N}{2} = 90°$$

In order to find the median let us first locate the point corresponding to 90 on Y-axis.

and from this draw a horizontal line to meet the ogive at P.

Now, through P, draw a vertical line to meet X-axis at M. The x-coordinate of M is 55.

$\therefore$ Median from the graph = 55

(ii) Number of students who scored distinction marks (75% and above)

$$= 180 - 160 = 20$$

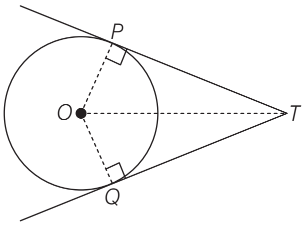
(iii) Number of students whose passed

$$= 180 - 62 = 118$$

10. (a) Given, two tangents TP and TQ are drawn from a point T to a circle with centre O.

To prove TP = TQ

Construction Join OP, OQ and OT



Proof

Consider the triangle OPT and OQT.

$$OP = OQ \quad \text{[radii]}$$

$$\angle OPT = \angle OQT = 90°$$

[$\because$ radius is perpendicular to the tangent at the point of contact]

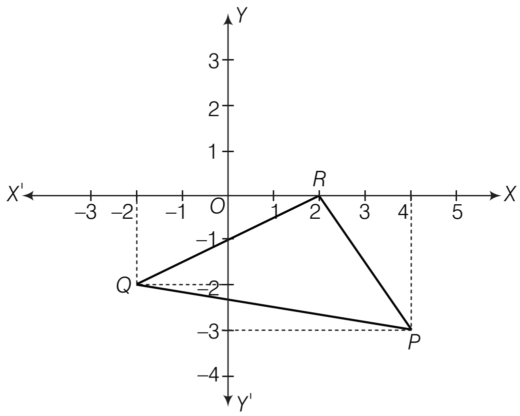
$$OT = OT \quad \text{[common]}$$

$\therefore$ By RHS criteria, $\Delta OPT \cong \Delta OQT$

$\therefore$ By CPCT,

$$TP = TQ$$ **Hence proved.**

(b)



(i) From the graph it is clear that,

$$p(4, -3), Q(-2, -2) \text{ and } R(2, 0)$$

(ii) Slope of line joining QR $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{0-(-2)}{2-(-2)} = \frac{2}{4} = \frac{1}{2}$$

Since, the required line is parallel to QR, therefore slope of required line is $\frac{1}{2}$.

$\therefore$ Equation of line passing through $P(4, -3)$ and having slope $\frac{1}{2}$ is

$$y - (-3) = \frac{1}{2}(x - 4)$$

$$\Rightarrow \quad 2y + 6 = x - 4$$

$$\Rightarrow \quad x - 2y - 10 = 0$$

11. (a) Given, dividend = ₹ 900, face value = ₹ 50 and market value = ₹ 80.

Let the number of shares be n.

Then, $50 \times \frac{20}{100} \times n = 900$

$$\therefore \quad n = \frac{900}{10} = 90$$

(i) Amount interested by Mr. Sharma $= 90 \times 80$

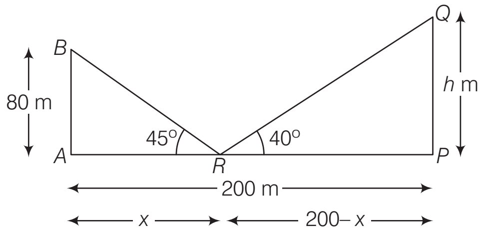
$= ₹ 7200$

(ii) Percentage return $= \frac{\text{Income}}{\text{Amount Interested}} \times 100$

$$= \frac{900}{7200} \times 100$$

$$= \frac{100}{8}\% = 12.5\%$$

(b) We have, AB and PQ as two poles.



Let $PQ = h$ m and $AR = x$ m.

So, $PR = (200 - x)$ m

Now, in right angled Δ ABR,

$$\tan 45^\circ = \frac{AB}{AR}$$

$$\Rightarrow \quad 1 = \frac{80}{x}$$

$$\Rightarrow \quad x = 80 \text{ m}$$

$$\therefore \quad PR = 200 - x$$

$$= 200 - 80 = 120 \text{ m}$$

Now, in right angled Δ PQR,

$$\tan 40^\circ = \frac{PQ}{PR}$$

$$\Rightarrow \quad 0.84 = \frac{PQ}{120} \text{ m}$$

$[\because \tan 40^\circ \approx 0.84]$

$$\Rightarrow \quad PQ = 0.84 \times 120$$

$$= 100.8 \approx 101 \text{ m}$$

(c) **Steps of construction**

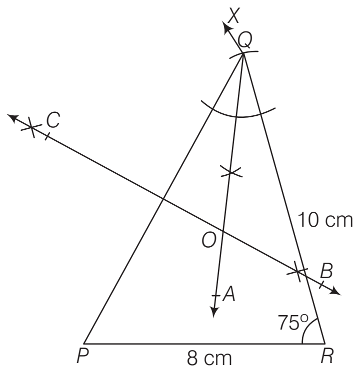
- Draw a line segment RP of length 8 cm.
- At R, draw a ray RX making an angle of = 75° with PR.
- Draw an arc RQ of length 10 cm.
- Join P to Q to form Δ PQR.

(i) Draw the angle bisector of $\angle PQR$ and name it QA.

QA is the required locus of points which are equidistant from QR and QP.

(ii) Draw the perpendicular bisector of PQ and name it *BC*.

BC is the required locus of points which are equidistant from P and Q.



(iii) Mark the point of intersection of QA and BC as O.

O is the required point where both the above two locus are satisfied.

FULLY SOLVED

ICSE
Examination Paper
2020

ICSE EXAMINATION PAPER 2020

MATHEMATICS (FULLY SOLVED)

GENERAL INSTRUCTIONS

1. You will not be allowed to write during the first 15 minutes. This time is to be spent in reading the questions paper.
2. The time given at the head of this paper is the time allowed for writing the answers.
3. Attempt **all questions** from **Section A** and **any 4 questions** from **Section B**.
4. All working, including rough work, must be clearly shown and must be done on the same sheet as the rest of the answer.
5. Omission of essential working will result in loss of marks.
6. The intended marks for questions or parts of questions are given in brackets [ ].
7. Mathematical tables are provided.

Time : 2.5 Hrs **Max. Marks : 80**

FULLY SOLVED

Section A [40 *Marks*]

1. (a) Solve the following quadratic equation.

$$x^2 - 7x + 3 = 0$$ [3]

Give your answer correct to two decimal places.

(b) Given $A = \begin{bmatrix} x & 3 \\ y & 3 \end{bmatrix}$. If $A^2 = 3I$, where I is the identity matrix of order 2, find x and y. [3]

(c) Using ruler and compass construct a ΔABC where $AB = 3$ cm, $BC = 4$ cm and $\angle ABC = 90°$. Hence construct a circle circumscribing the ΔABC. Measure and write down the radius of the circle. [4]

2. (a) Use factor theorem to factorise

$6x^3 + 17x^2 + 4x - 12$ completely. [3]

(b) Solve the following inequation and represent the solution set on the number line.

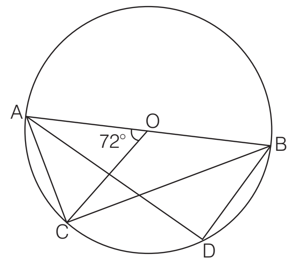
$$\frac{3x}{5} + 2 < x + 4 \le \frac{x}{2} + 5, x \in R$$ [3]

(c) Draw a Histogram for the given data, using a graph paper.

Weekly wages (in ₹)	Number of people
3000-4000	4
4000-5000	18
5000-6000	9
6000-7000	6
7000-8000	7
8000-9000	2
9000-10000	4

Estimate the mode from the graph. [4]

3. (a) In the figure below, O is the centre of the circle and AB is diameter. If $AC = BD$ and $\angle AOC = 72°$. Find



(i) $\angle ABC$ (ii) $\angle BAD$ (iii) $\angle ABD$ [3]

(b) Prove that $\dfrac{\sin A}{1 + \cot A} - \dfrac{\cos A}{1 + \tan A} = \sin A - \cos A$ [3]

(c) In what ratio is the line joining $P(5, 3)$ and $Q(-5, 3)$ divided by the Y-axis? Also find the coordinates of the point of intersection. [4]

4. (a) A solid spherical ball of radius 6 cm is melted and recast into 64 identical spherical marbles. Find the radius of each marble. [3]

(b) Each of the letters of the word 'AUTHORIZES' is written on identical circular disc and put in a bag. They are well shuffled. If a disc is drawn at random from the bag, what is the probability that the letter is

(i) a vowel?

(ii) one of the first 9 letters of the English alphabet which appears in the given word?

(iii) one of the last 9 letters of the English alphabet which appears in the given word? [3]

(c) Mr. Bedi visits the market and buys the following articles.

Medicines costing ₹ 950, GST @ 5%

A pair of shoes costing ₹ 3000, GST @ 18%

A laptop bag costing ₹ 1000 with a discount of 30%, GST @ 18%.

(i) Calculate the total amount of GST paid.

(ii) The total bill amount including GST paid by Mr. Bedi. [4]

Section B [40 *Marks*]

5. (a) A company with 500 shares of nominal value ₹ 120 declares and annual dividend of 15%. Calculate.

(i) the total amount of dividend paid by the company.

(ii) annual income of Mr. Sharma who holds 80 shares of the company.

If the return percent of Mr. Sharma from his shares is 10%, find the market value of each share. [3]

(b) The mean of the following data is 16. Calculate the value of f.

Marks	5	10	15	20	25
Number of Students	3	7	f	9	6

[3]

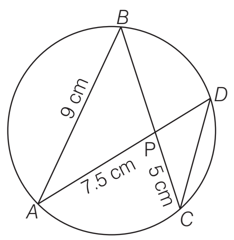
(c) The 4th, 6th the last term of a geometric progression are 10, 40 and 640 respectively. If the common ratio is positive, find the first term, common ratio and the number of terms of the series. [4]

6. (a) If $A = \begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$

Find $A^2 - 2AB + B^2$. [3]

(b) In the given figure $AB = 9$ cm, $PA = 7.5$ cm and $PC = 5$ cm.

Chords AD and BC intersect at P.



(i) Prove that $\Delta PAB \sim \Delta PCD$.

(ii) Find the length of CD.

(iii) Find area of ΔPAB : area of ΔPCD. [3]

(c) From the top of a cliff, the angle of depression of the top and bottom of a tower are observed to be 45° and 60° respectively. If the height of the tower is 20 m.

Find

(i) the height of the cliff.

(ii) the distance between the cliff and the tower. [4]

7. (a) Find the value of 'p' if the lines, $5x - 3y + 2 = 0$ and $6x - py + 7 = 0$ are perpendicular to each other. Hence find the equation of a line passing through $(-2, -1)$ and parallel to $6x - py + 7 = 0$. [3]

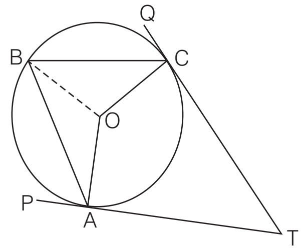
(b) Using properties of proportion find $x : y$, given

$$\frac{x^2 + 2x}{2x + 4} = \frac{y^2 + 3y}{3y + 9}.$$ [3]

(c) In the given figure TP and TQ are two tangents to the circle with centre O, touching at A and C respectively. If $\angle BCQ = 55°$ and $\angle BAP = 60°$, find

(i) $\angle OBA$ and $\angle OBC$ (ii) $\angle AOC$

(iii) $\angle ATC$



[4]

8. (a) What must be added to the polynomial $2x^3 - 3x^2 - 8x$, so that it leaves a remainder 10 when divided by $2x + 1$? [3]

(b) Mr. Sonu has a recurring deposit account and deposits ₹ 750 per month for 2 yr. If he gets ₹ 19125 at the time of maturity, find the rate of interest. [3]

(c) Use graph paper for this question.

Take 1 cm = 1 unit on both X and Y-axes.

(i) Plot the following points on your graph sheets

$A(-4, 0)$, $B(-3, 2)$, $C(0, 4)$, $D(4, 1)$ and $E(7, 3)$

(ii) Reflect the points B, C, D and E on the X-axis and name them as B', C', D' and E' respectively.

(iii) Join the points $A, B, C, D, E, E', D', C', B'$ and A in order.

(iv) Name the closed figure formed. [4]

9. (a) 40 students enter for a game of shot-put competition. The distance thrown (in metres) is recorded below

Distance in m	12-13	13-14	14-15	15-16	16-17	17-18	18-19
Number of students	3	9	12	9	4	2	1

Use a graph paper to draw an ogive for the above distribution.

Use a scale of 2 cm = 1 m on one axis and 2 cm = 5 students on the other axis.

Hence using your graph find

(i) the median

(ii) Upper Quartile

(iii) Number of students who cover a distance which is above $16\frac{1}{2}$ m. [6]

(b) If $x = \dfrac{\sqrt{2a+1}+\sqrt{2a-1}}{\sqrt{2a+1}-\sqrt{2a-1}}$, prove that $x^2 - 4ax + 1 = 0$. [4]

10. (a) If the 6th term of an A.P. is equal to four times its first term and the sum of first six terms is 75, find the first term and the common difference. [3]

(b) The difference of two natural number is 7 and their product is 450. Find the numbers. [3]

(c) Use ruler and compass for this question. Construct a circle of radius 4.5 cm. Draw a chord. $AB = 6$ cm.

(i) Find the locus of points equidistant from A and B. Mark the point where it meets the circles as D.

(ii) Join AD and find the locus of points which are equidistant from AD and AB. Mark the point where it meets the circle as C.

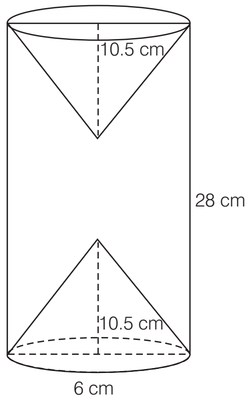
(iii) Join BC and CD. Measure and write down the length of side CD of the quadrilateral $ABCD$. [4]

11. (a) A model of a high rise building is made to a scale of 1 : 50.

(i) If the height of the model is 0.8 m, find the height of the actual building.

(ii) If the floor area of a flat in the building is 20 m^2. Find the floor area of that in the model. [3]

(b) From a solid wooden cylinder of height 28 cm and diameter 6 cm, two conical cavities are hollowed out. The diameter of the cones are also of 6 cm and height 10.5 cm. Taking $\pi = \frac{22}{7}$ find the volume of the remaining solid.



[3]

(c) Prove the identity $\left(\dfrac{1-\tan\theta}{1-\cot\theta}\right)^2 = \tan^2\theta$. [4]

SOLUTIONS

1. (a) Given, quadratic equation $x^2 - 7x + 3 = 0$

On comparing with $ax^2 + bx + c = 0$, we get

$$a = 1,\ b = -7 \text{ and } c = 3$$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{[by quadratic formula]}$$

$$= \frac{7 \pm \sqrt{(-7)^2 - 4 \times 1 \times 3}}{2 \times 1} = \frac{7 \pm \sqrt{49 - 12}}{2}$$

$$= \frac{7 \pm \sqrt{37}}{2}$$

$$= \frac{7 \pm 6.08}{2} = \frac{7 + 6.08}{2}, \frac{7 - 6.08}{2} = \frac{13.08}{2}, \frac{0.92}{2}$$

$$\Rightarrow \quad x = 6.54, 0.46$$

(b) Given, $A = \begin{bmatrix} x & 3 \\ y & 3 \end{bmatrix}$

Now, $A^2 = A \cdot A = \begin{bmatrix} x & 3 \\ y & 3 \end{bmatrix}\begin{bmatrix} x & 3 \\ y & 3 \end{bmatrix} = \begin{bmatrix} x^2 + 3y & 3x + 9 \\ xy + 3y & 3y + 9 \end{bmatrix}$

and $\quad 3I = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Since, $\quad A^2 = 3I$

$$\Rightarrow \begin{bmatrix} x^2 + 3y & 3x + 9 \\ xy + 3y & 3y + 9 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

On comparing corresponding elements both sides, we get

$$3x + 9 = 0 \Rightarrow x = -3$$

and $\quad 3y + 9 = 3 \Rightarrow 3y = -6 \Rightarrow y = -2$

$$\therefore \quad x = -3,\ y = -2$$

which satisfied $x^2 + 3y = 3$ and $xy + 3y = 0$.

(c) Given, $AB = 3$ cm, $BC = 4$ cm and $\angle ABC = 90°$

Steps of construction

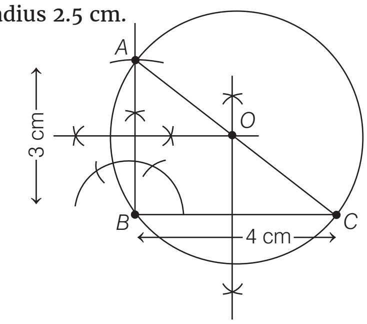
(i) Draw a line segment $BC = 4$ cm

(ii) Make an angle of 90° at B and cut-off $AB = 3$ cm from it.

(iii) Join AC. Thus, ΔABC is the required triangle.

(iv) Draw the perpendicular bisectors of BC and AB, which meets at point O.

(v) Draw a circle with centre O and radius OA, which passes through the all vertices of ΔABC. Hence, it is the required circumcircle of ΔABC with radius 2.5 cm.



2. (a) Let $f(x) = 6x^3 + 17x^2 + 4x - 12$

Using hit and trial method,

Putting $x = -2$, we get

$f(-2) = 6(-2)^3 + 17(-2)^2 + 4(-2) - 12$

$= -48 + 68 - 8 - 12 = 0$

So, by factor theorem, $(x+2)$ will be a factor of $f(x)$.
Now, to find other factor, let us divide $f(x)$ by $(x+2)$ using long division method

$$
\begin{array}{r|l}
x+2 & 6x^3 + 17x^2 + 4x - 12 \;(6x^2 + 5x - 6 \\
 & 6x^3 + 12x^2 \\
 & \underline{- \quad -} \\
 & 5x^2 + 4x \\
 & 5x^2 + 10x \\
 & \underline{- \quad -} \\
 & -6x - 12 \\
 & -6x - 12 \\
 & \underline{+ \quad +} \\
 & 0
\end{array}
$$

Clearly, $6x^2 + 5x - 6$ is another factor of $f(x)$.

$\therefore \quad f(x) = (x+2)(6x^2 + 5x - 6)$

Now, factor of $6x^2 + 5x - 6$

$= 6x^2 + 9x - 4x - 6$

$= 3x(2x+3) - 2(2x+3)$

$= (2x+3)(3x-2)$

Hence, $f(x) = (x+2)(2x+3)(3x-2)$.

(b) Given, $\frac{3x}{5} + 2 < x + 4 \le \frac{x}{2} + 5, \; x \in R$

$\Rightarrow \frac{3x}{5} + 2 < x + 4$ and $x + 4 \le \frac{x}{2} + 5$

[splitting into two inequations]

Now, consider $\frac{3x}{5} + 2 < x + 4$

$\Rightarrow \frac{3x}{5} + 2 - x < x + 4 - x$

[on subtracting x from both sides]

$\Rightarrow \frac{3x - 5x}{5} + 2 < 4$

$\Rightarrow -\frac{2x}{5} + 2 - 2 < 4 - 2$

[on subtracting 2 from both sides]

$\Rightarrow \quad -\frac{2x}{5} < 2 \Rightarrow \frac{x}{5} > -1$

[on dividing both sides by -2, the sign of in equality reverse]

$\Rightarrow \quad x > -5$ [on multiplying both sides by 5] ... (i)

and $\quad x + 4 \le \frac{x}{2} + 5$

$\Rightarrow \quad (x+4) - \frac{x}{2} \le \frac{x}{2} + 5 - \frac{x}{2}$

[on subtracting $\frac{x}{2}$ from both sides]

$\Rightarrow \quad \frac{x}{2} + 4 \le 5 \Rightarrow \frac{x}{2} + 4 - 4 \le 5 - 4$

[on subtracting 4 from both sides]

$\Rightarrow \quad \frac{x}{2} \le 1 \Rightarrow x \le 2$

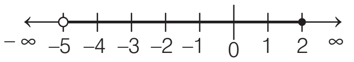
[on multiplying both sides by 2] ...(ii)

From Eqs. (i) and (ii), we get

$-5 < x \le 2$

$\therefore$ The solution set $= \{x : -5 < x \le 2, x \in R\} = (-5, 2]$

The graph of the solution set on the number line is shown by dark line.

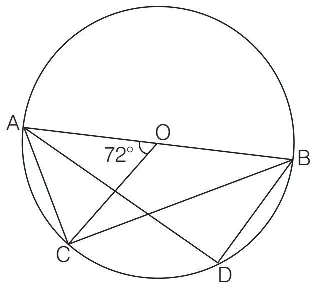


(c) Do same as Example 3 on Page 356.

Ans. Mode = 5428.57

3. (a) Given, O is the centre of the circle and AB is a diameter.

If $AC = BD$ and $\angle AOC = 72°$



(i) $\angle ABC = \frac{1}{2} \times \angle AOC = \frac{1}{2} \times 72° = 36°$

[$\because$ angle subtended by an arc at the remaining circumference is half the angle subtended at the centre]

(ii) $\angle BAD = \angle ABC = 36°$

[equal chords $AC = BD$ of a circle subtend equal angles at any point on the major (or minor) arcs of the circle]

(iii) In ΔABD, $\angle ABD + \angle ADB + \angle BAD = 180°$

[angle sum property of triangle]

$\Rightarrow \quad \angle ABD + 90° + 36° = 180°$

[$\angle ADB = 90°$, angle in a semi-circle is a right angle]

$\Rightarrow \quad \angle ABD + 126° = 180°$

$\Rightarrow \quad \angle ABD = 180° - 126° \Rightarrow \angle ABD = 54°$

(b) LHS $= \frac{\sin}{1 + \cot A} - \frac{\cos A}{1 + \tan A}$

$= \frac{\sin A}{1 + \frac{\cos A}{\sin A}} - \frac{\cos A}{1 + \frac{\sin A}{\cos A}}$

$= \frac{\sin^2 A}{\sin A + \cos B} - \frac{\cos^2 A}{\cos A + \sin A}$

$= \frac{\sin^2 A - \cos^2 A}{\sin A + \cos A}$

$= \frac{(\sin A + \cos A)(\sin A - \cos A)}{\sin A + \cos A}$

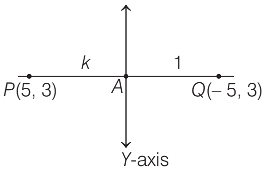
[$\because a^2 - b^2 = (a+b)(a-b)$]

$= \sin A - \cos A$

$=$ RHS **Hence proved.**

(c) Let line joining $P(5, 3)$ and $Q(-5, 3)$ divided by the Y-axis in the ratio $k : 1$ at the point A.

$\therefore$ Coordinates of $A\left(\frac{-5k+5}{k+1}, \frac{3k+3}{k+1}\right)$



Since, A lies on the Y-axis, therefore, x-coordinates of A is 0.

$\Rightarrow \quad \frac{-5k+5}{k+1} = 0 \Rightarrow -5k+5 = 0$

$\Rightarrow \quad -5k = -5 \Rightarrow k = 1$

$\therefore$ Required ratio $= k : 1 = 1 : 1$

$\therefore$ The coordinates of the point of intersection A are

$\left(\frac{-5 \times 1 + 5}{1+1}, \frac{3 \times 1 + 3}{1+1}\right) = (0, 3).$

4. (a) Let radius of each spherical marble be r cm.

According to question,

Volume of 64 identical spherical marbles

= Volume of solid spherical ball of radius 6 cm

$\Rightarrow \quad 64 \times \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times (6)^3 \Rightarrow 64 \times r^3 = 6^3$

$\Rightarrow \quad r^3 = \left(\frac{6}{4}\right)^3 \Rightarrow r = \frac{6}{4} = 1.5$ cm

(b) Given, word 'AUTHORIZES' so, total number of identical circular discs = 10

(i) Here vowels are A, U, O, I, E

Total number of vowel = 5

$\therefore$ Required probability $= \frac{5}{10} = \frac{1}{2}$

(ii) One of the first 9 letters of the English alphabet which appears in the given word are A, H, I, E

$\therefore$ Required probability $= \frac{4}{10} = \frac{2}{5}$

(iii) One of the last 9 letters of the English alphabets which appears in the given word are U, T, R, Z, S

$\therefore$ Required probability $= \frac{5}{10} = \frac{1}{2}$

(c) GST amount on Medicines $= 950 \times \frac{5}{100} = ₹ 47.5$

GST amount on a pair of shoes $= 3000 \times \frac{18}{100} = ₹ 540$

and GST amount on a laptop bag

$= 1000 \times \frac{70}{100} \times \frac{18}{100} = ₹ 126$

(i) Total amount of GST paid = ₹ (47.5 + 540 + 126)

= ₹ 713.5

(ii) The total bill amount including GST paid by Mr. Bedi

= Cost of articles + Total amount of GST

= 950 + 3000 + 700 + 713.5 = ₹ 5363.5

5. (a) Given, Nominal value of one share = ₹ 120

Annual dividend = 15%

(i) Total amount of dividend = Number of shares

$\times \frac{\text{Rate of dividend}}{100} \times$ Nominal value of one share

$= 500 \times \frac{15}{100} \times 120 = ₹\ 9000$

(ii) Annual income of Mr. Sharma

= 80 × Dividend on one share

$= 80 \times \frac{9000}{500} = ₹\ 1440$

Let the market value of one share be ₹ x.

The profit on one share = 10% of x = ₹ $\frac{x}{10}$

Since, the dividend paid on one share = ₹ $\frac{9000}{500}$

$\therefore \quad \frac{x}{10} = \frac{9000}{500} \Rightarrow x = ₹\ 180$

(b)

Marks (x_i)	Number of students (f_i)	$f_i x_i$
5	3	15
10	7	70
15	f	$15f$
20	9	180
25	6	150
Total	$\Sigma f_i = 25 + f$	$\Sigma f_i x_i = 415 + 15f$

Given, mean of the following data is 16

$\Rightarrow \quad \frac{\Sigma f_i x_i}{\Sigma f_i} = 16$

$\Rightarrow \quad \frac{415 + 15f}{25 + f} = 16$

$\Rightarrow \quad 415 + 15f = 400 + 16f \Rightarrow f = 415 - 400 = 15$

(c) Let the first term and common ratio of GP be a and r, respectively and total number of terms be n.

According to question,

4th term $= ar^{4-1} = 10 \Rightarrow ar^3 = 10$ ... (i)

6th term $= ar^{6-1} = 40 \Rightarrow ar^5 = 40$ ... (ii)

and last term $= ar^{n-1} = 640$ ... (iii)

On dividing Eq. (ii) by Eq. (i), we get

$\frac{ar^5}{ar^3} = \frac{40}{10} \Rightarrow r^2 = 4$

$\Rightarrow \quad r = 2$ [since common ratio is positive]

Now, from Eq. (i), $a(2)^3 = 10 \Rightarrow a = \frac{10}{8} = \frac{5}{4}$

On putting values of a and r in Eq. (iii), we get

$\left(\frac{5}{4}\right)(2)^{n-1} = 640$

$\Rightarrow \quad 2^{n-1} = 640 \times \frac{4}{5} = 512$

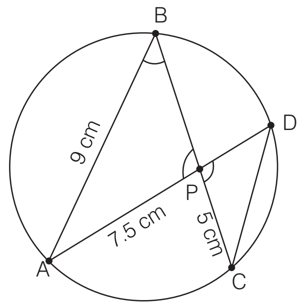
$\Rightarrow \quad 2^{n-1} = 2^9 \Rightarrow n - 1 = 9 \Rightarrow n = 10$

$\therefore$ First term $= \frac{5}{4}$, Common ratio = 2

and number of term of the series = 10.

6. (a) Do same as Example 11 on page 104. **Ans** $\begin{bmatrix} 51 & -20 \\ 54 & -17 \end{bmatrix}$

(b) Given, $AB = 9$ cm, $PA = 7.5$ cm and $PC = 5$ cm



(i) In ΔPAB and ΔPCD,

$\angle PBA = \angle PDC$

[$\because$ angle in the same segment of a circle are equal]

and $\angle APB = \angle CPD$ [vertically opposite angles]

by AA similarity criterion,

$\Delta PAB \sim \Delta PCD$

(ii) $\frac{AP}{CP} = \frac{AB}{CD}$ [Since, $\Delta PAB \sim \Delta PCD$ corresponding sides of similar triangles are proportional]

$\Rightarrow \frac{7.5}{5} = \frac{9}{CD} \Rightarrow CD = \frac{5 \times 9}{7.5} = 6$ cm

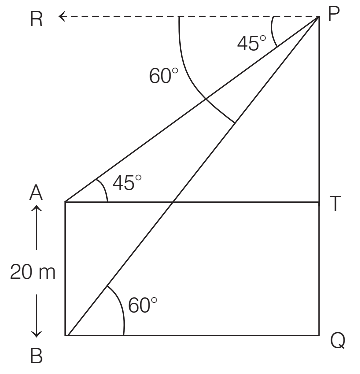
(iii) $\frac{\text{area of } \Delta PAB}{\text{area of } \Delta PCD} = \left(\frac{AB}{CD}\right)^2$

[$\because$ The ratio of areas of two similar triangles in equal to the square of the ratio of their corresponding sides]

$= \left(\frac{9}{6}\right)^2 = \frac{9}{4}$

$\Rightarrow$ area of ΔPAB : area of $\Delta PCD = 9 : 4$

(c) Given, height of the tower $AB = 20$ m,



Let height of the cliff be PQ m. Through A, draw a horizontal line to meet PQ at T.

Here, $\angle PAT = \angle APR = 45°$ [alternate angles]

and $\angle PBQ = \angle BPR = 60°$

In ΔPAT, $\tan 45° = \frac{PT}{AT} \Rightarrow 1 = \frac{PT}{AT}$

$\Rightarrow PT = AT$ ...(i)

In ΔPBQ,

$\tan 60° = \frac{PQ}{BQ}$

$\Rightarrow \sqrt{3} = \frac{PT + TQ}{BQ}$

$\Rightarrow \sqrt{3} = \frac{AT + 20}{BQ}$ [$\because PT = AT$, $TQ = AB = 20$ m]

$\Rightarrow \sqrt{3} = \frac{BQ + 20}{BQ}$ [$\because AT = BQ$]

$\Rightarrow \sqrt{3}\, BQ = BQ + 20 \Rightarrow BQ(\sqrt{3} - 1) = 20$

$\Rightarrow BQ = \frac{30}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$

$= \frac{20(\sqrt{3} + 1)}{2}$

$= 10(\sqrt{3} + 1)$ m $= 10(1.732 + 1)$m

$= 27.32$ m

Now, $PQ = PT + TQ = 27.32 + 20$

[$\because PT = AT = BQ = 27.32$ m and $TQ = 20$ m]

$= 47.32$ m

(i) The height of the cliff, $PQ = 47.32$ m

(ii) The distance between the cliff and the tower $BQ = 27.32$ m

7. (a) Given, equation of lines are

$5x - 3y + 2 = 0$ and $6x - py + 7 = 0$

$\Rightarrow -3y = -5x - 2$ and $-py = -6x - 7$

$\Rightarrow y = \frac{5}{3}x + \frac{2}{3}$ and $y = \frac{6}{p}x + \frac{7}{p}$

On comparing with $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively, we get

$m_1 = \frac{5}{3}$ and $\frac{6}{p}$

Since, given lines are perpendicular.

$\therefore m_1 \times m_2 = -1$

$\Rightarrow \frac{5}{3} \times \frac{6}{p} = -1 \Rightarrow \frac{10}{p} = -1 \Rightarrow p = -10$

When $p = -10$, then slope of the line $6x - py + 7 = 0$ is

$\frac{6}{-10}$ or $\frac{-3}{5}$.

$\therefore$ The equation of a line passing through $(-2, -1)$ and parallel to $6x - py + 7 = 0$ is

$y + 1 = \left(-\frac{3}{5}\right)(x + 2)$ [$\because y - y_1 = m(x - x_1)$]

$\Rightarrow 5y + 5 = -3x - 6 \Rightarrow 3x + 5y + 11 = 0$

(b) Given, $\frac{x^2 + 2x}{2x + 4} = \frac{y^2 + 3y}{3y + 9}$

Using componendo and dividendo properties,

$\frac{a}{b} = \frac{c}{d} = \frac{a + b}{a - b} = \frac{c + d}{c - d}$, we get

$\frac{x^2 + 2x + 2x + 4}{x^2 + 2x - 2x - 4} = \frac{y^2 + 3y + 3y + 9}{y^2 + 3y - 3y - 9}$

$$\Rightarrow \quad \frac{x^2+4x+4}{x^2-4} = \frac{y^2+6y+9}{y^2-9}$$

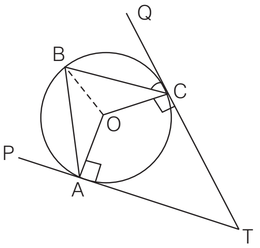
$$\Rightarrow \quad \frac{(x+2)^2}{(x+2)(x-2)} = \frac{(y+3)^2}{(y+3)(y-3)} \Rightarrow \frac{x+2}{x-2} = \frac{y+3}{y-3}$$

Again, applying componendo and dividendo property, we get

$$\frac{x+2+x-2}{x+2-x+2} = \frac{y+3+y-3}{y+3-y+3}$$

$$\Rightarrow \quad \frac{2x}{4} = \frac{2y}{6} \Rightarrow \frac{x}{2} = \frac{y}{3} \Rightarrow \frac{x}{y} = \frac{2}{3} \Rightarrow x : y = 2 : 3$$

(c) Given, TP and TQ are two tangents to the circle with centre O touching at A and C, respectively.



Since, $\angle BCO = 90° - \angle BCQ = 90° - 55° = 35°$ ...(i)

[$\because$ radius $\perp$ tangent]

and $\angle BAP = 60°$

$\therefore \quad \angle BAO = 90° - \angle BAP = 90° - 60° = 30°$ ...(ii)

[$\because$ radius $\perp$ tangent]

(i) In ΔBOC, $OC = OB$ [radii]

$\Rightarrow \quad \angle OBC = \angle OCB = 35°$ [using Eq. (i)]

[angles opposite to equal sides are equal]

In ΔAOB, $OA = OB$ [radii]

$\Rightarrow \quad \angle OBA = \angle OAB = 30°$ [using Eq. (ii)]

[angles opposite to equal sides are equal]

(ii) $\angle ABC = \angle ABO + \angle CBO = 30° + 35° = 65°$

$\therefore \quad \angle AOC = 2 \times ABC = 2 \times 65° = 130°$

(iii) In quadrilateral $AOCT$,

$$\angle AOC + \angle OAT + \angle OCT + \angle ATC = 360°$$

[$\because$ sum of all the angles of a quadrilateral is 360°]

$$\Rightarrow \quad 130° + 90° + 90° + \angle ATC = 360°$$

$$\Rightarrow \quad \angle ATC = 360° - 310° = 50°$$

8. (a) Do same as Q. 10 of Page 91. **Ans.** 7

(b) Given, money deposit per month $(P) =$ ₹ 750

Time $(n) = 2$ yr $= 2 \times 12 = 24$ months

and maturity value = ₹ 19125

Let the of rate of interest be r% per annum.

$$\therefore \text{ Maturity value} = Pn\left[1 + \frac{(n+1)r}{2400}\right]$$

$$\Rightarrow \quad 19125 = 750 \times 24\left(1 + \frac{25r}{2400}\right)$$

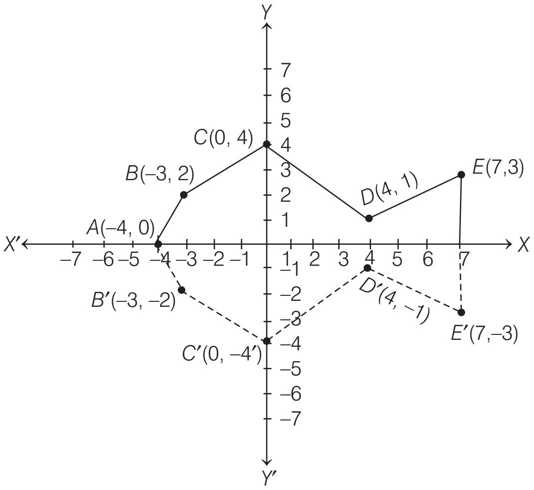
$$\Rightarrow \quad 19125 = 18000\left(1 + \frac{r}{96}\right)$$

$$\Rightarrow \quad 1 + \frac{r}{96} = \frac{19125}{18000} \Rightarrow \frac{r}{96} = \frac{19125 - 18000}{18000}$$

$$\Rightarrow \quad \frac{r}{96} = \frac{1125}{18000} \Rightarrow r = \frac{1125}{18000} \times 96 \Rightarrow r = 6$$

Hence, the required rate of interest is 6% per annum.

(c) (i) We plot the given points on the graph paper as shown below



(ii) The reflection of the points B, C, D and E on the X-axis are the points $B'(-3, -2)$, $C'(0, -4)$, $D'(4, -1)$ and $E'(7, -3)$

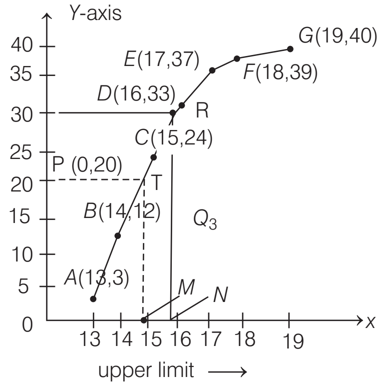
(iii) See figure

(iv) irregular decagon

9. (a) The cumulative frequency table for the continuous distribution is shown below.

Distance	12-13	13-14	14-15	15-16	16-17	17-18	18-19
Number of Students	3	9	12	9	4	2	1
Cumulative frequency (cf)	3	12	24	33	37	39	40

On the graph paper, we plot the following points $A(13, 3)$, $B(14, 12)$, $C(15, 24)$, $D(16, 33)$, $E(17, 37)$, $F(18, 39)$ and $G(19, 40)$. Join all these points by a free hand drawing. The required ogive is shown on the graph paper given below.



(i) For, median $\frac{n}{2} = \frac{40}{2} = 20$

Take a point P on Y-axis representing 20. Through P, draw a line PT parallel to X-axis which meets the ogive at T. Draw TM perpendicular to X-axis, meeting X-axis at M the abscissa of M, is 14.7 which is the required median.

(ii) Upper quartile, $\frac{3n}{4} = \frac{3 \times 40}{4} = 30$

Taking point Q on Y-axis representing frequency = 30 Through Q, draw a line QR parallel to X-axis, which meets the ogive at R. Draw RN perpendicular to X-axis, meeting X-axis at N. The abscissa of N is 15.67 which is required upper Quartile.

(iii) Number of students who cover a distance which is above $16\frac{1}{2}$ m are 5 i.e. $(2 + 2 + 1)$

(b) Given, $\frac{x}{1} = \frac{\sqrt{2a+1} + \sqrt{2a-1}}{\sqrt{2a+1} - \sqrt{2a-1}}$

On applying componendo and dividendo property, we get

$$\frac{x+1}{x-1} = \frac{\sqrt{2a+1} + \sqrt{2a-1} + \sqrt{2a+1} - \sqrt{2a-1}}{\sqrt{2a+1} + \sqrt{2a-1} - \sqrt{2a-1} + \sqrt{2a-1}}$$

$$\Rightarrow \quad \frac{x+1}{x-1} = \frac{\sqrt{2a+1}}{\sqrt{2a-1}}$$

On squaring both sides, we get

$$\frac{x^2+2x+1}{x^2-2x+1} = \frac{2a+1}{2a-1}$$

Again applying componendo and dividendo property, we get

$$\frac{x^2+2x+1+x^2-2x+1}{x^2+2x+1-x^2+2x-1} = \frac{2a+1+2a-1}{2a+1-2a+1}$$

$$\Rightarrow \quad \frac{2x^2+2}{4x} = \frac{4a}{2} \Rightarrow \frac{x^2+1}{2x} = 2a$$

$$\Rightarrow \quad x^2+1 = 4ax \Rightarrow x^2 - 4ax + 1 = 0$$ **Hence proved.**

10. (a) Let a be the first term and d be the common difference of the A.P.

According to question,

$$T_6 = 4a_1$$

$$\Rightarrow \quad a + (6-1)d = 4a \Rightarrow 3a = 5d \quad \text{...(i)}$$

and $$S_6 = 75$$

$$\Rightarrow \quad \frac{6}{2}[2a + (6-1)d] = 75 \quad \left[\because S_n = \frac{n}{2}[2a + (n-1)d]\right]$$

$$\Rightarrow \quad 3(2a+5d) = 75 \Rightarrow 2a + 5d = 25 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$2a + 3a = 25 \Rightarrow 5a = 25 \Rightarrow a = 5$$

and also, putting $a = 5$ in Eq. (i), we get

$$5d = 3 \times 5 = 15 \Rightarrow d = 3$$

$\therefore$ Required first term = 5 and common difference = 3

(b) Let two natural numbers are respectively x and y (such that $x > y$)

According to question,

$$x - y = 7 \Rightarrow x = 7 + y \quad \text{...(i)}$$

and $\quad xy = 450 \Rightarrow (7+y)y = 450 \quad$ [using Eq. (i)]

$$\Rightarrow \quad 7y + y^2 = 450$$

$$\Rightarrow \quad y^2 + 7y - 450 = 0 \Rightarrow y^2 + 25y - 18y - 450 = 0$$

$$\Rightarrow y(y+25) - 18(y+25) = 0 \Rightarrow (y-18)(y+25) = 0$$

$$\Rightarrow \quad y = 18 \quad [\because y \neq -25]$$

$$\therefore \quad x = 7 + 18 = 25$$

$\therefore$ Required natural number are 25 and 18.

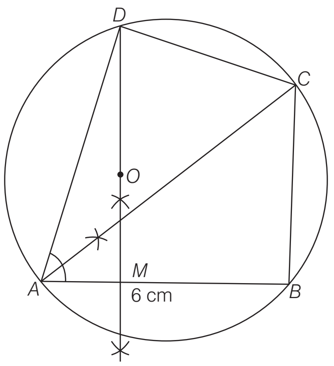
(c) Do similar as Example 8 on page 219.

Ans.

(i) Locus of points equidistant from A and B is the line DM.

(ii) Locus of points equidistant from AB and AD is the line AC.

(iii) Length CD = 5.2 cm



11. (a) Given, scale factor, $k = \frac{1}{50}$

$\therefore \frac{1}{k} = 50$

(i) The height of the actual building

$$= \frac{1}{k}[\text{height of the model}]$$

$$= 50 \times 0.8 = 40 \text{ m}$$

(ii) The floor area of that in the model = k^2 [the floor area of a flat in the building]

$$= \left(\frac{1}{50}\right)^2 (20) = \frac{20}{2500} = 0.008 \text{ m}^2$$

(b) Given, height of solid wooden cylinder is 28 cm and its diameter is 6 cm.

$\therefore$ Radius = 3 cm

Also, diameter of the cones are 6 cm

$\therefore$ Radius = 3 cm and height of cone = 10.5 cm

Volume of the remaining solid

= Volume of solid wooden cylinder − 2 × Volume of each cone

$$= \pi r^2 H - 2 \times \frac{1}{3}\pi r^2 h$$

$$= \pi r^2 \times 28 - 2 \times \frac{1}{3}\pi r^2 \times (10.5)$$

$$= \pi r^2\left[28 - \frac{2}{3} \times 10.5\right] = \frac{22}{7} \times 9\,[28 - 7]$$

$$= \frac{22 \times 9 \times 21}{7} = 594 \text{ cm}^3$$

(c) LHS $= \left(\frac{1 - \tan\theta}{1 - \cot\theta}\right)^2 = \left(\frac{1 - \tan\theta}{1 - 1/\tan\theta}\right)^2$

$$= \left(\frac{1 - \tan\theta}{\frac{\tan\theta - 1}{\tan\theta}}\right)^2 = \tan^2\theta\left(\frac{1 - \tan\theta}{1 - \tan\theta}\right)^2 = \tan^2\theta$$

= RHS **Hence proved.**

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SAMPLE PAPER ISSUED BY CISCE FOR ICSE CLASS X (SEMESTER I)

MATHEMATICS

GENERAL INSTRUCTIONS

1. All Questions are compulsory.
2. The marks intended for questions are given in brackets [ ].
3. Select the correct option for each of the following questions.

Time : 1 Hr 30 Min **Max. Marks : 40**

Section A (16 Marks)

[16×1]

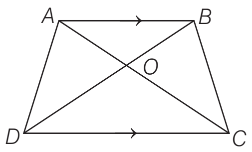
1. If matrix A is of order 3×2 and matrix B is of order 2×2, then the matrix AB is of order

(a) 3×2 (b) 3×1
(c) 2×3 (d) 1×3

2. The percentage share of SGST of total GST for an Intra-State sale of an article is

(a) 25% (b) 50%
(c) 75% (d) 100%

3. $ABCD$ is a trapezium with AB parallel to DC. Then, the triangle similar to ΔAOB is



(a) ΔADB (b) ΔACB
(c) ΔCOD (d) ΔCOB

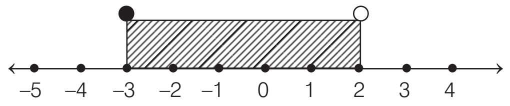
4. The mean proportion between 9 and 16 is

(a) 25 (b) 144
(c) 7 (d) 12

5. A man deposited ₹ 500 per month for 6 months and received ₹ 3300 as the maturity value. The interest received by him is

(a) 1950 (b) 300
(c) 2800 (d) None of these

6. The solution set representing the following number line is



(a) $\{x : x \in R, -3 \le x < 2\}$
(b) $\{x : x \in R, -3 < x < 2\}$
(c) $\{x : x \in R, -3 < x \le 2\}$
(d) $\{x : x \in R, -3 \le x \le 2\}$

7. The first three terms of an arithmetic progression (AP) are 1, 9, 17, then the next two terms are

(a) 25 and 35 (b) 27 and 37
(c) 25 and 33 (d) None of these

8. If $\Delta ABC \sim \Delta QRP$, then the corresponding proportional sides are

(a) $\frac{AB}{QR} = \frac{BC}{RP}$ (b) $\frac{AC}{QR} = \frac{BC}{RP}$

(c) $\frac{AB}{QR} = \frac{BC}{QP}$ (d) $\frac{AB}{PQ} = \frac{BC}{RP}$

9. If $x \in W$, then the solution set of the inequation $-x > -7$, is

(a) {8, 9, 10 ...}
(b) {0, 1, 2, 3, 4, 5, 6}
(c) {0, 1, 2, 3, ...}
(d) {– 8, – 9, – 10 ...}

10. The roots of the quadratic equation $4x^2 - 7x + 2 = 0$ are 1.390, 0.359. The roots correct to 2 significant figures are

(a) 1.39 and 0.36 (b) 1.3 and 0.35
(c) 1.4 and 0.36 (d) 1.390 and 0.360

11. 1.5, 3, x and 8 are in proportion, then x is equal to

(a) 6 (b) 4 (c) 4.5 (d) 16

12. If a polynomial $2x^2 - 7x - 1$ is divided by $(x + 3)$, then the remainder is

(a) -4 (b) 38
(c) -3 (d) 2

13. If 73 is the nth term of the arithmetic progression 3, 8, 13, 18 ..., then 'n' is

(a) 13 (b) 14
(c) 15 (d) 16

14. The roots of the quadratic equation $x^2 + 2x + 1 = 0$ are

(a) Real and distinct (b) Real and equal
(c) Distinct (d) Not real/imaginary

15. Which of the following statement is not true?

(a) All identity matrices are square matrix
(b) All null matrices are square matrix
(c) For a square matrix number of rows in equal to the number of columns
(d) A square matrix all of whose elements except those in the leading diagonal are zero is the diagonal matrix.

16. If $(x - 2)$ is a factor of the polynomial $x^3 + 2x^2 - 13x + k$, then 'k' is equal to

(a) -10 (b) 26
(c) -26 (d) 10

Section B (12 Marks) [6×2]

17. A man deposited ₹ 1200 in a recurring deposit account for 1 year at 5% per annum simple interest. The interest earned by him on maturity is

(a) 14790 (b) 390
(c) 4680 (d) 780

18. If $x^2 - 4$ is a factor of polynomial $x^3 + x^2 - 4x - 4$, then its factor are

(a) $(x - 2)(x + 2)(x + 1)$
(b) $(x - 2)(x + 2)(x - 1)$
(c) $(x - 2)(x - 2)(x + 1)$
(d) $(x - 2)(x - 2)(x - 1)$

19. The following bill shows the GST rates and the marked price of articles A and B

BILL : GENERAL STORE

Articles	Marked price	Rate of GST
A	₹300	12%
B	₹1200	5%

The total amount to be paid for the above bill is

(a) 1548 (b) 1596
(c) 1560 (d) 1536

20. The solution set for the linear inequation $-8 \le x - 7 < -4$, $x \in I$ is

(a) $\{x : x \in R, -1 \le x < 3\}$
(b) $\{0, 1, 2, 3\}$
(c) $\{-1, 0, 1, 2, 3\}$
(d) $\{-1, 0, 1, 2\}$

21. If $\frac{5a}{7b} = \frac{4c}{3d}$, then by componendo and dividendo

(a) $\frac{5a+7b}{5a-7b} = \frac{4c-3d}{4c+3d}$ (b) $\frac{5a-7b}{5a+7b} = \frac{4c+3d}{4c-3d}$
(c) $\frac{5a+7b}{5a-7b} = \frac{4c+3d}{4c-3d}$ (d) $\frac{5a+7b}{5a+7b} = \frac{4c-3d}{4c-3d}$

22. If $A = \begin{bmatrix} 2 & 0 \\ -1 & 7 \end{bmatrix}$, then A^2 is

(a) $\begin{bmatrix} 4 & 0 \\ 1 & 49 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 0 \\ -9 & 49 \end{bmatrix}$
(c) $\begin{bmatrix} 4 & 0 \\ 9 & 49 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 9 \\ -9 & 48 \end{bmatrix}$

Section C (12 Marks) [3×4]

23. The distance between station A and B by road is 240 km and by train it is 300 km. A car starts from station A with a speed x km/h whereas a train starts from station B with a speed 20 km/h more than the speed of the car.

(i) The time taken by car to reach station B is

(a) $\frac{240}{x}$ (b) $\frac{300}{x}$
(c) $\frac{20}{x}$ (d) $\frac{300}{x+20}$

(ii) The time taken by train to reach station A

(a) $\frac{240}{x}$ (b) $\frac{300}{x}$
(c) $\frac{20}{x}$ (d) $\frac{300}{x+20}$

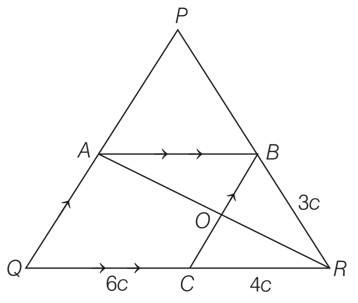
(iii) If the time taken by train is 1 h less than that taken by the car, then the quadratic equation formed is

(a) $x^2 + 80x - 6000 = 0$
(b) $x^2 + 80x - 48000 = 0$
(c) $x^2 + 240x - 1600 = 0$
(d) $x^2 - 80x + 4800 = 0$

(iv) The speed of the car is

(a) 60 km/h (b) 120 km/h
(c) 40 km/h (d) 80 km/h

24. In the given ΔPQR, $AB \| QR$, $QP \| CB$ and AR intersects CB at O.



Using the given diagram answer the following question

(i) The triangle similar to ΔARQ is

(a) ΔORC (b) ΔARP
(c) ΔOBR (d) ΔQRP

(ii) $\Delta PQR \sim \Delta BCR$ by axiom

(a) SAS (b) AAA
(c) SSS (d) AAS

(iii) If $QC = 6$ cm, $CR = 4$ cm, $BR = 3$ cm. The length of RP is

(a) 4.5 cm (b) 8 cm
(c) 7.5 cm (d) 5 cm

(iv) The ratio $PQ : BC$ is

(a) 2 : 3 (b) 3 : 2
(c) 5 : 2 (d) 2 : 5

25. Then nth term of an arithmetic progression (AP) is $(3n + 1)$

(i) The first three terms of this AP are

(a) 5, 6, 7 (b) 3, 6, 9
(c) 1,4, 7 (d) 4, 7, 10

(ii) The common difference of the AP is

(a) 3 (b) 1
(c) – 3 (d) 2

(iii) Which of the following is not a term of this AP?

(a) 25 (b) 27
(c) 28 (d) 31

(iv) Sum of the first 10 terms of this AP is

(a) 350 (b) 175
(c) – 95 (d) 70

ANSWER KEYS

1. (a)	2. (b)	3. (c)	4. (d)	5. (b)	6. (a)	7. (c)	8. (a)	9. (b)	10. (c)
11. (b)	12. (b)	13. (c)	14. (b)	15. (b)	16. (d)	17. (b)	18. (a)	19. (b)	20. (d)
21. (c)	22. (b)								

23. (i) → (a), (ii) → (d), (iii) → (b), (iv) → (c)
24. (i) → (a), (ii) → (b), (iii) → (c), (iv) → (c)
25. (i) → (d), (ii) → (a), (iii) → (b), (iv) → (b)

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SAMPLE PAPER ISSUED BY CISCE FOR ICSE CLASS X (SEMESTER II)

MATHEMATICS

GENERAL INSTRUCTIONS

1. Attempt all questions from Section A and any three questions from Section B.
2. All working, including rough work, must be clearly shown and must be done on the same sheet as the rest of the answer.
3. The intended marks for questions or part of questions are given in brackets [ ].
4. Mathematical table and graph papers are provided.

Time : 1 Hr 30 Min **Max. Marks : 40**

Section A (10 Marks)

*Attempt **all** questions from this Section.*

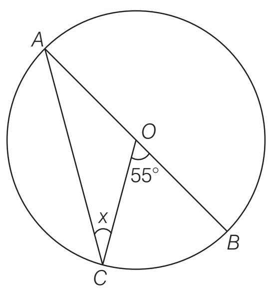
1. Choose the correct answers to the questions from the given options. [10 × 1]

(i) The point (3,0) is invariant under reflection in

(a) the origin (b) X-axis

(c) Y-axis (d) Both X and Y-axes

(ii) In the given figure, AB is a diameter of the circle with centre 'O'. If $\angle COB = 55°$, then the value of x is



(a) 27.5° (b) 55°

(c) 110° (d) 125°

(iii) If a rectangular sheet having dimensions 22 cm × 11 cm is rolled along its shorter side to form a cylinder. Then, the curved surface area of the cylinder so formed is

(a) 968 cm^2 (b) 424 cm^2

(c) 121 cm^2 (d) 242 cm^2

(iv) If the vertices of a triangle are (1, 3), (2, –4) and (–3, 1). Then, the coordinate of its centroid is

(a) (0, 0) (b) (0, 1)

(c) (1, 0) (d) (1, 1)

(v) $\tan\theta \times (\sqrt{1-\sin^2\theta})$ is equal to

(a) $\cos\theta$ (b) $\sin\theta$

(c) $\tan\theta$ (d) $\cot\theta$

(vi) The median class for the given distribution is

Class interval	1-5	6-10	11-15	16-20
Cumulative frequency	2	6	11	18

(a) 1–5 (b) 6–10

(c) 11–15 (d) 11–20

(vii) If the lines $7y = ax + 4$ and $2y = 3 - x$ are parallel to each other, then the value of a is

(a) –1 (b) $\frac{-7}{2}$

(c) $\frac{-2}{7}$ (d) 14

(viii) Volume of a cylinder is 330 cm^3. The volume of the cone having same radius and height as that of the given cylinder is

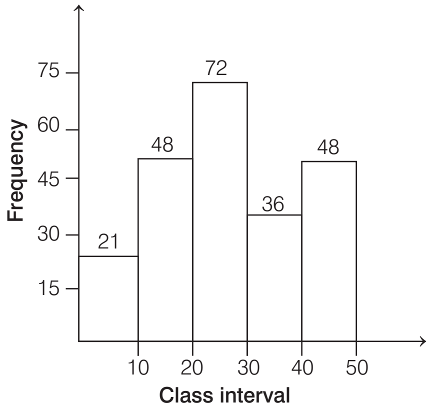
(a) 330 cm^3

(b) 165 cm^3

(c) 110 cm^3

(d) 220 cm^3

(ix) In the given graph, the modal class is the class with frequency



(a) 72 (b) 21
(c) 48 (d) 36

(x) If the probability of a player winning a game is 0.56, then the probability of his losing this game is

(a) 0.56 (b) 1
(c) 0.44 (d) 0

Section B (30 Marks)

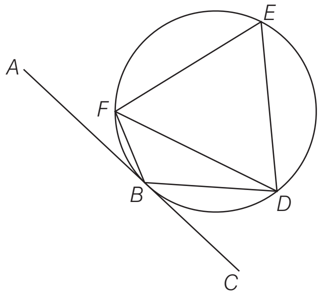
*Attempt any **three** questions from this Section.*

2. (i) Find the ratio, in which the X-axis divides internally the line joining points $A(6, -4)$ and $B(-3, 8)$. [2]

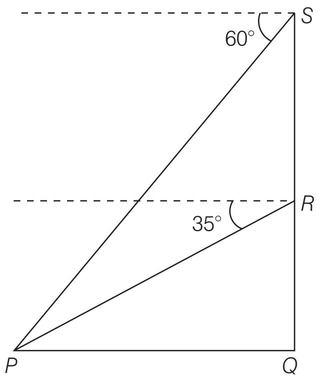
(ii) Three rotten apples are accidently mixed with twelve good ones. One apple is picked at random. What is the probability that it is a good one? [2]

(iii) In the given figure, AC is a tangent to circle at point B. ΔEFD is an equilateral triangle and $\angle CBD = 40°$. Find [3]

(a) $\angle BFD$. (b) $\angle FBD$. (c) $\angle ABF$.

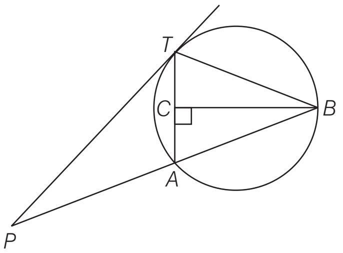


(iv) A drone camera is used to shoot an object P from two different positions R and S along the same vertical line QRS. The angle of depression of the object P from these two positions are 35° and 60° respectively, as shown in the diagram. If the distance of the object P from point Q is 50 m, then find the distance between R and S correct to the nearest metre. [3]



3. (i) In the given figure, PT is a tangent to the circle at T, chord BA is produced to meet the tangent at P. Perpendicular BC bisects the chord TA at C. If $PA = 9$ cm and $TB = 7$ cm, find the length of

(a) AB. (b) PT. [2]



(ii) How many solid right circular cylinders of radius 2 cm and height 3 cm can be made by melting a solid right circular cylinder of diameter 12 cm and height 15 cm? [2]

(iii) Prove that

$$\frac{\cos^2 A}{\cos A - \sin A} + \frac{\sin A}{1 - \cot A} = \sin A + \cos A.$$ [3]

(iv) Use graph paper for this question, take 2 cm = 10 marks along one axis and 2 cm = 10 students along the other axis.

The following table shows the distribution of marks in a 50 marks test in Mathematics

Marks	0-10	10-20	20-30	30-40	40-50
Number of students	6	10	13	7	4

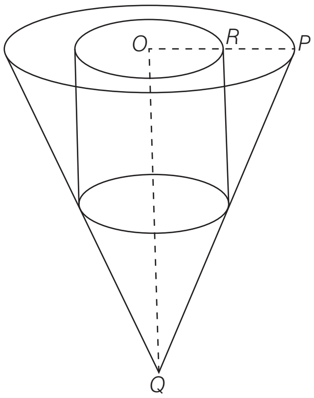
Draw the ogive for the above distribution and hence estimate the median marks. [3]

4. (i) Find the equation of the perpendicular dropped from the point $P(-1, 2)$ onto the line joining $A(1, 4)$ and $B(2, 3)$. [2]

(ii) Find the mean for the following distribution: [2]

Class interval	20-40	40-60	60-80	80-100
Frequency	4	7	6	3

(iii) A solid piece of wooden cone is of radius $OP = 7$ cm and height $OQ = 12$ cm. A cylinder whose radius and height equal to half of that of the cone is drilled out from this piece of wooden cone. Find the volume of the remaining piece of wood. $\left[\text{use, } \pi = \frac{22}{7}\right]$



[3]

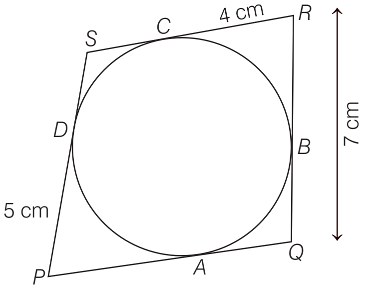
(iv) Use a graph sheet for this question, take 2 cm = 1 unit along both X and Y-axes

(a) Plot the points A (3,2) and B (5,0). Reflect point A on the Y-axis to A'. Write coordinates of A'.

(b) Reflect point B on the Y-axis to B'. Write the coordinates of B'.

(c) Name the closed figure A' B' AB. [3]

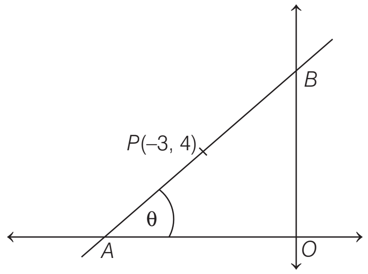
5. (i) In the given figure, the sides of the quadrilateral $PQRS$ touches the circle at A, B, C and D. If $RC = 4$ cm, $RQ = 7$ cm and $PD = 5$ cm. Find the length of PQ.



[2]

(ii) Prove that $\dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = 1 - \sin\theta\cos\theta$. [2]

(iii) In the given diagram, $OA = OB$, $\angle OAB = \theta$ and the line AB passes through point $P(-3, 4)$.



Find

(a) slope and inclination (θ) of the line AB

(b) equation of the line AB [3]

(iv) Use graph paper for this question. Estimate the mode of the given distribution by plotting a histogram. [3]

[take 2 cm = 10 marks along one axis and 2 cm = 5 students along the other axis]

Daily wages (in ₹)	30-40	40-50	50-60	60-70	70-80
Number of workers	6	12	20	15	9

6. (i) A box contains tokens numbered 5 to 16. A token is drawn at random. Find the probability that the token drawn bears a number divisible by

(a) 5

(b) Neither by 2 nor by 3 [2]

(ii) Point M (2, b) is the mid-point of the line segment joining points P (a, 7) and Q (6, 5). Find the values of a and b. [2]

(iii) An aeroplane is flying horizontally along a straight line at a height of 3000 m from the ground at a speed of 160 m/s. Find the time it would take for the angle of elevation of the plane as seen from a particular point on the ground to change from 60° to 45°. Give your answer correct to the nearest second. [3]

(iv) Given that the mean of the following frequency distribution is 30, find the missing frequency f [3]

Class interval	0-10	10-20	20-30	30-40	40-50	50-60
Frequency	4	6	10	f	6	4

ANSWER KEYS

1. (i) (b)
(ii) (a)
(iii) (d)
(iv) (a)
(v) (b)
(vi) (c)
(vii) (b)
(viii) (c)
(ix) (a)
(x) (c)

2. (i) 1: 2
(ii) $\frac{4}{5}$
(iii) (a) 40° (b) 120° (c) 20°
(iv) 52 m (approx)

3. (i) (a) 7 cm (b) 12 cm
(ii) 45
(iv) 24

4. (i) $x - y + 3 = 0$
(ii) 58
(iii) 385 cm^3
(iv) (a) $(-3, 2)$ (b) $(-5, 0)$ (c) trapezium

5. (i) 8 cm
(ii) $1 - \sin\theta\cos\theta$
(iii) $x - y + 7 = 0$
(iv) 21

6. (i) (a) $\frac{1}{4}$ (b) $\frac{1}{3}$
(ii) $a = -2,\ b = 6$
(iii) 8 s
(iv) 10

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ICSE Examination Paper Semester I

2021-22

ICSE EXAMINATION PAPER 2021-22

MATHEMATICS (SEMESTER I)

GENERAL INSTRUCTIONS

1. All Questions are compulsory.
2. The marks intended for questions are given in brackets [ ].
3. Select the correct option for each of the following questions.

Time : 1 Hr 30 Min **Max. Marks : 40**

Section A (16 Marks)

[16×1]

1. If $(x+2)$ is a factor of the polynomial $x^3 - kx^2 - 5x + 6$, then the value of k is

(a) 1 (b) 2
(c) 3 (d) – 2

2. The solution set of the inequation $x - 3 \geq -5$, $x \in R$ is

(a) $\{x : x > -2, x \in R\}$
(b) $\{x : x \leq -2, x \in R\}$
(c) $\{x : x \geq -2, x \in R\}$
(d) $\{-2, -1, 0, 1, 2\}$

3. The product AB of two matrices A and B is possible if

(a) A and B have the same number of rows.
(b) the number of columns of A is equal to the number of rows of B
(c) the number of rows of A is equal to the number of columns of B
(d) A and B have the same number of columns.

4. If 70, 75, 80, 85 are the first four terms of an Arithmetic Progression, then the 10th terms is

(a) 35 (b) 25
(c) 115 (d) 105

5. The selling price of a shirt excluding GST is ₹ 800. If the rate of GST is 12%, then the total price of the shirt is

(a) ₹ 704 (b) ₹ 96
(c) ₹ 896 (d) ₹ 848

6. Which of the following quadratic equations has 2 and 3 as its roots?

(a) $x^2 - 5x + 6 = 0$
(b) $x^2 + 5x + 6 = 0$
(c) $x^2 - 5x - 6 = 0$
(d) $x^2 + 5x - 6 = 0$

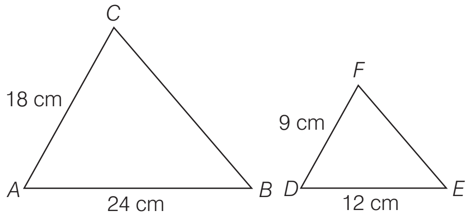
7. If x, 5.4, 5, 9 are in proportion then x is

(a) 3 (b) 9.72
(c) 25 (d) 25/3

8. Mohit opened a recurring deposit account in a bank for 2 yr. He deposits ₹ 1000 every month and receives ₹ 25500 on maturity. The interest he earned in 2 yr is

(a) ₹ 13500 (b) ₹ 3000
(c) ₹ 24000 (d) ₹ 1500

9. In the given figure $AB = 24$ cm, $AC = 18$ cm, $DE = 12$ cm, $DF = 9$ cm and $\angle BAC = \angle EDF$.



Then, $\Delta ABC \sim \Delta DEF$ by the condition

(a) AAA
(b) SAS
(c) SSS
(d) AAS

10. If $A = \begin{bmatrix} 5 & 10 \\ 3 & -4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then AI is equal to

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 10 \\ -3 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 10 \\ 3 & -4 \end{bmatrix}$ (d) $\begin{bmatrix} 15 & 15 \\ -1 & -1 \end{bmatrix}$

11. The polynomial $x^3 - 2x^2 + ax + 12$ when divided by $(x + 1)$ leaves a remainder 20, then 'a' is equal to

(a) -31 (b) 9

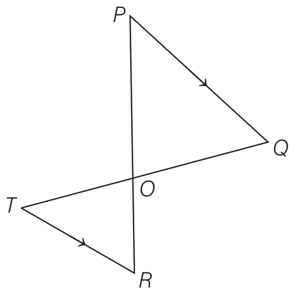
(c) 11 (d) -11

12. In an Arithmetic Progression (AP), if first term is 5, common difference is -3 and the nth term is -7, then n is equal to

(a) 5 (b) 17

(c) -13 (d) 7

13. In the given figure PQ is parallel to TR, then by using condition of similarity



(a) $\frac{PQ}{RT} = \frac{OP}{OT} = \frac{OQ}{OR}$

(b) $\frac{PQ}{RT} = \frac{OP}{OR} = \frac{OQ}{OT}$

(c) $\frac{PQ}{RT} = \frac{OR}{OP} = \frac{OQ}{OT}$

(d) $\frac{PQ}{RT} = \frac{OP}{OR} = \frac{OT}{OQ}$

14. If a, b, c and d are proportional, then $\frac{a+b}{a-b}$ is equal to

(a) $\frac{c}{d}$ (b) $\frac{c-d}{c+d}$

(c) $\frac{d}{c}$ (d) $\frac{c+d}{c-d}$

15. The first four terms of an Arithmetic Progression (AP), whose first term is 4 and common difference is -6, are

(a) $4, -10, -16, -22$

(b) $4, 10, 16, 22$

(c) $4, -2, -8, -14$

(d) $4, 2, 8, 14$

16. One of the roots of the quadratic equation $x^2 - 8x + 5 = 0$ is 7.3166. The root of the equation correct of 4 significant figures is

(a) 7.3166 (b) 7.317

(c) 7.316 (d) 7.32

Section B (12 Marks) [6×2]

17. $(x + 2)$ and $(x + 3)$ are two factors of the polynomial $x^3 + 6x^2 + 11x + 6$. If this polynomial is completely factorised the result is

(a) $(x - 2)(x + 3)(x + 1)$

(b) $(x + 2)(x - 3)(x - 1)$

(c) $(x + 2)(x + 3)(x - 1)$

(d) $(x + 2)(x + 3)(x + 1)$

18. The sum of the first 20 terms of the Arithmetic Progression 2, 4, 6, 8, ... is

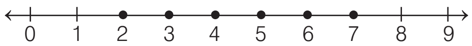
(a) 400 (b) 840

(c) 420 (d) 800

19. The solution set on the number line of the linear inequation.

$$2y - 6 < y + 2 \le 2y,\ y \in N \text{ is}$$

(a)


(b)


(c)


(d)


20. If x, y, z are in continued proportion, then $(y^2 + z^2) : (x^2 + y^2)$ is equal to

(a) $z : x$

(b) $x : z$

(c) zx

(d) $(y + z) : (x + y)$

21. The marked price of an article is ₹ 5000. The shopkeeper gives a discount of 10%. If the rate of GST is 12%, then the amount paid by the customer including GST is

(a) ₹ 5040 (b) ₹ 6100
(c) ₹ 6272 (d) ₹ 6160

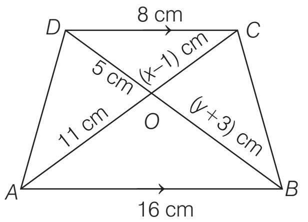
22. If $A = \begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, then $5A - BC$ is equal to

(a) $\begin{bmatrix} -5 & -23 \\ 1 & 17 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 23 \\ 1 & 17 \end{bmatrix}$
(c) $\begin{bmatrix} -2 & 8 \\ -3 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 23 \\ -1 & 17 \end{bmatrix}$

Section C (12 Marks) [3× 4]

23. In the given figure $ABCD$ is a trapezium, which DC is parallel to AB.

$AB = 16$ cm and $DC = 8$ cm, $OD = 5$ cm, $OB = (y + 3)$ cm, $OA = 11$ cm and $OC = (x - 1)$ cm.



Using the given information answer the following questions.

(i) From the given figure name the pair of similar triangles.
(a) ΔOAB, ΔOBC
(b) ΔCOD, ΔAOB
(c) ΔADB, ΔACB
(d) ΔCOD, ΔCOB

(ii) The corresponding proportional sides with respect to the pair of similar triangles obtained in (i)
(a) $\frac{CD}{AB} = \frac{OC}{OA} = \frac{OD}{OB}$
(b) $\frac{AD}{BC} = \frac{OC}{OA} = \frac{OD}{OB}$
(c) $\frac{AD}{BC} = \frac{BD}{AC} = \frac{AB}{DC}$
(d) $\frac{OD}{OB} = \frac{CD}{CB} = \frac{OC}{OA}$

(iii) The ratio of the sides of the pair of similar triangles is
(a) 1 : 3 (b) 1 : 2
(c) 2 : 3 (d) 3 : 1

(iv) Using the ratio of sides of the pair of similar triangles the values of x and y are respectively
(a) $x = 4.6$, $y = 7$
(b) $x = 7$, $y = 7$
(c) $x = 6.5$, $y = 7$
(d) $x = 6.5$, $y = 2$

24. Two cars X and Y use 1 L of diesel to travel x km and $(x + 3)$ km respectively. If both the cars covered a distance of 72 km, then

(i) the number of litres of diesel used by car X is
(a) $\frac{72}{x-3}$ L (b) $\frac{72}{x+3}$ L
(c) $\frac{72}{x}$ L (d) $\frac{12}{x}$ L

(ii) The number of litres of diesel used by car Y is
(a) $\frac{72}{x-3}$ L (b) $\frac{72}{x+3}$ L
(c) $\frac{72}{x}$ L (d) $\frac{12}{x+3}$ L

(iii) If car X used 4 L of diesel more than car Y in the journey, then
(a) $\frac{72}{x-3} - \frac{12}{x} = 4$
(b) $\frac{72}{x+3} - \frac{72}{x} = 4$
(c) $\frac{72}{x} - \frac{72}{x+3} = 4$
(d) $\frac{72}{x-3} - \frac{12}{x+3} = 4$

(iv) The amount of diesel used by the car X is
(a) 6 L (b) 12 L
(c) 18 L (d) 24 L

25. Joseph has a recurring deposit account in a bank for two years at the rate of 8% per annum simple interest.

(i) If at the time of maturity Joseph receives ₹ 2000 as interest, then the monthly instalment is
(a) ₹ 1200 (b) ₹ 600
(c) ₹ 1000 (d) ₹ 1600

(ii) The total amount deposited in the bank
(a) ₹ 25000 (b) ₹ 24000
(c) ₹ 26000 (d) ₹ 23000

(iii) The amount Joseph receives on maturity is
(a) ₹ 27000
(b) ₹ 25000
(c) ₹ 26000
(d) ₹ 28000

(iv) If the monthly instalment is ₹ 100 and the rate of interest is 8%, in how many months Joseph will receive ₹ 52 as interest?
(a) 18 (b) 30
(c) 12 (d) 6

ANSWER KEYS

1. (b)	*2. (c)*	*3. (b)*	*4. (c)*	*5. (c)*	*6. (a)*	*7. (a)*	*8. (d)*	*9. (b)*	*10. (c)*
11. (d)	*12. (a)*	*13. (b)*	*14. (d)*	*15. (c)*	*16. (b)*	*17. (d)*	*18. (c)*	*19. (b)*	*20. (a)*
21. (a)	*22. (d)*								

23. (i) → (b), (ii) → (a), (iii) → (b), (iv) → (c)
24. (i) → (c), (ii) → (b), (iii) → (c), (iv) → (b)
25. (i) → (c), (ii) → (b), (iii) → (c), (iv) → (c)